Investigating Multivariate Tail Dependence in Currency Carry Trade Portfolios via Copula Models

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Abstract

This dissertation investigates the well-known financial puzzle of the currency carry trade, which is yet to be satisfactorily explained. It is one of the most robust financial puzzles in international finance and has attracted the attention of academics and practitioners alike for the past 25 years.

The currency carry trade is the investment strategy that involves selling low interest rate currencies in order to purchase higher interest rate currencies, thus profiting from the interest rate differentials. Assuming foreign exchange risk is uninhibited and the markets have rational risk-neutral investors, then one would not expect profits from such strategies. That is uncovered interest rate parity (UIP); the parity condition in which exposure to foreign exchange risk, with unanticipated changes in exchange rates, should result in an outcome that changes in the exchange rate should offset the potential to profit from such interest rate differentials. The two primary assumptions required for interest rate parity are related to capital mobility and perfect substitutability of domestic and foreign assets. Given foreign exchange market equilibrium, the interest rate parity condition implies that the expected return on domestic assets will equal the exchange rate-adjusted expected return on foreign currency assets. However, it has been shown empirically, that investors can actually earn arbitrage profits by borrowing in a country with a lower interest rate, exchanging for foreign currency, and investing in a foreign country with a higher interest rate, whilst allowing for any losses (or gains) from exchanging back to their domestic currency at maturity. Therefore trading strategies that aim to exploit the interest rate differentials can be profitable on average.

This research comprises of a comprehensive review of the literature surrounding the forward premium puzzle, a mathematical background to copulas and a review of their various uses in the literature to model dependence, followed by an investigation of the forward premium puzzle via analysis of the multivariate tail dependence in currency carry trades. A dataset of daily closes on spot and one month forward contracts for 20 currencies from 2000 to 2013 was used to investigate the behaviour of carry portfolios, formed by sorting on the forward premium (a proxy to the interest rate differential to US dollar). A rigorous statistical modelling approach is proposed, which captures the specific statistical features of both the individual currency log-return distributions as well as the joint features, such as the dependence structures prevailing between the exchange rates. The individual currency returns were transformed to standard uniform margins after fitting appropriately heavy tailed marginal models, namely log-normal and log generalised gamma models. To analyse the tail dependence present in the carry portfolios - mixture copula models, consisting of weighted Clayton, Frank and Gumbel components, were fitted on a rolling daily basis to the previous six months of transformed log returns. Extracting and interpreting the multivariate tail dependence present in the rolling daily baskets provided significant evidence that the average excess returns earned from the carry trade strategy can be attributed to compensation for not only individual currency tail risk, but also exposure to significant risk of large portfolio losses due to joint adverse movements.

The main contribution of this dissertation is therefore to provide a rationale for the unintuitive excess returns seen empirically in the currency carry trade via the presence of multivariate tail dependence and therefore increased portfolio crash risk. This is a novel and promising approach. A further contribution of this research is the identification of significant periods of carry portfolio construction and unwinding through the analysis of multivariate tail dependence in mixture copula models.

The research contained in this dissertation has been presented at the peer reviewed IMA Conference on Mathematics in Finance (8th - 9th April 2013, Edinburgh Conference Centre, Heriot-Watt University), as a poster at the Mathematics of Financial Risk Management Workshop (28th March 2013, Isaac Newton Institute for Mathematical Sciences, Cambridge) and will be presented at the forthcoming "Risk Management Reloaded" conference at the Technical University of Munich (9th - 13th September 2013). This research also forms the base of an extended paper to be presented at the Computational and Financial Econometrics (CFE) 2013 conference in December.

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Chapter 1

Introduction

This chapter presents an overview of the dissertation. The motivation for researching the currency carry trade puzzle is presented, as well as a discussion of previous related work. Subsequently, this chapter describes the objective of this research, i.e. to investigate multivariate tail dependence in currency carry portfolios.

1.1 Motivation

The main motivation of this dissertation is to investigate the well-known forward premium puzzle and the associated currency carry trade. The currency carry trade is the investment strategy that involves selling low interest rate currencies in order to purchase higher interest rate currencies, thus profiting from the interest rate differentials. Assuming foreign exchange risk is uninhibited and the markets have rational risk-neutral investors, then one would not expect profits from such strategies. That is uncovered interest rate parity (UIP); the parity condition in which exposure to foreign exchange risk, with unanticipated changes in exchange rates, should result in an outcome that changes in the exchange rate should offset the potential to profit from such interest rate differentials. The two primary assumptions required for interest rate parity are related to capital mobility and perfect substitutability of domestic and foreign assets. Given foreign exchange market equilibrium, the interest rate parity condition implies that the expected return on domestic assets will equal the exchange rate-adjusted expected return on foreign currency assets. However, it has been shown empirically, that investors can actually earn arbitrage profits by borrowing in a country with a lower interest rate, exchanging for foreign currency, and investing in a foreign country with a higher interest rate, whilst allowing for any losses (or gains) from exchanging back to their domestic currency at maturity. Therefore trading strategies that aim to exploit the interest rate differentials can be profitable on average.

The intention of this dissertation is therefore to reinterpret the currency carry trade puzzle in light of heavy tailed marginal models coupled with multivariate tail dependence features. To achieve this analysis of the multivariate extreme tail dependence several parametric models are developed and detailed model comparison is performed.

This research thus demonstrates that tail dependencies among specific sets of currencies provide other justifications to the carry trade excess return and also allows one to detect construction and unwinding periods of such carry portfolios.

1.2 Related Work

The currency carry trade is one of the most robust financial puzzles in international finance and has attracted the attention of academics and practitioners alike for the past 25 years. Numerous empirical studies [Engel, 1996; Fama, 1984; Hansen and Hodrick, 1980; Lustig and Verdelhan, 2007] have previously demonstrated the excess returns resulting from carry trade strategies.

Such a confounding puzzle has understandably resulted in a vast and varied literature, in which a number of theories have been proposed to justify the phenomenon.

Fama [1984] initially proposed a time varying risk premium within the forward rate relative to the associated spot rate - concluding that, under rational markets, most of the variation in forward rates was due to the variation in risk premium.

Weitzman [2007] demonstrates through a Bayesian approach that the uncertainty about the variance of the future growth rates combined with a thin-tailed prior distribution would generate the fat-tailed distribution required to solve the forward premium puzzle. This could be compared to the argument retained by

Menkhoff et al. [2012] who demonstrate that high interest rate currencies tend to be negatively related to the innovations in global FX volatility, which is considered as a proxy for unexpected changes in the FX market volatility. Menkhoff et al. [2012] show that sorting currencies by their beta with global FX volatility innovations yields portfolios with large differences in returns, and also similar portfolios to those obtained when sorting by forward discount. Another risk factor shown to be significant, although to a much lesser degree, is liquidity risk.

Burnside et al. [2007] presents an alternative model to a pure risk factor model, in which "adverse selection problems between market makers and traders rationalizes a negative covariance between the forward premium and changes in exchange rates". Here, the authors suggest that the foreign exchange market should not be considered as a Walrasian market and that market makers face a worse adverse selection problem when an agent wants to trade against a public information signal, i.e. to place a contrarian bet as an informed trader.

Another hypothesis, proposed by Farhi and Gabaix [2008], consists of justifying this puzzle through the inclusion of a mean reverting risk premium. According to their model a risky country, which is more sensitive to economic extreme events, represents a high risk of currency depreciation and has thus to propose, in order to compensate this risk, a higher interest rate. Then, when the risk premium reverts to the mean, their exchange rate appreciates while they still have a high interest rate which thus replicates the forward rate premium puzzle.

The causality relation between the interest rate differential and the currency shocks can be presented the other way around as detailed in Brunnermeier and Pedersen [2009]. In this paper, the authors assume that the currency carry trade mechanically attracts investors and more specifically speculators who accordingly increase the probability of a market crash. Tail events among currencies would thus be caused by speculators' need to unwind their positions when they get closer to funding constraints.

This recurrent statement of a relation between tail events and forward rate premium [Brunnermeier et al., 2008; Farhi and Gabaix, 2008] has led to the proposal in this dissertation of a rigorous measure and estimation of the tail thickness at the level of the marginal distribution associated to each exchange rate. Moreover, the question of the link between the currency's marginal distribution and the associated interest rate differential leads to the consideration more globally of the joint dependence structures between the individual marginal cdf tails with respect to their respective interest rate differential.

The approach adopted in this dissertation is a statistical framework with a high degree of sophistication, however its fundamental reasoning and justification is indeed analogous in nature to the ideas considered when investigating the "equity risk premium puzzle" coined by Mehra and Prescott [1985] in the late 80's. The equity risk premium puzzle effectively refers to the fact that demand for government bonds which have lower returns than stocks still exists and generally remains high. This poses a puzzle for economists to explain why the magnitude of the disparity between the returns on each of these asset classes, stocks versus bonds, known as the equity risk premium, is so great and therefore implies an implausibly high level of investor risk aversion. In the seminal paper written by Rietz [1988], the author proposes to explain the "equity risk premium puzzle" [Mehra and Prescott, 1985] by taking into consideration the low but still significant probability of a joint catastrophic event.

Analogously in this dissertation, an exploration is presented of the highly leveraged arbitrage opportunities in currency carry trades that arise due to violation of the UIP. However, it is conjectured that if the assessment of the risk associated with such trading strategies was modified to adequately take into account the potential for joint catastrophic risk events accounting for the non-trivial probabilities of joint adverse movements in currency exchange rates, then such strategies may not seem so profitable relative to the risk borne by the investor. A rigorous probabilistic model is proposed in order to quantify this phenomenon and potentially detect when liquidity in FX markets may dry up. This probabilistic measure of dependence can then be very useful for risk management of such portfolios but also for making more tractable the valuation of structured products or other derivatives indexed on this specific strategy. To be more specific, the principal contribution of this dissertation is indeed to model the dependencies between exchange rates using a flexible family of mixture copulae comprised of Archimedean members. This probabilistic approach allows the joint distribution of the vectors of random variables, in this case vectors of exchange rates log-returns in each basket of currencies, to be expressed as functions of each marginal distribution and the copula function itself.

Whereas in the literature mentioned earlier, the tail thickness resulting from the carry trade has been either treated individually for each exchange rate or through the measurement of distribution moments that may not be adapted to a proper estimation of the tail dependencies. In this dissertation, it is proposed instead to build, on a daily basis, a set of portfolios of currencies with regards to the interest rate differentials of each currency with the US dollar. Using a mixture of copula functions, a measure of the tail dependencies within each portfolio is extracted and finally the results are interpreted. Among the outcomes of this study, it is demonstrated that during the crisis periods, the high interest rate currencies tend to display very significant upper tail dependence. Accordingly, it can thus be concluded that the appealing high return profile of a carry portfolio is not only compensating the tail thickness of each individual component probability distribution but also the fact that they tend to occur simultaneously and lead to a portfolio particularly sensitive to the risk of drawdown. Furthermore, it is also shown that high interest rate currency portfolios can display periods during which the tail dependence gets inverted demonstrating when periods of construction of the aforementioned carry positions are being undertaken by investors.

1.3 Investigating Multivariate Tail Dependence in Currency Carry Trade Portfolios via Copula Models

This dissertation includes an investigation of the forward premium puzzle via analysis of the multivariate tail dependence in currency carry trades. A dataset of daily closes on spot and one month forward contracts for 20 currencies from 2000 to 2013 was used to investigate the behaviour of carry portfolios, formed by sorting on the forward premium (a proxy to the interest rate differential to US dollar). A rigorous statistical modelling approach is proposed, which captures the specific statistical features of both the individual currency log-return distributions as well as the joint features, such as the dependence structures prevailing between the exchange rates with regards to their rates differential. The individual currency returns were transformed to standard uniform margins after fitting appropriately heavy tailed marginal models, namely log-normal and log generalised gamma models. To analyse the tail dependence present in the carry portfolios - mixture copula models, consisting of weighted Clayton, Frank and Gumbel components, were fitted on a rolling daily basis to the previous six months of transformed log returns. Extracting and interpreting the multivariate tail dependence present in the rolling daily baskets provided significant evidence that the average excess returns earned from the carry trade strategy can be attributed to compensation for not only individual currency tail risk, but also exposure to significant risk of large portfolio losses due to joint adverse movements.

1.4 Dissertation Structure

This dissertation is structured as follows. A mathematical background and literature review for the field of copula modelling is provided in chapter 2. Chapter 3 presents the theory of the forward premium puzzle and reviews the literature surrounding the puzzle and the associated currency carry trade. Chapter 4 presents the investigation of the forward premium puzzle using real world data. An analysis of multivariate tail dependence in currency carry portfolios is presented, along with a detailed discussion of the results. Finally, conclusions and future research are given in chapter 5 .

Chapter 2

Copula Literature Review

In this chapter, the origins and mathematical background of copulae are reviewed, before discussing the development of copula modelling in the fields of financial mathematics and insurance. Classical measures of dependence are detailed, followed by the concept of tail dependence. Some key copula families are then presented.

2.1 Origins

The explosion of interest in copula modelling over the past few decades can largely be attributed to their flexibility and usefulness in a wide range of practical applications, particularly in the world of finance and insurance, see Genest et al. [2009].

The first mathematical use of the word copula can be traced back to Abel Sklar's theorem in 1959, Sklar [1959], in which one-dimensional distribution functions are *joined together* by a copula function to form multivariate distribution functions. However, the roots of copula theory can in fact be traced back further to Hoeffding's work on 'standardised distributions' on the square $[-\frac{1}{2}, \frac{1}{2}]^2$ in the 1940s, Hoeffding [1994a,b]. A more detailed history of the origins and development of copula theory can be found in the introduction of the excellent monograph Nelsen [2006]. Personal recollections by the founders of the field can be found in Schweizer [1991] and Sklar [1996].

So, why are we interested in copulae? As Fisher notes in his article in the Encyclopedia of Statistical Sciences, Fisher [1997], "Copulas [are] of interest to statisticians for two main reasons:

1. as a way of studying scale-free measures of dependence.
2. as a starting point for constructing families of bivariate distributions, sometimes with a view to simulation."

The most natural place to begin this literature review is with Sklar's introduction of the copula function in his famous theorem, Sklar [1959].

A copula is specified according to the following definition.

Definition 1. ***Copula***
A d-dimensional copula is a multivariate cumulative distribution function C with uniform $[0,1]$ *margins.* *i.e.* $C : [0,1]^d \to [0,1]$.

One of the main attractions for practitioners for the use of copula models is the separation of a multivariate distribution into its marginal distributions and the dependence structure between the margins. Sklar's theorem (2.1) provides the foundation to the study of copulae by proving that any multivariate distribution with continuous margins has a unique copula representation.

Sklar's Theorem (1959)
Consider a d-dimensional cdf H with marginals $F_1, \ldots, F_d$. *There exists a copula C, s.t.*

$$H(x_1, ..., x_d) = C(F_1(x_1), \ldots, F_d(x_d)) \tag{2.1}$$

for all $x_i \in (-\infty, \infty), i \in 1, \ldots, d$. *Furthermore, if* F_i *is continuous for all* $i = 1, \ldots, d$ *then C is unique; otherwise C is uniquely determined only on* $RanF_1 \times \cdots \times RanF_d$, *where* $RanF_i$ *denotes the range of the cdf* F_i.

An intuitive pictorial representation of the transformation of marginal distributions to standard uniform margins can be seen in Figure 2.1, as shown in Meucci [2011]. Here, it can be seen that using the individual empirical CDFs,

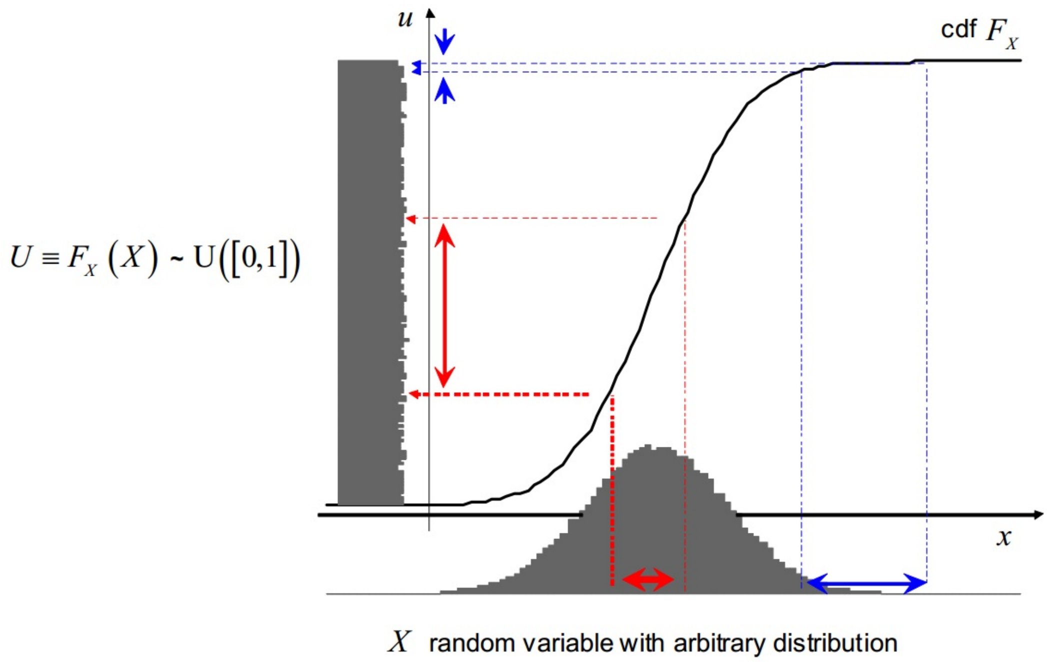


Figure 2.1: Transforming marginal distributions into standard uniform [0,1] margins. (Source: Meucci [2011])

an arbitrary data sample can be transformed to have approximately standard uniform margins.

Copula models therefore provide a mechanism to model the marginal behaviour of each currency and then separately to focus on developing hypotheses regarding the possible dependence structures between the log returns of the forward exchange rates of the currencies in the portfolios which can be tested through parameterization of a model via a copula and then a process of model selection.

There already exists an extensive literature on copulae, with publications gathering pace over recent years. Excellent textbooks on the topic include Aglio et al. [1991]; Joe [1997]; Nelsen [2006]. A number of gentle introductions to the world of copulae are available, such as Bouyé et al. [2000]; Embrechts et al. [2003]; Frees and Valdez [1998]; Genest and Favre [2007]; Meucci [2011]; Schmidt [2006].

2.2 Impact of Copula Modelling on Financial Mathematics

The explosion of interest in copulae, beginning in the eighties, was in most part due to advances in quantitative risk management methodology in the financial and insurance world. The creation of more complex derivative products and new guidelines on regulation (see Chapter 1 of Embrechts et al. [2005]) contributed heavily to the need for risk management developments.

A Notable paper from this era is Embrechts et al. [2002], in which the authors argue for copula approaches over linear correlation for the modelling of dependency for risk management. In particular, the authors point out the pitfalls of using linear correlation in the non-Gaussian world of finance and insurance. Hence, *beyond elliptical multivariate models* we have the following fallacies:

- Fallacy 1 : Marginal distributions and correlation determine the joint distribution.
- Fallacy 2 : Given marginal distributions F_1 and F_2 for X and Y, all linear correlations between -1 and 1 can be attained through suitable specification of the joint distribution.
- Fallacy 3 : The worst case VaR (quantile) for a linear portfolio $X+Y$ occurs when $\rho(X,Y)$ is maximal, i.e. X and Y are comonotonic.

Another hugely important paper, on the topic of credit portfolio default modelling, is Li [1999], in which Li proposes the use of copulae to specify the joint distribution of survival times (time until default of a financial instrument) with given marginal distributions (credit curves - giving all the marginal conditional default probabilities over a number of years). However, Li presents the Gaussian copula as the industry standard approach of the time (see Gupton et al. [1997]). It was the use and abuse of this Gaussian copula by the credit rating agencies (Moody's, Standard & Poor's and Finch) and the derivatives departments of investment banks that allowed the CDS (Credit Default Swap) market to balloon out to \$62 trillion in 2007 from \$920 billion in 2001. The CDO (Collateralised Debt Obligation) market saw a similar explosion, from \$275 billion in 2000 to $4.7 trillion by 2006. Li's formula came under much criticism at the time, notably Salmon [2012], for causing the collapse of the global economy. A more detailed analysis of the development and use of the Gaussian copula in this context is given in MacKenzie and Spears [2012], showing the unjustified blame placed on Li. Donnelly and Embrechts [2010] examines the (well-known) shortcomings of the Gaussian copula - explaining the overly simplistic nature of the model for credit derivatives. The authors present a clear analysis of the challenges of applying mathematical models to the constantly changing real world of finance.

The paper of Schönbucher and Schubert [2001] allows for a much more general specification of the dependency between default events than previous works. The modelling framework introduced here is a continuous-time dynamic model, with defaults and default probabilities evolving consistently within the model. The Clayton and Gumbel copulae are proposed to model the default dependency, allowing for more realistic default contagion.

On the topic of portfolio allocation, Patton [2004] explores asymmetries in the dependence structure of stocks across different market conditions. Patton notes that "stock returns appear to be more dependent during market downturns than during market upturns", hence violating the assumption of elliptically distributed asset returns. Dependence models that allow for, but do not impose, greater dependence during bear markets than bull markets are considered. The author finds substantial evidence that skewness and asymmetric dependence are important considerations in portfolio allocation. In particular, the portfolios based on the more flexible copula dependence models outperform both the equally weighted portfolio and the portfolio based on the bivariate normal model.

Hong et al. [2007] introduces a test for asymmetric dependence and then goes on to propose a Bayesian framework for modelling asymmetry via a mixture model of normal and Clayton copulae. The authors conclude that "incorporating assets' asymmetric characteristics can add substantial economic value in portfolio decisions."

During this period of time there was a huge rise in the application of copula models across many fields, such as hydrology - Genest and Favre [2007], climate research - Schoelzel et al. [2008] and neuroscience - Onken et al. [2009] to name but a few.

Amidst all of this new found excitement for copulae there were some outspoken critics. Most notably was Mikosch [2006a], who cited a concern that copulae were being viewed as *the* solution to all problems in stochastic dependence modelling, whereas in his view "copulas do not contribute to a better understanding of multivariate extremes". There were numerous responses from leaders in the copula field to Mikosch's attack, such as de Haan [2006]; Embrechts [2006]; Genest and Rémillard [2006]; Joe [2006]; Lindner [2006]; Peng [2006] and Segers [2006] - leading to a rejoinder by Mikosch, see Mikosch [2006b]. Embrechts [2009] sums up the responses best in his personal review of copulae shortly after:

> "Copulas form a most useful concept for a lot of applied modeling, they do not yield, however, a panacea for the construction of useful and well-understood multivariate dfs, and much less for multivariate stochastic processes. But none of the copula experts makes these claims."

It is useful at this point to discuss the pros and cons of the copula modelling framework.

PROS:

- Separating out the modelling of the marginals and the dependence structure allows for more flexibility in the complete multivariate model.
- The dependence structure as summarized by a copula is invariant under increasing and continuous transformations of the marginals.
- The tail characteristics within the dependence structure can be explicitly modelled using well-known and interpretable parametric models, e.g. Archimedean copulae.
- High dimensional copulae can be reduced to the composition of lower dimensional building block copulae, e.g. pair-copula constructions, to create extremely flexible models of complex dependence structures.

CONS:

- Which copula to choose? Sometimes it is not easy to say which parametric copula fits a dataset best, since some copulae may provide a better fit near the center and others near the tails. However, by focusing on models with suitable characteristics for the application at hand and using goodness-of-fit tests, e.g. AIC, BIC or CIC, one can overcome this issue.
- As with any statistical model, ignorance on the behalf of practitioners can lead to dangerous oversimplification and reliance on inappropriate models.

Thus, when applying these models in practice it is of the utmost importance to carefully consider the assumptions one is making. The key focus in this research is on combining suitable marginal models, i.e. with the capacity to mode skewness and tail-heaviness flexibly, with a model of the dependence structure that captures the upper and lower multivariate tail characteristics asymmetrically.

In the context investigated in this dissertation, i.e. currency carry trade portfolios, the application of copula models is a novel approach to describe the rationale of the forward premium puzzle.

2.3 Classical Measures of Dependence

Measuring the dependence between random variables has long been of interest to statisticians and practitioners alike. A history of the development of dependency measures can be found in Mari and Kotz [2001]. It is important to note that, in general, the dependence structure between two random variables can *only* be captured in full by their joint probability distribution, and thus any scalar quantity extracted from this structure must be viewed as such. Scarsini [1984] gives the following intuitive definition of dependence:

> "Dependence is a matter of association between X and Y along any measurable function, i.e. the more X and Y tend to cluster around the graph of a function, either $y = f(x)$ or $x = g(y)$, the more they are dependent."

2.3.1 Linear Correlation

The most well-known measure of dependence, Pearson's Product Moment Correlation Coefficient, was developed by Karl Pearson, see Pearson [1896], building on Sir Francis Galton's approach using the median and semi-interquartile range, see Galton [1889].

Pearson's correlation coefficient is a measure of how well the two random variables can be described by a linear function and is defined as follows:

Definition 2. ***Pearson's Correlation Coefficient***

$$\rho := \frac{Cov[X,Y]}{\sqrt{Var[X]Var[Y]}} \tag{2.2}$$

Hence perfect linear dependence gives $\rho = +1$ or $\rho = -1$. The major weakness of linear correlation is its non-invariance under non-linear monotonic transformations of the random variables.

2.3.2 Rank Correlation

Rank correlation measures the relationship between the *rankings* of variables, i.e after assigning the labels "first", "second", "third", etc. to different observations of a particular variable. The coefficient lies in the interval [1, 1], where +1 indicates the agreement between the two rankings is perfect, i.e. the same; -1 indicates the disagreement between the two rankings is perfect, i.e. one ranking is the reverse of the other; 0 indicates the rankings are completely independent. Due to this scale-invariance, rank correlations thus provide an approach for fitting copulae to data.

The choice of dependence measure is influenced by the type of dependence one seeks to capture, such as lower left quadrant, upper right quadrant etc. However, in non-trivial multivariate distributions it isn't possible to capture all of the possible combinations of dependence patterns within a single dependence measure.

2.3.2.1 Spearman's Rho

Charles Spearman introduced the nonparametric measure of dependence, Spearman's rank correlation coefficient, in Spearman [1904]. This measure assesses how well the dependence between two random variables can be described by a monotonic function. As such it is equivalent to the Pearson's correlation coefficient between the ranked variables, defined as follows:

Definition 3. ***Spearman's rank correlation coefficient***

$$\rho := \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_i (x_i - \bar{x})^2 (y_i - \bar{y})^2}} \tag{2.3}$$

where x_i, y_i *are the ranks.*

Spearman's rank correlation can be directly derived from the copula describing the dependence between random variables X_1 and X_2:

$$\rho(X_1, X_2) := 12 \int_0^1 \int_0^1 \left(C\left(u_1, u_2\right) - u_1 u_2\right) du_1 du_2 \tag{2.4}$$

2.3.2.2 Kendall's Tau

Maurice Kendall developed the τ rank correlation coefficient in Kendall [1938], although Gustav Fechner proposed a similar measure in the context of time series in 1897, see Kruskal [1958].

Let (X_1, Y_1) and (X_2, Y_2) be two independent pairs of random variables from a joint distribution function F, then Kendall's rank correlation is given by

Definition 4. ***Kendall's Tau***

$$\tau := \mathbb{P}\left[\left(X_1 - X_2\right)\left(Y_1 - Y_2\right) > 0\right] - \mathbb{P}\left[\left(X_1 - X_2\right)\left(Y_1 - Y_2\right) < 0\right] \tag{2.5}$$

Similarly, Kendall's rank correlation can be directly derived from the copula describing the dependence between random variables X_1 and X_2:

$$\tau := 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1 \tag{2.6}$$

The exact non-linear transformations between the copula parameter ρ and Kendall's rank correlation τ for the Clayton, Frank and Gumbel copulae can be seen in Table 2.1.

Table 2.1: Kendall's tau and tail dependence coefficients.

Family	τ	λ_L	λ_U
Clayton	$\frac{\rho}{\rho+2}$	$2^{-\frac{1}{\rho}}$	0
Frank	$1+\frac{4D_1{}^{1}(\rho)-1}{\rho}$	0	0
Gumbel	$\frac{(\rho-1)}{\rho}$	0	$2-2^{\frac{1}{\rho}}$

Figures 2.2 and 2.3 illustrate the non-linear relationship between the Clayton copula parameter and the Kendall's Tau measure of dependence. Figure 2.2 shows a contour plot for a Clayton copula with $\rho = 8$ and thus $\tau = 0.8$, whereas Figure 2.3 shows a contour plot for a Clayton copula with $\rho = 38$ and thus $\tau = 0.95$. For such a large increase in the copula parameter there is a much smaller increase in Kendall's Tau and also the observable dependence between the variables, as shown by the contour plots, is more similar than perhaps one would expect.

Spearman's ρ and Kendall's τ share a lot of common properties, however "Spearman's ρ is a measure of average quadrant dependence, while Kendall's τ is a measure of average likelihood ratio dependence", see Fredricks and Nelsen [2007]. In layman's terms it can be seen that Kendall's τ penalises rank displacements by the distance of the displacement, whilst Spearman's ρ penalises by the square of the distance. Also, as Newson [2002] notes, "confidence intervals for Spearman's ρ are less reliable and less interpretable than confidence intervals for Kendall's τ-parameters".

[1] $D_1 = \int_0^{\rho} \frac{t}{exp(t)-1} dt/\rho$ is the *Debye function of order one.*

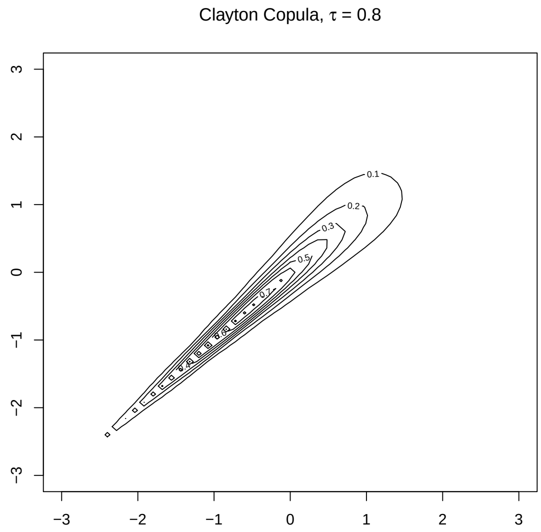


Figure 2.2: Contour plot of Clayton copula with Kendall's $\tau = 0.8$ and copula parameter $\rho = 8$.

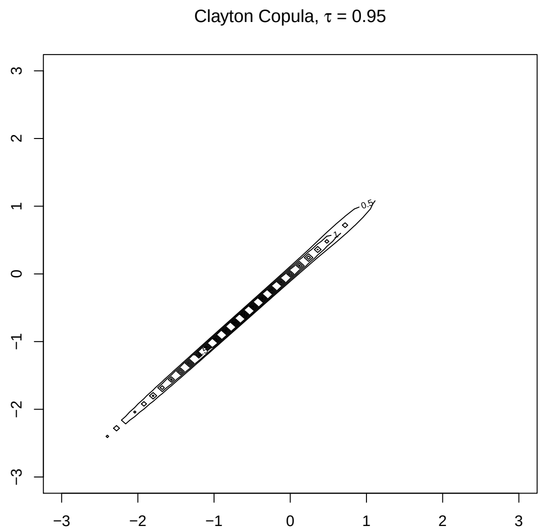


Figure 2.3: Contour plot of Clayton copula with Kendall's $\tau = 0.95$ and copula parameter $\rho = 38$.

2.4 Tail Dependence

In order to examine the dependence behaviour in the extremes of multivariate distributions we use the concept of tail dependence. The bivariate tail dependence coefficient is defined as the conditional probability that a random variable exceeds a certain threshold given that the other random variable in the joint distribution has exceeded this threshold.

Definition 5. ***Bivariate tail dependence***

For random variables X_1 and X_2 with cdfs $F_i, i = 1, 2$ and copula C. We define the coefficient of upper tail dependence by:

$$\lambda_u := \lim_{u \nearrow 1} \mathbb{P}\left(X_2 > F_2^{-1}(u) \,|X_1 > F_1^{-1}(u)\right) = \lim_{u \nearrow 1} \frac{1 - 2u + C(u, u)}{1 - u} \tag{2.7}$$

and similarly we define the coefficient of lower tail dependence by:

$$\lambda_l := \lim_{u \searrow 0} \mathbb{P}\left(X_2 \leq F_2^{-1}(u) \,|X_1 \leq F_1^{-1}(u)\right) = \lim_{u \searrow 0} \frac{C(u, u)}{u} \tag{2.8}$$

This concept has been recently extended to the multivariate setting by De Luca and Rivieccio [2012]. Now one may accurately interpret the tail dependence present between sub-vector partitions of the multivariate random vector with regard to joint tail dependence behaviours. In the context of the applications I consider in this dissertation, this allows us to examine the probability that any subvector of the log return forward exchange rates for the basket of currencies will exceed a certain threshold given that the log return forward exchange rates for the remaining currencies in the basket have exceeded this threshold, in particular thresholds that are placing an interest in the tails of the multivariate distribution. The interpretation of such results is then directly relevant to assessing the chance of large adverse movements in multiple currencies which could potentially increase the risk associated with currency carry trade strategies significantly, compared to risk measures which only consider the marginal behaviour in each individual currency.

Definition 6. ***Multivariate tail dependence***

Let $X = (X_1, ..., X_d)^T$ be a d dimensional random vector with marginal distribution functions $F_1, ..., F_d$ and copula C. We define the coefficient of multivariate upper tail dependence by:

$$\begin{aligned}\lambda_u^{1,\ldots,h|h+1,\ldots,d} &= \lim_{\nu \to 1-} P\left(X_1 > F^{-1}(\nu), \ldots, X_h > F^{-1}(\nu) | X_{h+1} > F^{-1}(\nu), \ldots, X_d > F^{-1}(\nu)\right) \\ &= \lim_{\nu \to 1-} \frac{\bar{C}_n(1-\nu, \ldots, 1-\nu)}{\bar{C}_{n-h}(1-\nu, \ldots, 1-\nu)}\end{aligned} \tag{2.9}$$

where $\bar{C}$ is the survival copula of C.

Similarly we define the coefficient of multivariate lower tail dependence by:

$$\begin{aligned}\lambda_l^{1,\ldots,h|h+1,\ldots,d} &= \lim_{\nu \to 0+} P\left(X_1 < F^{-1}(\nu), \ldots, X_h < F^{-1}(\nu) | X_{h+1} < F^{-1}(\nu), \ldots, X_d < F^{-1}(\nu)\right) \\ &= \lim_{\nu \to 0+} \frac{C_n(\nu, \ldots, \nu)}{C_{n-h}(\nu, \ldots, \nu)}\end{aligned} \tag{2.10}$$

Here, h is the number of variables conditioned on (from the d considered).

2.4.1 Asymptotic Independence

In the case where the extremes of marginal distributions are asymptotically independent one would find the tail dependence coefficient to be zero. Thus applying extreme value models based on non-zero tail dependence to these cases leads to the over-estimation of probabilities of extreme joint events. Examining this class of distributions at finite levels, i.e. non-asymptotic, allows for a more useful measure of extremal dependence. Coles et al. [1999] defines a new quantity $\bar{\chi}$ as given by Equation 2.11.

Definition 7. $\bar{\chi}$ **-** ***Measure of Extremal Dependence***

$$\bar{\chi} := \frac{2 \, log \, Pr(U > u)}{log \, Pr(U > u, V > v)} - 1 = \frac{2 \, log(1-u)}{log \, \bar{C}(u,u)} - 1 \tag{2.11}$$

where $-1 < \bar{\chi}(u) \leq 1$ for all $0 \leq u \leq 1$.

Hence, $\bar{\chi}$ increases with dependence strength and equals 1 for asymptotically dependent variables. For Gaussian models of dependence the measure $\bar{\chi}$ is equal to the correlation, providing a benchmark for interpretation in general models of dependence. Coles et al. [1999] thus argues that using this new measure in addition to the tail dependence measure gives a more complete summary of extremal dependence.

2.5 Decomposing Multivariate Distributions

A statistician faced with the task of modelling a multivariate distribution has a multitude of techniques at his disposal. The simplest possible choice one could make is to assume all of the random variables are independent and hence only the marginals need to be modelled and combined to form the multivariate model. Whilst simple, this approach neglects any dependence between the variables and thus is often a very poor model.

A multivariate distribution may be decomposed in all manner of ways, for example via conditional distributions, factor models, tree representations etc. Barber [2012] is a good resource for exploring the possible methods of decomposing multivariate distributions.

The copula modelling framework provides an intuitive method of constructing a multivariate model by carefully considering the marginal models and then the dependence structure between the random variables in two distinct stages. If the dimension of the multivariate model is not too high (e.g. $d = 5$), then it is reasonable to assume that one d-dimensional copula will be sufficiently flexible to model the characteristics of the dependence structure.

In the application considered in this research a mixture of d-dimensional copulae has been considered to provide a model with asymmetric tail dependence and the capability of capturing negative dependence between the currencies. Since the carry portfolios only contain four currencies, this mixture of 4-dimensional copulae has sufficient flexibility to accurately model the overall dependence structure, and in particular the upper and lower tails.

In cases of much higher dimensional distributions one should consider vine copula models, since standard multivariate copulae do not accommodate different dependency structures between pairs of variables. Vine copulae use bivariate copulae (not necessarily from the same parametric family) and a nested set of trees to build up the overall dependence structure more flexibly. Clearly there is a trade-off with the number of parameters here. Kurowicka and Joe [2011] provides an excellent overview of this burgeoning topic. Some key papers include [Aas et al., 2009; Bedford and Cooke, 2002; Berg and Aas, 2009].

It is worth noting that one key challenge to be tackled in copula modelling is the construction of dynamic models that capture the time-varying nature of dependence in the real world, such as in finance.

2.6 Copula Families

There is a vast collection of different parametric copulae in the literature, each with associated dependence features. The monograph Nelsen [2006] provides a detailed mathematical background of many important copulae. There are many useful papers reviewing the different families of copulae available to the practitioner, such as [Bouyé et al., 2000; Durante and Sempi, 2010; Schmidt, 2006; Trivedi and Zimmer, 2007].

Genest and Neslehova [2007] discusses the issues associated with modelling multivariate distributions with discrete margins, such as in count data. As discussed in Sklar's theorem (2.1), the copula representation of a multivariate distribution is only guaranteed to be unique when the marginal distributions are continuous. This does not present a problem in this dissertation as all of the marginals considered for this application are continuous.

Amongst the most popular copulae are elliptical copulae and Archimedean copulae.

2.6.1 Elliptical Copulae

In general, elliptical copulae arise naturally from their respective elliptical distributions following Sklar's theorem. Although elliptical copulae have no closed form, they have the property that the dependence structure is fully described by the correlation. An elliptical distribution is defined as follows:

Definition 8. *Elliptical distribution*

The density function of an elliptical distributions (if it exists) is given by:

$$f(x) = |\Sigma|^{-\frac{1}{2}} g\left[(x-\mu)^T \Sigma^{-1}(x-\mu)\right] \quad , \quad x \in \mathbb{R}^n \tag{2.12}$$

where Σ *(dispersion) is a symmetric positive semi-definite matrix,* $\mu \in \mathbb{R}^n$ *(location) and* g *(density generator) is a* $[0,\infty) \to [0,\infty)$ *function.*

2.6.1.1 Gaussian Copula

The Gaussian copula has long been favoured by practitioners due to its simplicity. The bivariate Gaussian copula is defined as follows:

Definition 9. *Bivariate Gaussian copula*

$$C^{Gaussian}(u,v) := \Phi_\rho\left(\Phi^{-1}(u), \Phi^{-1}(v)\right) \quad , \tag{2.13}$$

where

$$\Phi_\rho(x,y) := \int_{-\infty}^{x}\int_{-\infty}^{y} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\frac{2\rho st - s^2 - t^2}{2(1-\rho^2)} dsdt$$

and Φ *denotes the standard normal cdf.*

A random sample from a Gaussian copula with $\rho = 0.8$ can be seen in Figure 2.4. The copula density plot for the Gaussian copula with $\rho = 0.3$ can be seen in Figure 2.5. It is important to note the lack of tail dependence in the Gaussian copula, i.e. in the lower left and upper right corners of the unit square. Hence the Gaussian copula is a very restrictive model of dependence in the real world, since it does not allow for variables to become highly concordant in the extremes, e.g. default contagion.

2.6.1.2 t-Copula

Student's t-copula retains much of the simplicity of the Gaussian copula, such as in simulation and calibration, but also allows for the modelling of tail dependence between variables. The behaviour of the model at the four corners is quite different from that of the Gaussian copula, while towards the center they are more

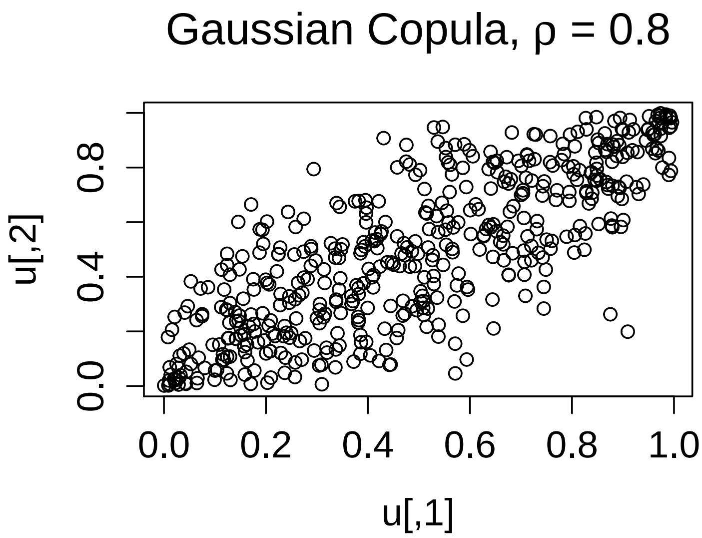


Figure 2.4: Scatterplot of 500 random samples from a Gaussian copula with $\rho = 0.8$.

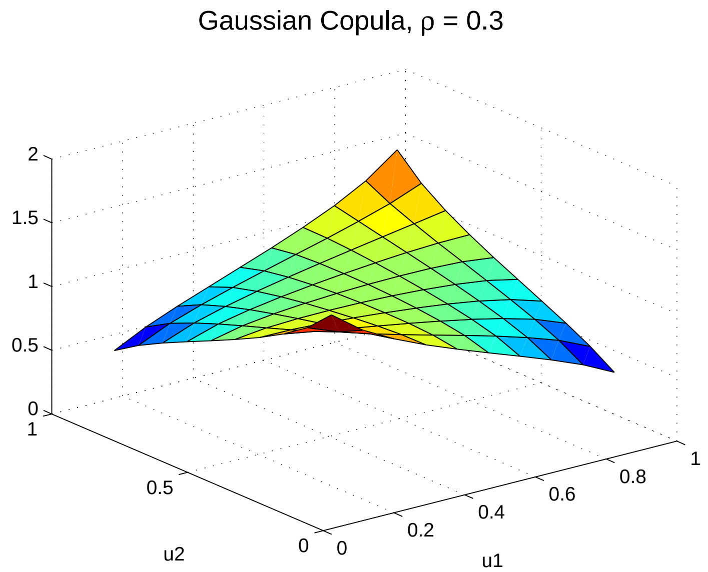


Figure 2.5: Density plot of Gaussian copula with $\rho = 0.3$.

similar, as can be seen in Figure 2.6 and more clearly in the copula density plot in Figure 2.7 with different parameters. Hence, although having the same correlation as the Gaussian copula, the extreme events are much more likely under the t-copula. This copula has often been referred to as the “desert island copula” by Dr. Paul Embrechts due to its excellent fit to multivariate financial return data. However, it does not allow for asymmetry in the tails, i.e. differing upper and lower tail dependence in a portfolio of currencies. The Student's t-Copula is defines as:

Definition 10. ***Student's t-Copula***

$$C^t(u_1, u_2; \nu, \rho) := \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left(1 + \frac{s^2 - 2\rho st + t^2}{\nu(1-\rho^2)}\right)^{-\frac{\nu+2}{2}} dsdt \tag{2.14}$$

where $t_\nu^{-1}(u_i)$ denotes the inverse cdf of the standard univariate t-distribution with ν degrees of freedom.

In practice the use of a standard t copula comes under fire since it has only a single parameter for the degrees of freedom. This may restrict the flexibility in modelling the tail dependence structure in a multivariate case. The most advanced solution in the literature in this regard is Luo and Shevchenko [2010], in which the authors propose a modified grouped t-copula, “where each group consists of one risk factor only, so that a priori grouping is not required”, i.e. each group has only one member and an individual degrees of freedom parameter associated with it.

2.6.2 Archimedean Copulae

Archimedean copulae are not derived from multivariate distributions, but can be stated explicitly in a simple form. Many Archimedean copulae have been proposed in the literature, see Nelsen [2006], with many further copulae available as extensions and combinations of these base copulae. Archimedean copulae are attractive to researchers and practitioners due to their directly interpretable tail dependence features and parsimonious representations.

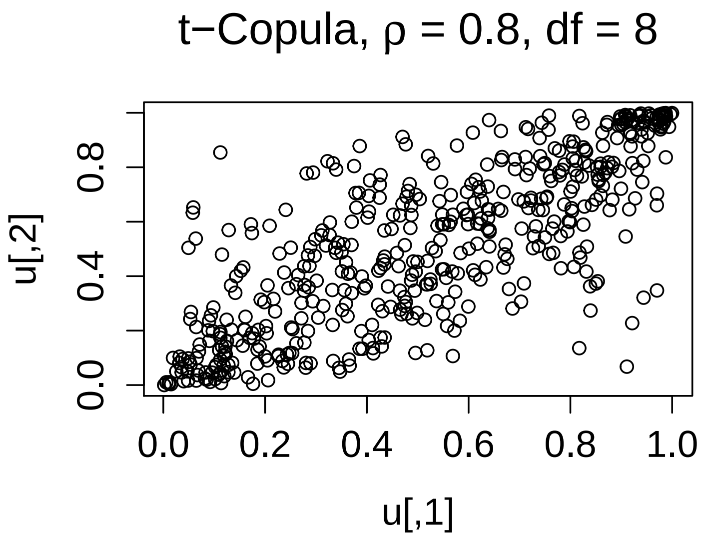


Figure 2.6: Scatterplot of 500 random samples from a t-copula with $\rho = 0.8$, degrees of freedom $= 8$.

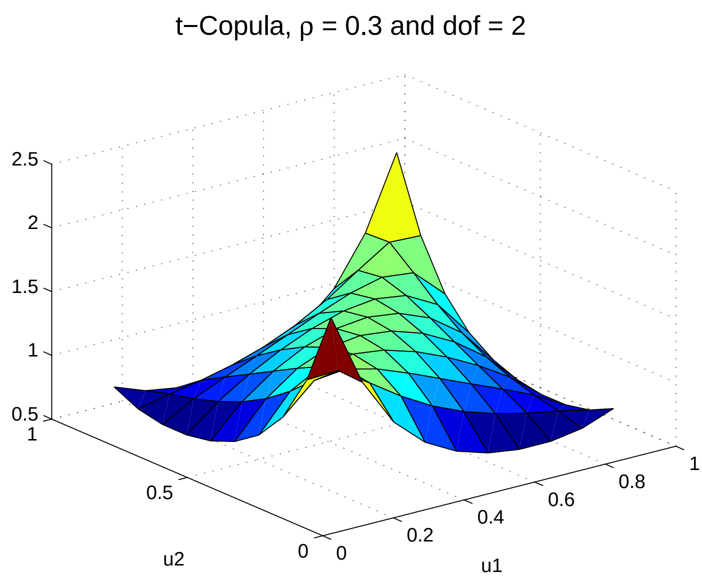


Figure 2.7: Density plot of t-copula with $\rho = 0.3$, degrees of freedom $= 2$.

A function ψ is said to generate an Archimedean copula if it satisfies the properties below.

Definition 11. ***Archimedean Generator***
An Archimedean generator is a continuous, decreasing function $\psi : [0, \infty] \rightarrow [0, 1]$ *which satisfies the following conditions:*

1. $\psi(0) = 1$

2. $\psi(\infty) = lim_{t \rightarrow \infty} \psi(t) = 0$

3. ψ *is strictly decreasing on* $[0, inf\{t : \psi(t) = 0\}]$

Definition 12. ***Archimedean Copula***
A d-dimensional copula C is called Archimedean if for some generator ψ *it can be represented as:*

$$C(\boldsymbol{u}) = \psi\{\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)\} = \psi\{t(\boldsymbol{u})\} \quad \forall \boldsymbol{u} \in [0, 1]^d \tag{2.15}$$

where $\psi^{-1} : [0, 1] \rightarrow [0, \infty]$ *is the inverse generator with* $\psi^{-1}(0) = inf\{t : \psi(t) = 0\}$.

Note the shorthand notation $t(\mathbf{u}) = \psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)$ that will be used throughout this section.

As we will see later, it is necessary to have formulas for computing the copula densities if one seeks to fit these models using a maximum likelihood approach. Equation 2.16 provides such a formula in a generic form for each member of the family of Archimedean copulae.

Definition 13. ***Archimedean Copula Density***
McNeil and Nešlehová [2009] prove that an Archimedean copula C admits a density c if and only if $\psi^{(d-1)}$ *exists and is absolutely continuous on* $(0, \infty)$*. When this condition is satisfied, the copula density c is given by*

$$c(\boldsymbol{u}) \;=\; \frac{\partial^d C(u_1, \ldots, u_d)}{\partial u_1 \ldots \partial u_d} \;=\; \psi^{(d)}\{t(\boldsymbol{u})\} \prod_{j=1}^{d} (\psi^{-1})'(u_j) \;, \quad \boldsymbol{u} \in (0, 1)^d \tag{2.16}$$

There are many possible copula models that could be considered in the modelling of the multivariate dependence features of the currency portfolios. The intention of this analysis was to work with well known models which have well understood tail dependence features and are relatively parsimonious with regard to the number of parameters specifying the copula. I obtain flexible dependence relationships by combining such components into mixture models that allow for a range of flexible tail dependence relationships to be studied.

In particular, I will focus on the well-known class of Archimedean copulae, as defined in (2.15), since they provide a parsimonious approach that allows for the modelling of various tail dependence characteristics.

2.6.2.1 One-parameter Archimedean Members:

In this section I describe three of the one parameter multivariate Archimedean family copula models which have become popular model choices and are widely used for estimation. This is primarily due to there directly interpretable features. I select these three component members, the Clayton, Frank and Gumbel models, for our mixture models since they each contain differing tail dependence characteristics.

Clayton provides lower tail dependence, as seen in the random sample in Figure 2.8 and the copula density plot in Figure 2.9. The Gumbel copula provides upper tail dependence, as seen in the random sample in Figure 2.12 and the copula density plot in Figure 2.13. The Frank copula also provides dependence in the unit cube with elliptical contours with semi-major axis oriented at either $\pi/4$ or $3\pi/4$ depending on the sign of the copula parameter in the estimation. Therefore the Frank model component will allow me to capture parsimoniously potential negative dependence relationships between the currencies in the portfolio under study, as seen in Figure 2.10 and the copula density plot in Figure 2.11.

Formulas for these copulae, as well as their respective generators, inverse generators and the d-th derivatives of their generators (required for the density evaluation) are given in Table 2.2. The explicit formulas for the d-th derivatives for all of the copulae in Table 2.2 were derived in Hofert et al. [2012].

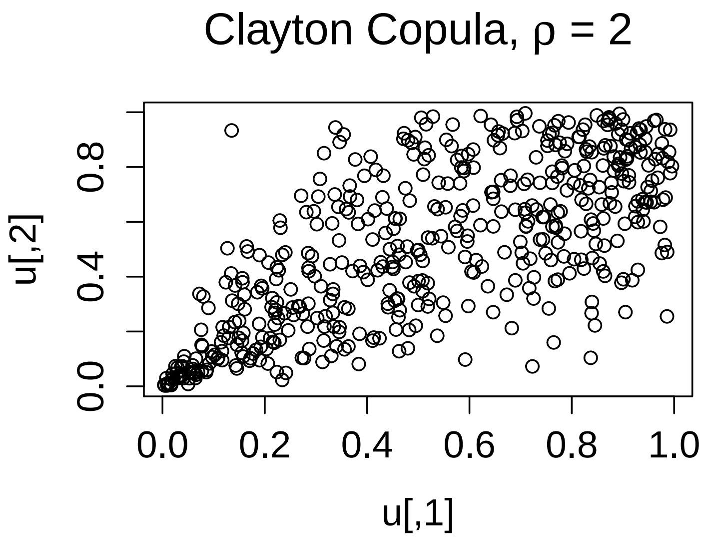


Figure 2.8: Scatterplot of 500 random samples from a Clayton copula with $\rho = 2$.

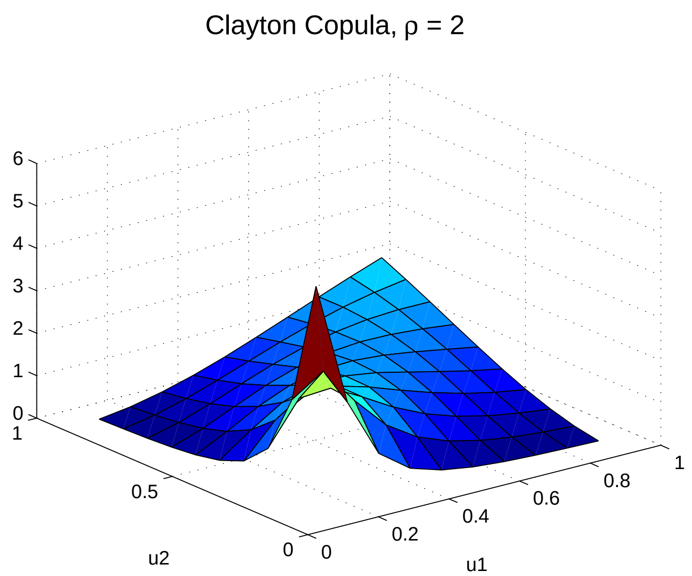


Figure 2.9: Density plot of a Clayton copula with $\rho = 2$.

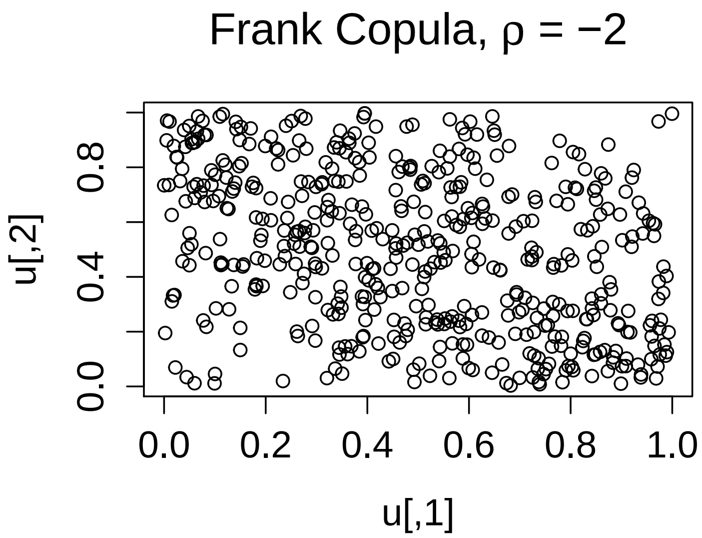


Figure 2.10: Scatterplot of 500 random samples from a Frank copula with $\rho = -2$. The variables show negative dependence here.

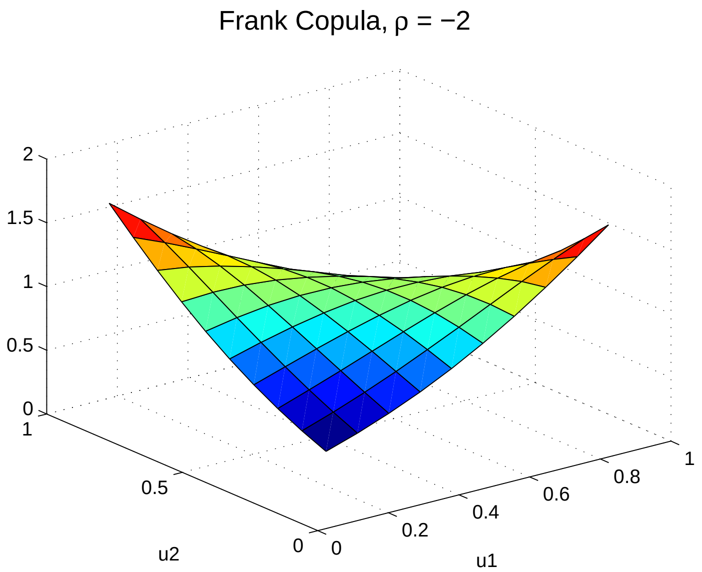


Figure 2.11: Density plot of a Frank copula with $\rho = 2$.

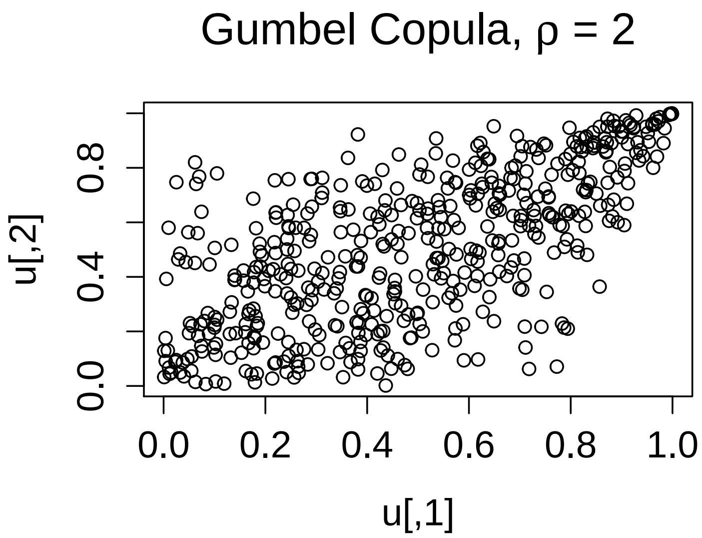


Figure 2.12: Scatterplot of 500 random samples from a Gumbel copula with $\rho = 2$.

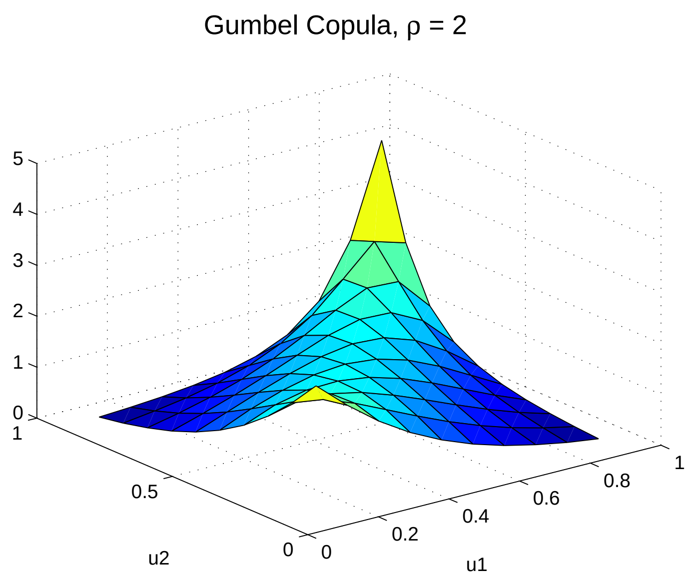


Figure 2.13: Density plot of a Gumbel copula with $\rho = 2$.

2.6.2.2 Two-parameter Archimedean Members via Outer Power Transforms

In this section I also consider more flexible generalizations of the single parameter Archimedean members discussed above. To achieve these generalizations I consider the outer-power transforms of the Clayton, Frank and Gumbel members, as discussed below which is based on a result in Feller [1971].

Definition 14. ***Outer power copula***
The copula family generated by $\tilde{\psi}(t) = \psi(t^{\frac{1}{\beta}})$ is called an outer power family, where $\beta \in [1, \infty)$ and $\psi \in \Psi_\infty$ (the class of completely monotone Archimedean generators).

The proof of this follows from Feller [1971], i.e. the composition of a completely monotone function with a non-negative function that has a completely monotone derivative is again completely monotone. Such copula model transforms were also studied in Nelsen [1997], where they are referred to as a beta family associated with the inverse generator ψ^{-1}.

As has been noted above, in performing the estimation of these transformed copula models via likelihood based inference it will be of great benefit to be capable of performing evaluation pointwise of the copula densities. In the case of the outer power transformed models, this will require the utilization of a specific multivariate chain rule result widely known as the Faà di Bruno's Formula, see Faa di Bruno [1857] and discussions in for example Constantine and Savits [1996] and Roman [1980]. To understand how such a result is required consider the following remark.

Remark: *The generator derivatives for the outer power transforms can be calculated using the base generator derivatives and the following multi-dimensional extension to the chain rule for the outer power versions. The densities for the outer power copulae in Table 2.2 can thus be calculated using equation 2.16.*

Before stating Faà di Bruno's Formula for differentiation of multivariate composite functions via a generalized chain rule, it will be convenient notationally to present such results with respect to Bell polynomials. Therefore we recall the definition of such polynomials below, which are widely used in combinatorics analysis, see Mihoubi [2008] for details.

Definition 15. ***Bell Polynomial***
The Bell polynomial with arguments n and k is given by

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{j_1!j_2!\cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}} \tag{2.17}$$

where the sum is taken over all sequences j_1, j_2, j_{n-k+1} of non-negative integers such that $j_1 + j_2 + \cdots = k$ and $j_1 + 2j_2 + 3j_3 + \cdots = n$.

These polynomials are then utilised to simplify the expressions for the differentiation of multivariate composite functions in Faà di Bruno's Formula as detailed next.

Faà di Bruno's Formula: Riordan [1946]
If f and g are functions with a sufficient number of derivatives, then

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{k=0}^{n} f^{(k)}(g(x)) \cdot B_{n,k}\left(g'(x), g''(x), ..., g^{n-k+1}(x)\right) \tag{2.18}$$

where $B_{n,k}$ are the Bell polynomials, defined above.

2.6.2.3 Two-parameter Archimedean Members via Inner Power Transforms

Definition 16. ***Inner power copula***
The copula family generated by $\tilde{\psi}(t) = \psi^{\frac{1}{\alpha}}(t)$ is called an inner power family, where $\alpha \in (0, \infty)$ and $\psi \in \Psi_\infty$ (the class of completely monotone Archimedean generators).

Inner power transforms produce a family of generators associated with the base generator, e.g. the Clayton generator is the inner power transform of the base generator $\psi(t) = (1 + t)^{-1}$. The lower tail dependence of the transformed copula is $\lambda_L^{1/\alpha}$, whilst the upper tail dependence remains unchanged.

Inner power copula model transforms were also studied in Nelsen [1997], where they are referred to as an alpha family associated with the inverse generator ψ^{-1}.

2.6.3 Multivariate Archimedean Copula Tail Dependence

As discussed in Section 2.4, it is important to be able to accurately interpret the tail dependence present between sub-vector partitions of the multivariate random vector with regard to joint tail dependence behaviours. Below I give the explicit generalised multivariate expressions for Archimedean copulae, equations 2.19 and 2.20, derived in De Luca and Rivieccio [2012].

Definition 17. ***Generalized Archimedean Upper Tail Dependence***
Let $X = (X_1, ..., X_d)^T$ be a d dimensional random vector with marginal distribution functions $F_1, ..., F_d$. The coefficient of upper tail dependence is defined as:

$$\begin{aligned}\lambda_u^{1,\ldots,h|h+1,\ldots,d} &= \lim_{\nu\to 1-} P\left(X_1 > F^{-1}(\nu), ..., X_h > F^{-1}(\nu) | X_{h+1} > F^{-1}(\nu), ..., X_d > F^{-1}(\nu)\right) \\ &= \lim_{t\to 0^+} \frac{\sum_{i=1}^{d}\left(\binom{d}{d-i} i(-1)^i \left[\psi^{-1'}(it)\right]\right)}{\sum_{i=1}^{d-h}\left(\binom{d-h}{d-h-i} i(-1)^i \left[\psi^{-1'}(it)\right]\right)}\end{aligned} \tag{2.19}$$

where $\psi^{-1'}$ is the derivative of the inverse generator. Here, h is the number of variables conditioned on (from the d considered).

Definition 18. ***Generalized Archimedean Lower Tail Dependence***
Let $X = (X_1, ..., X_d)^T$ be a d dimensional random vector with marginal distribution functions $F_1, ..., F_d$. The coefficient of lower tail dependence is defined as:

$$\begin{aligned}\lambda_l^{1,\ldots,h|h+1,\ldots,d} &= \lim_{\nu\to 0+} P\left(X_1 < F^{-1}(\nu), ..., X_h < F^{-1}(\nu) | X_{h+1} < F^{-1}(\nu), ..., X_d < F^{-1}(\nu)\right) \\ &= \lim_{t\to\infty} \frac{d}{d-h} \frac{\psi^{-1'}(dt)}{\psi^{-1'}((d-h)t)}\end{aligned} \tag{2.20}$$

where $\psi^{-1'}$ is the derivative of the inverse generator. Here, h is the number of variables conditioned on (from the d considered).

2.6.4 Mixtures of Archimedean Copulae

In order to add additional flexibility to the possible dependence features available for the currency portfolios, I decided to utilize mixtures of copula models. The advantage of this approach is that I can consider asymmetric dependence relationships in the upper tails and the lower tails in the multivariate model. In addition I can perform a type of model selection purely by incorporating into the estimation the mixture weights associated with each dependence hypothesis. That is the data can be utilised to decide the strength of each dependence feature as interpreted directly through the estimated mixture weight attributed to the feature encoded in the particular mixture component from the Archimedean family.

In particular I have noted that mixture copulae can be used to model asymmetric tail dependence, i.e. by combining the one-parameter or two-parameter families discussed above or indeed by any combination of copulae. This is possible since a linear convex combination of 2 copulae is itself a copula, see discussions on this result in Nelsen [2006].

Definition 19. ***Mixture Copula***
A mixture copula is a linear weighted combination of copulae of the form:

$$C_M(\boldsymbol{u};\Theta) = \sum_{i=1}^{N} \lambda_i C_i(\boldsymbol{u};\theta_i) \tag{2.21}$$

where $0 \leq \lambda_i \leq 1 \;\; \forall i = 1, ..., N$ *and* $\sum_{i=1}^{N} \lambda_i = 1$

Thus we can combine a copula with lower tail dependence, a copula with positive or negative dependence and a copula with upper tail dependence to produce a more flexible copula capable of modelling the multivariate log returns of forward exchange rates of a basket of currencies. For this reason in this analysis I will use the Clayton-Frank-Gumbel mixture model. In addition to the C-F-G mixture model I will also investigate a mixture of outer power versions of the base copula Clayton, Frank and Gumbel.

Remark *We note that the tail dependence of a mixture copula can be obtained as* *the linear weighted combination of the tail dependence of each component in the mixture weighted by the appropriate mixture weight, as discussed in for example Nelsen [2006] and Peters et al. [2012]*

Table 2.2: Archimedean copula generator functions, inverse generator functions and generator function d-th derivatives.

Family	ψ	ψ^{-1}	$(-1)^d\psi^{(d)}$
Clayton	$(1+t)^{-\frac{1}{\rho}}$	$(s^{-\rho}-1)$	$\frac{\Gamma\left(d+\frac{1}{\rho}\right)}{\Gamma\left(\frac{1}{\rho}\right)}(1+t)^{-\left(d+\frac{1}{\rho}\right)}$
OP-Clayton	$\left(1+t^{\frac{1}{\beta}}\right)^{-\frac{1}{\rho}}$	$(s^{-\rho}-1)^{\beta}$	$\frac{\sum_{k=1}^{d} {}^{*}a_{dk}^{G}(\frac{1}{\beta})\frac{\Gamma\left(k+\frac{1}{\rho}\right)}{\Gamma\left(\frac{1}{\rho}\right)}\left(1+t^{\frac{1}{\beta}}\right)^{-\left(k+\frac{1}{\rho}\right)}\left(t^{\frac{1}{\beta}}\right)^{k}}{t^d}$
Frank	$-\frac{1}{\rho}\ln\left[1-e^{-t}(1-e^{-\rho})\right]$	$-\ln\frac{e^{-s\rho}-1}{e^{-\rho}-1}$	$\frac{1}{\rho}{}^{\dagger}Li_{-(d-1)}\{(1-e^{-\rho})e^{-t}\}$
OP-Frank	$-\frac{1}{\rho}\ln\left[1-e^{-t^{\frac{1}{\beta}}}(1-e^{-\rho})\right]$	$\left[-\ln\frac{e^{-s\rho}-1}{e^{-\rho}-1}\right]^{\beta}$	$\frac{\sum_{k=1}^{d} a_{dk}^{G}\left(\frac{1}{\beta}\right)\frac{1}{\rho}Li_{-(k-1)}\left\{(1-e^{-\rho})e^{-t^{\frac{1}{\beta}}}\right\}\left(t^{\frac{1}{\beta}}\right)^{k}}{t^d}$
Gumbel	$e^{-t^{\frac{1}{\rho}}}$	$(-\ln s)^{\rho}$	$\frac{\psi_\rho(t)}{t^d}{}^{\ddagger}P_{d,\frac{1}{\rho}}^{G}\left(t^{\frac{1}{\rho}}\right)$
OP-Gumbel	$e^{-t^{\frac{1}{\beta\rho}}}$	$(-\ln s)^{\rho\beta}$	$\frac{\sum_{k=1}^{d} a_{dk}^{G}\left(\frac{1}{\beta}\right)\frac{\psi_\rho\left(t^{\frac{1}{\beta}}\right)}{t^{\frac{k}{\beta}}}P_{k,\frac{1}{\rho}}^{G}\left(t^{\frac{1}{\rho\beta}}\right)\left(t^{\frac{1}{\beta}}\right)^{k}}{t^d}$

Remark: The densities for the one-parameter copulae in Table 2.2 can be calculated using equation 2.16. For details of the results contained in this table see Hofert et al. [2012].

$^{*}a_{dk}^{G}(\frac{1}{\rho}) = \frac{d!}{k!}\sum_{i=1}^{k}\binom{k}{i}\binom{i/\rho}{d}(-1)^{d-i}\ , \quad k \in 1, ..., d$

$^{\dagger}Li_s(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^s}$

$^{\ddagger}P_{d,\frac{1}{\rho}}^{G}\left(t^{\frac{1}{\rho}}\right) = \sum_{k=1}^{d} a_{dk}^{G}\left(\frac{1}{\rho}\right)(t^{\frac{1}{\rho}})^k$

Chapter 3

Carry Trade Literature Review

In this chapter, the forward premium puzzle is presented and then the literature surrounding the puzzle and the associated currency carry trade is reviewed . The novel approach of analysing both individual tail thickness and joint tail dependence, as proposed in this dissertation, is discussed.

3.1 The Forward Premium Puzzle

This phenomenon introduced initially by Hansen and Hodrick [1980], Hansen and Hodrick [1983], Fama [1984] and Engel [1984] is directly linked to the arbitrage relation existing between the spot and the forward prices of a given currency, namely the Covered Interest Parity. This relation states that the price of a forward rate can be expressed according to the relationship:

$$F_t^T = e^{(r_t - r_t^f)(T-t)} S_t \tag{3.1}$$

where F_t^T and S_t denote respectively the forward and the spot prices at time t. While r_t and r_t^f represent the local risk free rate* and the foreign risk free rate. I denote by T the maturity of the forward contract considered. It is worth emphasizing that under the absence of an arbitrage hypothesis, this relation is directly resulting from the replication of the forward contract payoff using a self

*I mean by local risk free rate the interest rate prevailing in the reference country which would be for instance the dollar for an American investor.

financed strategy. Moreover, it has been demonstrated empirically the validity of this arbitrage relation in the currency market [Akram et al., 2008; Juhl et al., 2006] when we consider daily data. Then, if we take the logarithm of expression (3.1) we thus obtain the following relation:

$$\begin{aligned} f_t^T - s_T &= (r_t - r_t^f)\tau - (s_T - s_t) \\ &= (r_t - r_t^f)\tau - \Delta s_T \end{aligned} \tag{3.2}$$

where τ is the time to maturity from calendar day t given by $(T - t)$ while f_t^T and s_t denote respectively the log-values of the forward and the spot prices.

This expression notably represents the excess return received by an investor who has invested one unit of local currency* in a forward contract F_t^T of maturity T at time t and has held this position until expiration to convert it back to his reference currency with an exchange rate equal to S_T. We can notice that at the trade settlement the profit or the loss in local currency for this investor equals the differential of interest rates on a prorata temporis basis, plus the differential of exchange rate between t and T.

Finally, if we assume the forward price is a martingale under the risk neutral probability [Musiela and Rutkowski, 2011] then its value equals:

$$E_{\mathbb{Q}}[S_T|\mathcal{F}_t] = F_t^T \tag{3.3}$$

where $\mathcal{F}_t$ is the filtration associated to the stochastic process S_t. Replacing the expression (3.3) in the relation (3.1) leads to the formula:

$$E_{\mathbb{Q}}\left[\frac{S_T}{S_t}\middle|\mathcal{F}_t\right] = \frac{F_t^T}{S_t} = e^{(r_t - r_t^f)(T-t)} \tag{3.4}$$

Which leads then to the UIP hypothesis since according to this expression, under the risk neutral probability the expected variation of the exchange rate S_t should equal the differential of interest rates between the two countries. Thus if an investor buys a forward contract the profit or the loss resulting at the contract

*Which corresponds in this case to the US dollar since I consider the position of an American investor.

maturity should accordingly be equal to zero as the exchange rate at maturity S_T should be equal to $S_t e^{(r_t - r_t^f)(T-t)}$ which was the price paid initially for the forward contract under the hypothesis of absence of arbitrage opportunities.

3.2 Currency Carry Trade

Numerous empirical studies [Engel, 1996; Fama, 1984; Hansen and Hodrick, 1980; Lustig and Verdelhan, 2007] have previously demonstrated, that investors can actually earn arbitrage profits by borrowing in a country with a lower interest rate, exchanging for foreign currency, and investing in a foreign country with a higher interest rate, whilst allowing for any losses (or gains) from exchanging back to their domestic currency at maturity. Therefore, trading strategies that aim to exploit the interest rate differentials can be profitable on average. This is notably the case for the currency carry trade which is thus the simple investment strategy of selling a low interest rate currency forward and then buying a high interest rate currency forward. The idea is that the interest rate returns will outweigh any potential adverse moves in the exchange rate. Historically the Japanese yen and Swiss franc have been used as "funding currencies", since they have maintained very low interest rates for a long period. The currencies of developing nations, such as the South African rand and Brazilian real have been typically used as "investment currencies". Whilst this sounds like an easy money making strategy there is of course a downside risk. This risk comes in the form of currency crashes in periods of global FX volatility and liquidity shortages. A prime example of this is the sharp yen carry trade reversal in 2007.

3.3 A Review of the Literature

If the UIP relationship held then there should indeed not be any yield difference between a risk-free investment in a reference currency and a risk-free investment in another currency after converting it back to the reference currency. Accordingly, the depreciation of a currency relative to another should be equal to the risk free interest rates differential between them. However, Hansen and Hodrick [1980], Hansen and Hodrick [1983] and Fama [1984] among other recent articles [Lustig and Verdelhan, 2007; Lustig et al., 2011; Menkhoff et al., 2012], demonstrate that this relation is not observed empirically in markets data and that the "currency carry trade" strategy discussed above can even benefit from this flaw.

Over the last few decades there have been many theories proposed for the justification of this phenomenon. Fama [1984] initially proposed a time varying risk premium within the forward rate relative to the associated spot rate - concluding that, under rational markets, most of the variation in forward rates was due to the variation in risk premium.

Weitzman [2007] demonstrates through a Bayesian approach that the uncertainty about the variance of the future growth rates combined with a thin-tailed prior distribution would generate the fat-tailed distribution required to solve the forward premium puzzle. This could be compared to the argument retained by Menkhoff et al. [2012] who demonstrate that high interest rate currencies tend to be negatively related to the innovations in global FX volatility, which is considered as a proxy for unexpected changes in the FX market volatility. Menkhoff et al. [2012] show that sorting currencies by their beta with global FX volatility innovations yields portfolios with large differences in returns, and also similar portfolios to those obtained when sorting by forward discount. Another risk factor shown to be significant, although to a much lesser degree, is liquidity risk.

Burnside et al. [2007] presents an alternative model to a pure risk factor model, in which "adverse selection problems between market makers and traders rationalizes a negative covariance between the forward premium and changes in exchange rates". Here, the authors suggest that the foreign exchange market should not be considered as a Walrasian market and that market makers face a worse adverse selection problem when an agent wants to trade against a public information signal, i.e. to place a contrarian bet as an informed trader.

Another hypothesis, proposed by Farhi and Gabaix [2008], consists of justifying this puzzle through the inclusion of a mean reverting risk premium. According to their model a risky country, which is more sensitive to economic extreme events, represents a high risk of currency depreciation and has thus to propose, in order to compensate this risk, a higher interest rate. Then, when the risk premium reverts to the mean, their exchange rate appreciates while they still have a high interest rate which thus replicates the forward rate premium puzzle.

The causality relation between the interest rate differential and the currency shocks can be presented the other way around as detailed in Brunnermeier and Pedersen [2009]. In this paper, the authors indeed assume that the currency carry trade mechanically attracts investors and more specifically speculators who accordingly increase the probability of a market crash. Tail events among currencies would thus be caused by speculators' need to unwind their positions when they get closer to funding constraints.

This recurrent statement of a relation between tail events and forward rate premium [Brunnermeier et al., 2008; Farhi and Gabaix, 2008] has led to the proposal in this dissertation of a rigorous measure and estimation of the tail thickness at the level of the marginal distribution associated to each exchange rate. Moreover, the question of the link between the currency's marginal distribution and the associated interest rates differential leads to the consideration more globally of the joint dependence structures between the individual marginal cdf tails with respect to their respective interest rate differential.

3.3.1 Research Contribution: Tail Dependence and Forward Premium Puzzle

The approach adopted in this dissertation is a statistical framework with a high degree of sophistication, however its fundamental reasoning and justification is indeed analogous in nature to the ideas considered when investigating the "equity risk premium puzzle" coined by Mehra and Prescott [1985] in the late 80's. The equity risk premium puzzle effectively refers to the fact that demand for government bonds which have lower returns than stocks still exists and generally remains high. This poses a puzzle for economists to explain why the magnitude of the disparity between the returns on each of these asset classes, stocks versus bonds, known as the equity risk premium, is so great and therefore implies an implausibly high level of investor risk aversion. In the seminal paper written by Rietz [1988], the author proposes to explain the "equity risk premium puzzle" [Mehra and Prescott, 1985] by taking into consideration the low but still significant probability of a joint catastrophic event.

Analogously in this dissertation, an exploration is presented of the highly leveraged arbitrage opportunities in currency carry trades that arise due to violation of the UIP. However, it is conjectured that if the assessment of the risk associated with such trading strategies was modified to adequately take into account the potential for joint catastrophic risk events accounting for the non-trivial probabilities of joint adverse movements in currency exchange rates, then such strategies may not seem so profitable relative to the risk borne by the investor. A rigorous probabilistic model is proposed in order to quantify this phenomenon and potentially detect when liquidity in FX markets may dry up. This probabilistic measure of dependence can then be very useful for risk management of such portfolios but also for making more tractable the valuation of structured products or other derivatives indexed on this specific strategy. To be more specific, the principal contribution of this dissertation is indeed to model the dependencies between exchange rates using a flexible family of mixture copulae comprised of Archimedean members. This probabilistic approach allows the joint distribution of the vectors of random variables, in this case vectors of exchange rates log-returns in each basket of currencies, to be expressed as functions of each marginal distribution and the copula function itself.

Whereas in the literature mentioned earlier, the tail thickness resulting from the carry trade has been either treated individually for each exchange rate or through the measurement of distribution moments that may not be adapted to a proper estimation of the tail dependencies. In this dissertation, it is proposed instead to build, on a daily basis, a set of portfolios of currencies with regards to the interest rate differentials of each currency with the US dollar. Using a mixture of copula functions, a measure of the tail dependencies within each portfolio is extracted and finally the results are interpreted. Among the outcomes of this study, it is demonstrated that during the crisis periods, the high interest rate currencies tend to display very significant upper tail dependence. Accordingly, it can thus be concluded that the appealing high return profile of a carry portfolio is not only compensating the tail thickness of each individual component probability distribution but also the fact that they tend to occur simultaneously and lead to a portfolio particularly sensitive to the risk of drawdown. Furthermore, it is also shown that high interest rate currency portfolios can display periods during which the tail dependence gets inverted demonstrating when periods of construction of the aforementioned carry positions are being undertaken by investors.

Chapter 4

Investigating Multivariate Tail Dependence in Currency Carry Trade Portfolios via Copula Models

This chapter presents the investigation of the forward premium puzzle using real world data. Analysis of multivariate tail dependence in currency carry trade portfolios is detailed and the results discussed.

4.1 Data Description and Portfolios Construction

In this section, I describe the set of data used for this empirical study and describe the macro-economic specificities associated to some of the currencies I considered. Furthermore, I present the method I retained in this dissertation to build the portfolios that are combined later on to build a carry trade position.

4.1.1 Data Description

I consider for this empirical analysis a set of 20 currency exchange rates relative to the USD. I indeed considered the point of view of an American investor as this is generally the hypothesis retained in the literature [Brunnermeier et al., 2008; Menkhoff et al., 2012]. However the same analysis could be carried out from any other investor standpoint as the phenomenon I will describe does not only depend on a specific currency but more on two sets of currencies. These sets of currencies correspond to the high interest rate currencies which are used to obtain the highest return (named the "investment currencies") and the low interest rate currencies which allows for borrowing at a low cost the amount of money necessary for this investment (named the "financing currencies").

The time series analysed range from 04/01/2000 to 02/01/2013 and comprise the following currencies: Euro (EUR), Turkish lira (TRY), Japanese yen (JPY), British pound sterling (GBP), Australian dollar (AUD), Canadian dollar (CAD), Norwegian krone (NOK), Swiss franc (CHF), Swedish krona (SEK), Mexican peso (MXN), Polish zloty (PLN), Malaysian ringgit (MYR), Singaporean dollar (SGD), Indian rupee (INR), South African rand (ZAR), New Zealand dollar (NZD), Thai baht (THB), South Korean won (KRW), Taiwanese dollar (TWD), Brazilian real (BRL). I have been provided, on a daily basis, with the settlement prices for each currency exchange rate as well as the simultaneous price for the associated 1 month forward contract. Due to differing market closing days, e.g. national holidays, there was missing data for a couple of currencies and for a small number of days. For missing prices, the previous day's closing prices were retained.

The reason why I based this analysis upon a constant maturity 1 month forward is twofold. Firstly, I do not try in this investigation to replicate as realistically as possible a currency carry trade portfolio to see if there is a recurrently high average return. The main inconvenience of such analysis comes from the loss of data points. As a matter of fact, to build a carry portfolio, the position has to be held until the maturity of the forward contract which leads in this case to retain only one point for each month. However, in this case I have at my disposal one point per day which makes this analysis of individual tails and their interdependencies more robust. Secondly, tail behaviour of monthly data is naturally different from the tail behaviour of daily data, one reason for this difference is that individual currencies can display a mean reversion in the mid-term and thus reduce the amplitude of the movement.

Among the currencies under scrutiny, some of them have displayed very large variations in the last decade mainly for macro-economic reasons. Therefore, I considered it insightful to mention some of the most meaningful. The Brazilian real displays in its time series two important periods of shocks, the first in 2001 and the second in 2002. Naturally the first of them was due to the terrorist attacks against the world trade center in September. However the Brazilian real has been also impacted by the market's concerns of a contagion after the rumours of default of the Argentinian government. The second shock on the Brazilian real in 2002 was related to the potential election of the Workers' Party leader Luiz Inacio Lula da Silva which prompted concern he might spark a default by overspending to meet promises of spurring growth and employment. In 2001, the South African rand slumped 29% after the events of September 11 and the market's concern of a global recession and a slump in commodity prices to which the South African economy is particularly exposed to. As a third example of a shock in an instrumental currency in a carry trade strategy we note the 30% daily loss of the Turkish lira on the 22nd of February 2001. This was due to Turkey's decision to abandon the defense of their currency in order to reduce the cost of financing lira-denominated debt. It is worth mentioning that I did not remove these data points from the time series given that different events may have impacted the other exchange rates at a different time but this analysis does not focus only on the tail events associated to a particular currency but more on the events impacting simultaneously a set of currencies.

4.1.2 Currency Portfolios Formation

As described earlier, the currency carry trade results from the differential of interest rates prevailing in different countries. By borrowing a certain amount of money in low interest rate countries and investing it in high interest rate countries, a recurrent profit can be generated given that the UIP condition is on average not satisfied. In order to differentiate the "financing currencies" from the "investment currencies", I start by classifying each currency relative to its differential of risk free rate with the US dollar. We note the following basic explanation of the high rates and low rates. In general countries that are considered 'safe' can borrow at a lower interest rate, which may explain why historically the US dollar or Swiss franc interest rates were low [Gourinchas and Rey, 2007] while the Turkish lira rates were historically high as this country is not considered as financially secure.

Moreover I demonstrated in expression (3.4) that the differential of interest rates between two countries can be estimated through the ratio of the forward contract price and the spot price. Accordingly, instead of considering the differential of risk free rates between the reference and the foreign countries, I build the respective baskets of currencies with respect to the ratio of the forward and the spot prices for each currency. On a daily basis I compute this ratio for each currency and then build five portfolios of four currencies each. The first portfolio gathers the four currencies with the highest positive differential of interest rate with the US dollar. The selected currencies over the period 04/01/2000 to 02/01/2013 for the high interest rate basket are displayed in Figure 4.1. These currencies are thus representing the "investment" currencies, through which we invest the money to benefit from the currency carry trade. The last portfolio will gather the four currencies with the highest negative differential (or at least the lowest differential) of interest rate. As with the high interest rate basket, I also display the low interest interest rate currency selections in Figure 4.2. These currencies are thus representing the "financing" currencies, through which we borrow the money to build the currency carry trade.

Conditionally to this classification I investigate then the joint distribution of each group of currencies to understand the impact of the currency carry trade, embodied by the differential of interest rates, on currencies returns. In our analysis I concentrate on the high interest rate basket (investment currencies) and the low interest rate basket (funding currencies), since typically when implementing a carry trade strategy one would go short the low basket and go long the high basket.

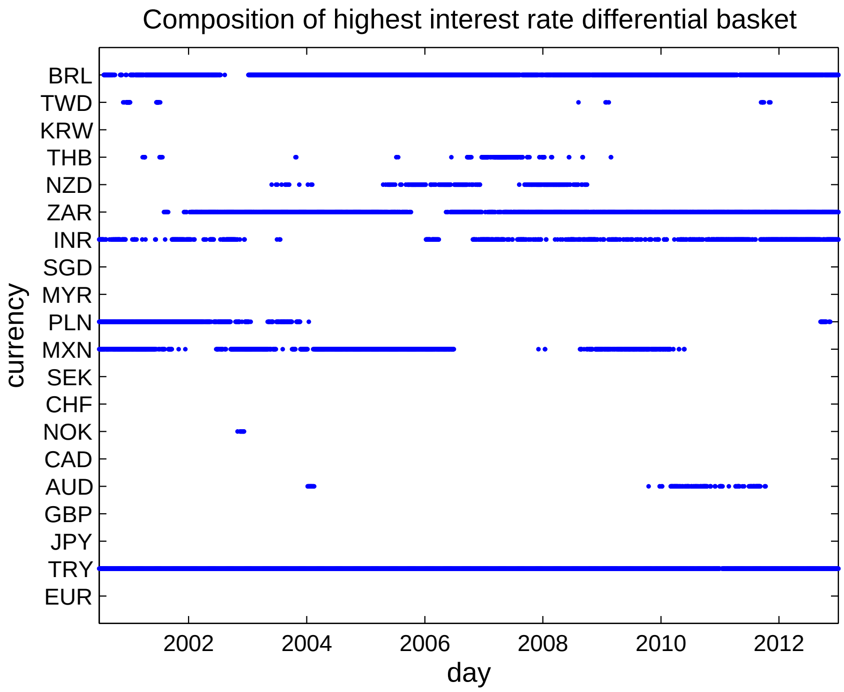


Figure 4.1: Basket 5 (highest IR) composition.

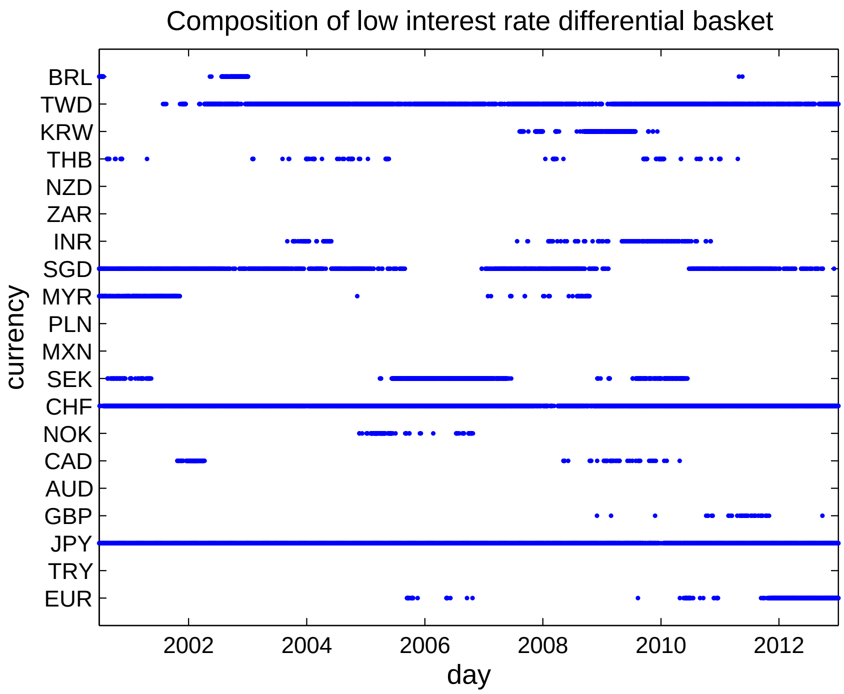


Figure 4.2: Basket 1 (lowest IR) composition.

4.2 Likelihood Based Estimation of the Mixture Copula Models

Let me begin this section with a discussion on the choices I make for the marginal distributions for each of the currencies specified in the baskets constructed for the high interest rate differentials and also the baskets for the low interest rate differentials.

In modelling parametrically the marginal features of the log return forward exchange rates, I wanted flexibility to capture a broad range of skew-kurtosis relationships as well as potential for sub-exponential heavy tailed features. In addition, I wished to keep the models to a selection which is efficient to perform inference and easily interpretable. I therefore considered a first analysis utilizing log-normal distributions for the monthly forward exchange rate returns, which would be equivalent to specification of a Normality assumption on the distribution for the log return forward exchange rates. This model is given by the following parametric density, for a random variable $X \sim F(x; \mu, \sigma)$, in Equation 4.1 below.

$$f_X(x; \mu, \sigma) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \tag{4.1}$$

with the shape parameter $\sigma^2 > 0$ and the log-scale parameter $\mu \in \mathbb{R}$ and the support $x \in (0, \infty)$.

I found when analysing the goodness-of-fit for this log-normal model on each of the assets in the 20 currencies considered, over both 6 month and 1 year sliding windows, that the fit of the log-normal model would be systematically rejected as a suitable model for a couple of currencies. In the majority of cases over these sliding windows (locally stationary time series) the log-normal model was more than adequate. However, since some of the currencies that were rejecting this fit were appearing regularly in the high interest rate baskets I also decided to consider a more flexible three parameter model for the marginal distributions given by the Log-Generalized-Gamma distribution (l.g.g.d.), see details in Lawless [1980] and Consul and Jain [1971].

The l.g.g.d. is a parametric model based on the generalized gamma distribution which is highly utilized in lifetime modelling and survival analysis. The density for the generalized gamma distribution and the l.g.g.d are given respectively by Equations 4.2 and 4.3.

$$f_X(x;k,\alpha,\beta) = \frac{\beta}{\Gamma(k)}\frac{x^{\beta k-1}}{\alpha^{\beta k}} \exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right) \tag{4.2}$$

with parameter ranges $k > 0$, $\alpha > 0$ and $\beta > 0$ and a support of $x \in (0,\infty)$. Then the log transformed g.g.d. random variable $Y = \ln X$ is given by the density of the l.g.g.d. as follows.

$$f_Y(y;k,u,b) = \frac{1}{b\Gamma(k)} \exp\left[k\left(\frac{y-u}{b}\right) - \exp\left(\frac{y-u}{b}\right)\right] \tag{4.3}$$

with $u = \log(\alpha)$, $b = \beta^{-1}$ and the support of the l.g.g.d. distribution is $y \in \mathbb{R}$.

This more flexible three parameter model is particularly interesting in the context of the marginal modelling we are considering since the log-normal model is nested within the g.g.d. family as a limiting case. In addition the g.g.d. also includes the exponential model ($\beta = k = 1$), the Weibul distribution with ($k = 1$) and the Gamma distribution with ($\beta = 1$). Next I discuss how one can perform inference for the multivariate currency basket models using these marginal models and the mixture copula discussed previously.

4.2.1 Two Stages: Inference For the Margins

The inference function for margins (IFM) technique introduced in Joe [2005] provides a computationally faster method for estimating parameters than Full Maximum Likelihood, i.e. simultaneously maximising all model parameters and produces in many cases a more stable likelihood estimation procedure. An alternative approach to copula model parameter estimation that is popular in the literature is known as the Maximum Partial Likelihood Estimator (MPLE) detailed in Genest et al. [1995].

The procedure I adopt for likelihood based estimation is the two stage estimation known as Inference on the Margins which is studied with regard to the asymptotic relative efficiency of the two-stage estimation procedure compared with maximum likelihood estimation in Joe [2005] and in Hafner and Manner

[2010]. It can be shown that the IFM estimator is consistent under weak regularity conditions. However, it is not fully efficient for the copula parameters. Nevertheless, it is widely used for its ease of implementation and efficiency in large data settings such as the models I consider in this study.

To complete this discussion on general IFM, before providing the MLE estimation expressions, we first note that in this study I fit copula models to the high interest rate (IR) basket and the low IR basket updated for each day in the period 04/01/2000 to 02/01/2013 using log return forward exchange rates at one month maturities for data covering both the previous 6 months and previous year as a sliding window analysis on each trading day in this period. Next I discuss briefly the marginal MLE estimations for the log-normal and the l.g.g.d. models.

4.2.1.1 Stage 1: Fitting the Marginal Distributions via MLE

In the first step I fit the marginal distributions to either the log-normal model or the l.g.g.d model. In the case of the log-normal model this is achieved effortlessly since we may utilise the well-known analytic expressions for the MLE estimates:

$$\begin{aligned}\hat{\mu}_j &= \frac{1}{N}\sum_j \log(x_j) \\ \hat{\sigma}_j &= \sqrt{\frac{1}{N}\sum_j \log(x_j)^2 - \hat{\mu}_j^2}\end{aligned} \tag{4.4}$$

In the case of the l.g.g.d. distribution the estimation for the three model parameters can be significantly more challenging due to the fact that a wide range of model parameters, especially for k can produce similar resulting density shapes, see discussions in Lawless [1980]. To overcome this complication and to make the estimation efficient it is proposed to utilise a combination of profile likelihood methods over a grid of values for k and perform profile likelihood based MLE estimation for each value of k, then for the other two parameters b and u. The differentiation of the profile likelihood for a given value of k produces the system of two equations given by

$$\begin{aligned} \exp(\tilde{\mu}) &= \left[\frac{1}{n}\sum_{i=1}^{n}\exp\left(\frac{y_i}{\tilde{\sigma}\sqrt{k}}\right)\right]^{\tilde{\sigma}\sqrt{k}} \\ \frac{\sum_{i=1}^{n} y_i \exp\left(\frac{y_i}{\tilde{\sigma}\sqrt{k}}\right)}{\sum_{i=1}^{n}\exp\left(\frac{y_i}{\tilde{\sigma}\sqrt{k}}\right)} &- \overline{y} - \frac{\tilde{\sigma}}{\sqrt{k}} = 0 \end{aligned} \tag{4.5}$$

with n the number of observations, $y_i = \log x_i$ and the parameter transformations $\tilde{\sigma} = \frac{b}{\sqrt{k}}$ and $\tilde{\mu} = u + b \ln k$. The second equation is solved directly via a simple root search for the estimation of $\tilde{\sigma}$ and then substitution into the first equation provides the estimation of $\tilde{\mu}$. Note, for each value of k we select in the grid, we get the pair of parameter estimates $\tilde{\mu}$ and $\tilde{\sigma}$, which can then be plugged back into the profile likelihood to make it purely a function of k, with the estimator for k then selected as the one with the maximum likelihood score.

4.2.1.2 Stage 2: Fitting the Mixture Copula via MLE

In order to fit the Clayton-Frank-Gumbel model the copulae parameters ($\rho_{Clayton}$, $\rho_{Frank}, \rho_{Gumbel}$) and the copulae mixture parameters ($\lambda_{Clayton}, \lambda_{Frank}, \lambda_{Gumbel}$) are estimated using maximum likelihood on the data after conditioning on the selected marginal distribution models and their corresponding estimated parameters obtained in Stage 1. These models are utilised to transform the data using the cdf function with the mle parameters ($\hat{\mu}$ and $\hat{\sigma}$) if the log-normal model is used or ($\hat{k}$, $\hat{u}$ and $\hat{b}$) if the l.g.g.d is considered.

Therefore, in this second stage of MLE estimation we aim to estimate either the one parameter mixture of C-F-G components with parameters $\underline{\theta} = (\rho_{Clayton}, \rho_{Frank}, \rho_{Gumbel}, \lambda_{Clayton}, \lambda_{Frank}, \lambda_{Gumbel})$ or the two parameter mixture of outer power transformed mixture components OC-OF-OG components with parameters $\underline{\theta} = (\rho_{Clayton}, \rho_{Frank}, \rho_{Gumbel}, \lambda_{Clayton}, \lambda_{Frank}, \lambda_{Gumbel}, \beta_{Clayton}, \beta_{Frank}, \beta_{Gumbel})$. This is achieved in each case by the conditional maximum likelihood. To achieve this we need to maximise the log likelihood expressions for the mixture copula models, which in this framework are given generically by the following function for which we need to find the mode,

$$l(\underline{\theta}) = \sum_{i=1}^{n} log\ c^{C-F-G}(F_1(X_{i1};\hat{\mu}_1,\hat{\sigma}_1),\ldots,F_d(X_{id};\hat{\mu}_d,\hat{\sigma}_d)) + \sum_{i=1}^{n}\sum_{j=1}^{d} log\ f_j(X_{ij};\hat{\mu}_j,\hat{\sigma}_j) \quad (4.6)$$

with respect to the parameter vector $\underline{\theta}$.

For example in the case of the Clayton-Frank-Gumbel mixture copula we need to maximise on the log-scale the following expression.

$$\begin{aligned} l(\underline{\theta}) = \sum_{i=1}^{n} log\ \Big[\lambda_C * \Big(c^{C}_{\rho_C}\left(F_1\left(X_{i1};\hat{\mu}_1,\hat{\sigma}_1\right)\ldots,F_d\left(X_{id};\hat{\mu}_d,\hat{\sigma}_d\right)\right)\Big) \\ + \lambda_F * \Big(c^{F}_{\rho_F}\left(F_1\left(X_{i1};\hat{\mu}_1,\hat{\sigma}_1\right)\ldots,F_d\left(X_{id};\hat{\mu}_d,\hat{\sigma}_d\right)\right)\Big) \\ + \lambda_G * \Big(c^{G}_{\rho_G}\left(F_1\left(X_{i1};\hat{\mu}_1,\hat{\sigma}_1\right)\ldots,F_d\left(X_{id};\hat{\mu}_d,\hat{\sigma}_d\right)\right)\Big)\Big] \end{aligned} \quad (4.7)$$

This optimization is achieved via a gradient descent iterative algorithm which was found to be quite robust given the likelihood surfaces considered in these models with the real data. To illustrate this point, at this stage it is instructive to present some examples of the shapes of the profile likelihoods that are being optimized over for some of the important copula model parameters in the C-F-G mixture example for a 6 month window of data randomly selected from the data set for both the high interest rate basket and the low interest rate basket. Example plots of the profile likelihood for the 6-dimensional optimisation space for two different example days can be seen in Figures 4.3 and 4.4.

4.2.2 Goodness-of-Fit Tests

In this section I briefly comment on the model selection aspects of the analysis I undertook. As mentioned I first undertook a process of fitting the marginal log-normal model to all of the 20 currencies considered in the analysis over a sliding window of 6 months and 1 year. For each of these fits I then performed a formal hypothesis test in which I postulated that the null distribution is the log-normal model and then look for evidence in the data to reject this hypothesis at a level of significance of 5%. To undertake this test I considered the standard

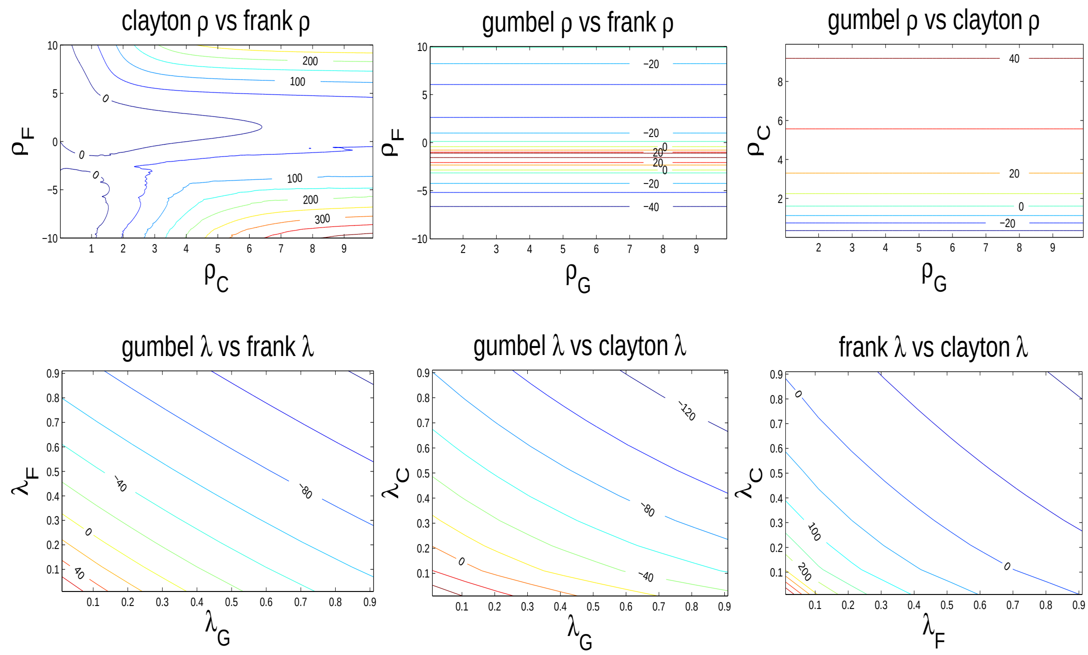


Figure 4.3: Profile likelihood plots for C-F-G mixture model.

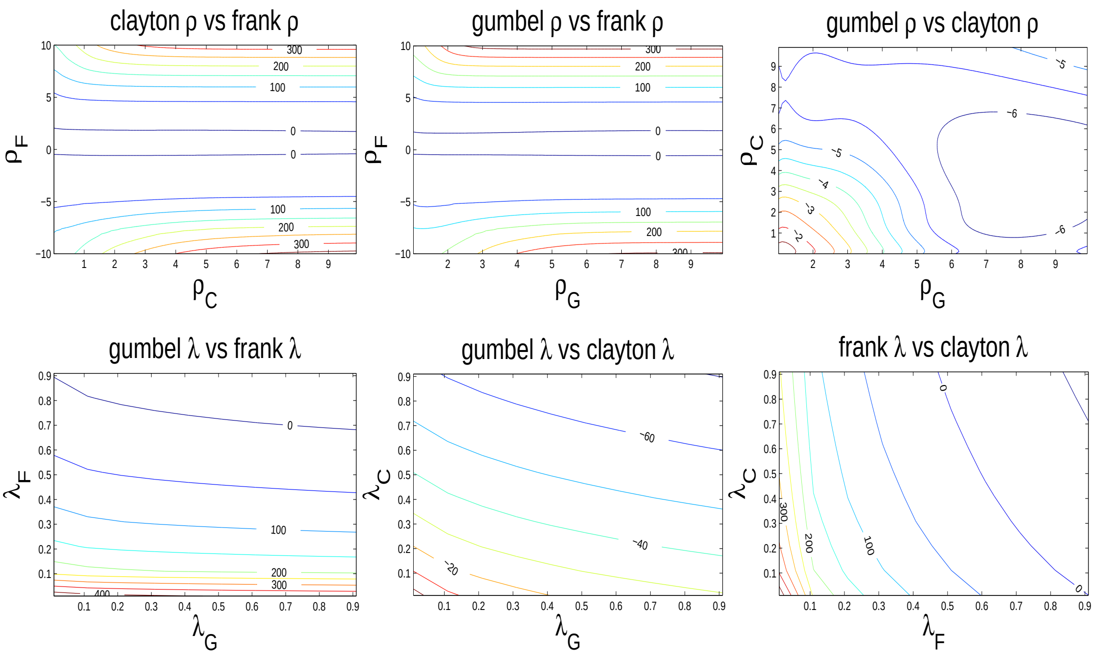


Figure 4.4: Profile likelihood plots for C-F-G mixture model.

Kolmogorov-Smirnov test. As I will present in the results I found strong evidence to reject the null systematically for a few important developing countries' marginal models, hence I also undertook estimation of the l.g.g.d. models for all of the 20 currencies. I am particularly interested in this case in the optimal choice of the model parameter k which as it asymptotically gets large $k \to \infty$ will produce a log-normal model. I found as expected the estimated model fits were significantly improved when fitting the l.g.g.d. models for the cases in which the log-normal was rejected by the K-S test. In addition the estimated k parameter in the periods of rejection of the log-normal hypothesis were estimated at values significantly lower than the upper bound in the search space. I assessed the optimal choice of marginal model between the log-normal and the l.g.g.d. models then via a standard information criterion based on the Akaike Information Criterion (AIC).

In terms of the selection of the copula mixture models, between the mixture of one parameter C-F-G model versus the two parameter mixtures of OC-OF-OG models, I again used a scoring via the AIC. We note that there are also alternative information criterion developed for copula models to assess the joint suitability of the copula model incorporating both the marginal and the joint copula structure which are modifications of the AIC, adjusting the penalty term for the approach adopted in the estimation, see for example the Copula-Information-Criterion (CIC) in Grønneberg [2010] for details. The results are presented for this comparison in Figure 4.5 in the top panel for the high interest rate basket and in Figure 4.5 in the lower panel for the low interest rate basket, over time based on the 6 month sliding window.

To further analyse this comparison of optimal copula mixtures I plot the AIC differentials for each of the currency baskets in Figure 4.6.

Figures 4.5 and 4.6 show it is not unreasonable to consider the C-F-G model for this analysis, since the mean difference between the two AIC scores for the models is 2.05 in favour of the C-F-G. However, we do note that the OP.C-OP.F-G model seems to fit better during crisis periods.

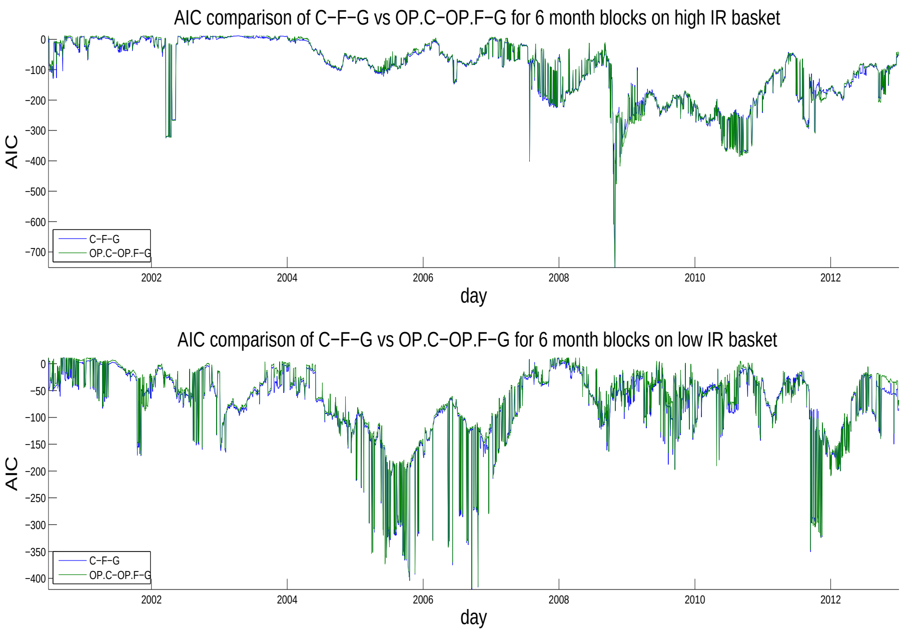


Figure 4.5: AIC comparison of C-F-G vs OP.C-OP.F-G for 6 month blocks on high and low IR baskets.

4.3 Results and Analysis

In this section I present a detailed analysis of the estimation of the marginal distributional models and the mixture copula models for both the high interest rate basket and the low interest rate basket. Firstly, I investigate the properties of the marginal distributions of the exchange rate log-returns for the 20 currencies. I then interpret the time-varying dependence characteristics of the fitted copula models to the high interest rate basket and the low interest rate basket across the period 04/01/2000 to 02/01/2013. Note, all results presented below are for the case in which I considered a 6 month sliding window, results for the 1 year sliding window were similar in nature and so are omitted.

4.3.1 Modelling the Marginal Exchange Rate Log-Returns

In order to model the marginal exchange rate log-returns I first fit log-normal models to each of the 20 currencies considered in the analysis, updating the fits

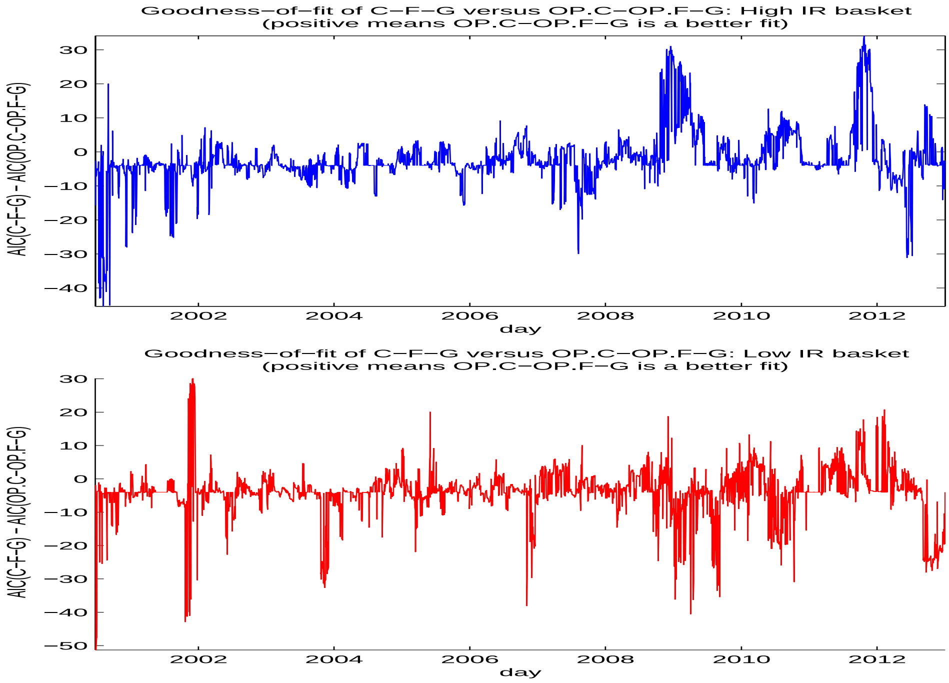


Figure 4.6: AIC comparison of C-F-G vs OP.C-OP.F-G for 6 month blocks on high and low IR baskets.

for every trading day in the period 04/01/2000 to 02/01/2013 based on the 6 months sliding window. The log-normal model was selected due to the fact it has a positive support, represents a range of skew-kurtosis characteristics and can display sub-exponential tail features (ie. heavy tailed features) should such attributes be present in the data. I assessed the quality of the fits for each currency using a standard Kolmogorov-Smirnov goodness-of-fit test, at the 5% significance level. A summary of the results of this analysis are presented in Table 4.1 which shows the proportion of rejections of the null hypothesis, that the marginal distribution is log-normal for each of the currencies on a given 6 months block of trading days.

We learn from this analysis that the majority of the currencies demonstrate reasonable marginal distribution fits under a log-normal family, however there are a few notable exceptions. Specifically the Turkish lira, Malaysian ringgit, Indian rupee, Thai baht, South Korean won and Taiwanese dollar demonstrated sustained periods in the analysis in which the log-normal model would be unsuitable to capture the features of the time series adequately. This is significant in this analysis since these currencies actually correspond to the currencies that have a strong presence in the high interest rate baskets, as seen in Figure 4.1. Therefore, they will play an important role in the multivariate analysis of the currency carry trade. As such, it is important to accurately model the features of each of these particular currencies' marginal distributions, before undertaking the multivariate mixture copula analysis, I proposed to generalize the marginal model analysis to a more flexible three parameter family of models given by the log generalized gamma distribution, as discussed in Section 4.2.

The log-generalised gamma distribution (l.g.g.d.) should improve the fit for all currencies since it allows for more flexibility in the tails of the distribution and a wider range of skew-kurtosis relationships when compared to the log-normal model family. In addition, as we note in Section 4.2, for those currencies in which the log-normal model was a suitable fit, then they will still obtain such distributional characteristics since the log-normal model is a limiting case of the l.g.g.d. as k tends to infinity. Hence, we can still incorporate the log-normal model for the currencies that were a good fit.

The maximum likelihood parameters $(\hat{\mu}, \hat{\sigma}, \hat{k})$ of the fitted l.g.g.d. margins for each of the currencies can be seen in Figures 4.7, 4.8 and 4.9. These plots demonstrate the time varying attributes of the marginal distributions for each currency, illustrating interesting changes in tail behaviour and skewness-kurtosis characteristics over time, especially in heightened periods of volatility in some of these currencies. In particular, there are three standout periods (2003, 2009 and 2012) of heightened μ and σ parameter values across most of the currencies. Hence, during these periods the exchange rate log-returns may demonstrate heavier tails, and increased volatility in the parameter estimates. In addition, we observe that a few important currencies for the currency carry trade analysis demonstrate sustained differences in their marginal distribution attributes relative to the other currencies. An important example of this is the μ estimates in Figure 4.7 for the TRY, the NZD and the BRL. Similar significant differences between these particular currencies and the rest of the currencies are observed in the estimates of σ in Figure 4.8.

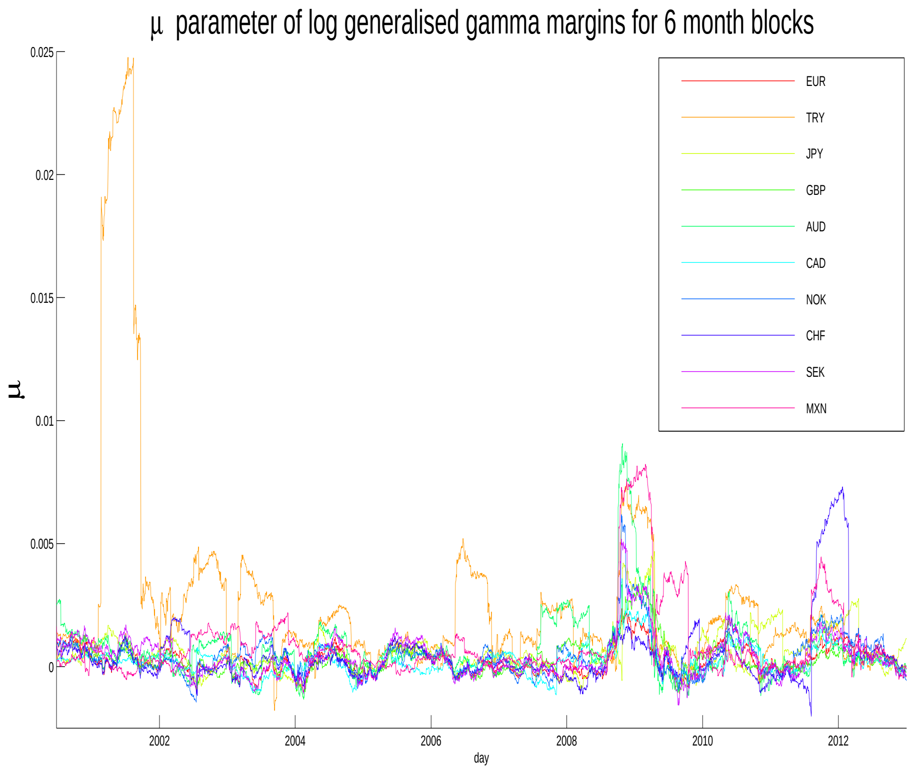
As the value of the parameter k in the l.g.g.d. gets large I expect the log-normal fit to be a suitable model structure for the marginal distributions. As illustrated in the K-S test results certain currencies systematically did not have a suitable fit with the log-normal model. Examples of this are clear when we consider the estimates of k in Figure 4.9. Again we see systematically smaller values for the estimate of k in the TRY and the BRL. We see clearly in Figure 4.9 the periods of time during which the currencies display non log-normal behaviour. The most prominent example being the Turkish lira (orange), which shows consistently low values of k. As we noted in Section 4.2, for small values of $k \approx 1$ we obtain Weibull like tail behaviour and in addition, in the cases when $\sigma \approx 1$ jointly with small values of k, I expect the light tailed exponential models to be suitable. As a consequence of this analysis and comparison of AIC results I proceeded with the joint estimation utilising the l.g.g.d. marginal models for every currency.

A noticeable period for the Turkish lira is early in 2001 during which low values of the parameter k clearly provides evidence of heavy tail log-returns distribution for this specific currency. As mentioned earlier in this investigation the Turkish government's decision in February 2001 to stop draining reserves to bolster its currency led the same day to a 30% devaluation of the Turkish lira relative to the dollar.

Table 4.1: Proportion of rejections of the null hypothesis that the sample is from a log-normal distribution, measured using a k-s test at the 5% level.

Block length	**EUR**	**TRY**	**JPY**	**GBP**	**AUD**	**CAD**	**NOK**	**CHF**	**SEK**	**MXN**
6 month	0.001	0.198	0.043	0.000	0.023	0.000	0.000	0.031	0.012	0.032
Year	0.000	**0.553**	0.107	0.007	0.120	0.018	0.006	0.084	0.018	0.128

Block length	**PLN**	**MYR**	**SGD**	**INR**	**ZAR**	**NZD**	**THB**	**KRW**	**TWD**	**BRL**
6 month	0.018	0.494	0.000	0.234	0.025	0.012	0.221	0.130	0.192	0.086
Year	0.094	**0.651**	0.071	**0.549**	0.124	0.113	**0.504**	0.350	0.381	0.403



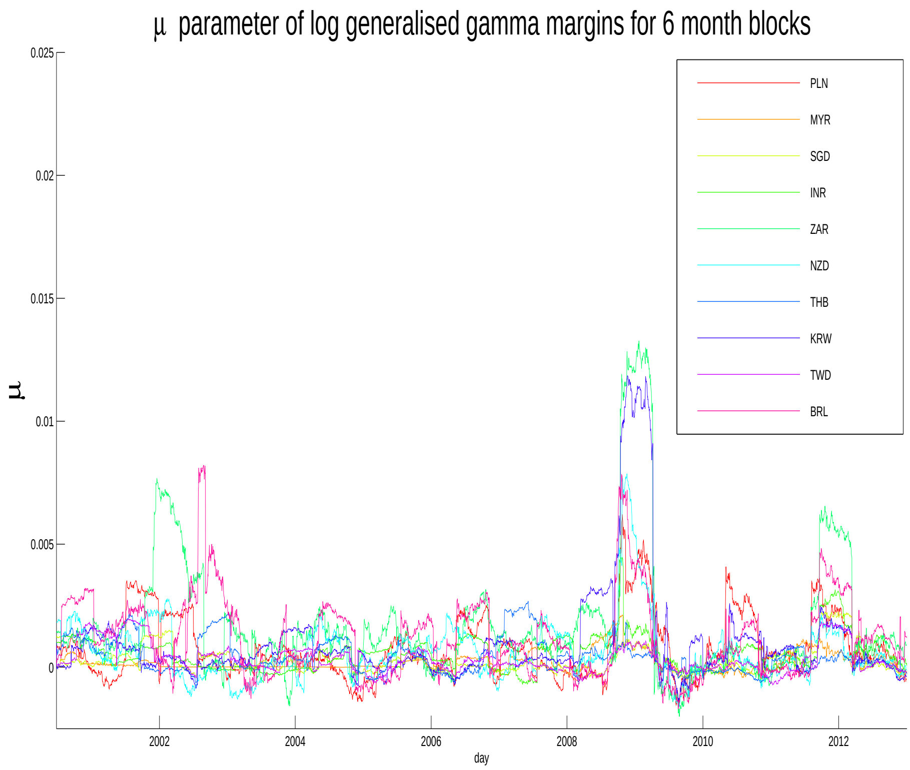
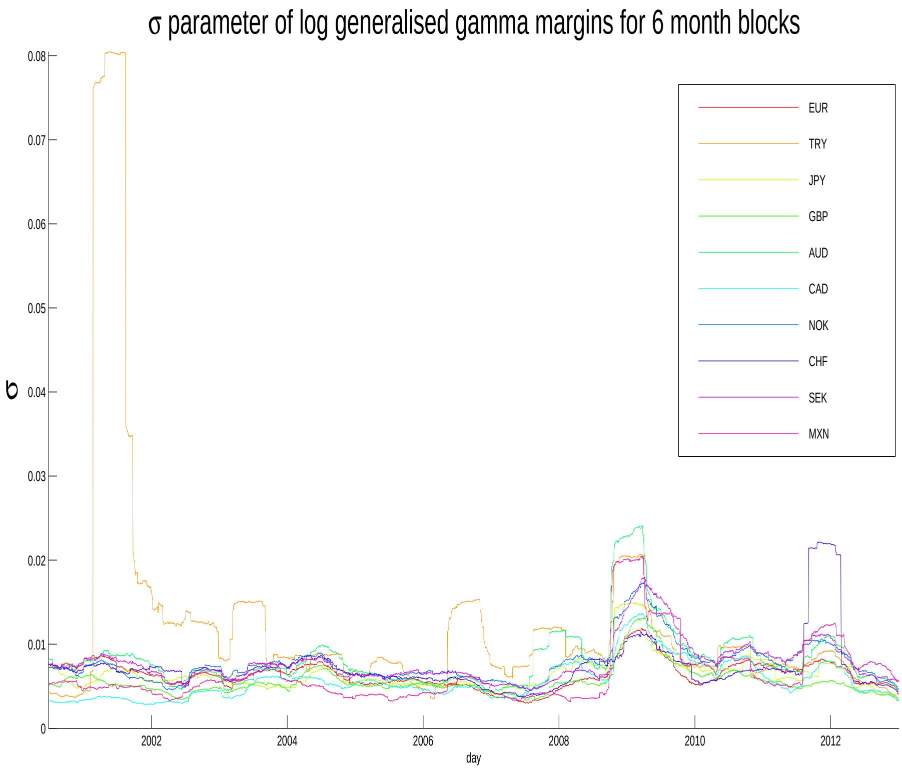


Figure 4.7: μ parameter of log generalised gamma margins using 6 month blocks



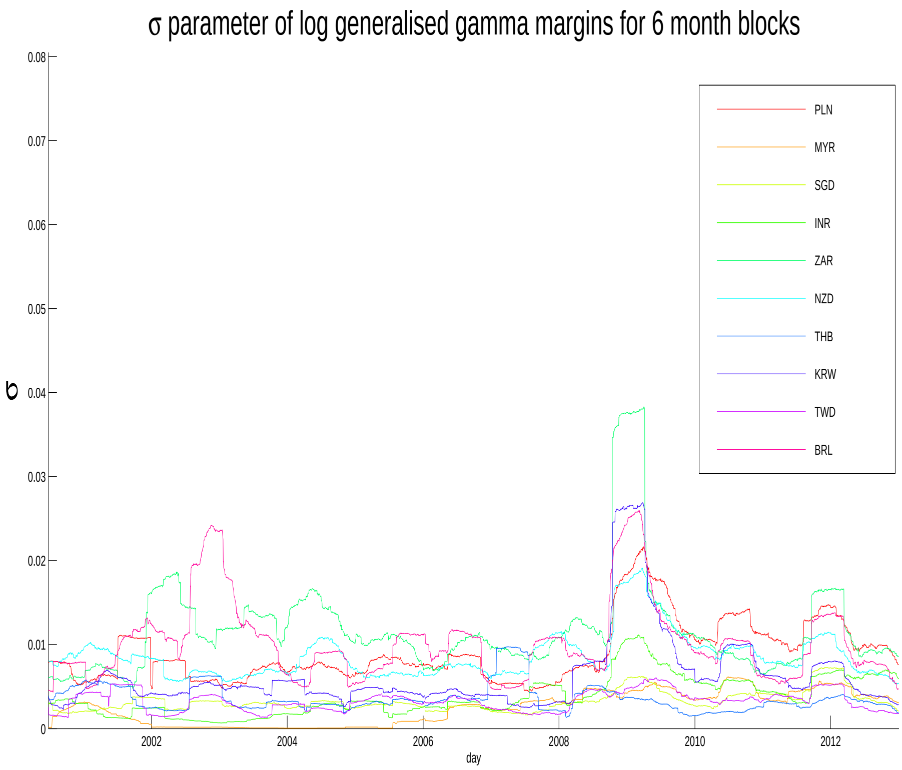
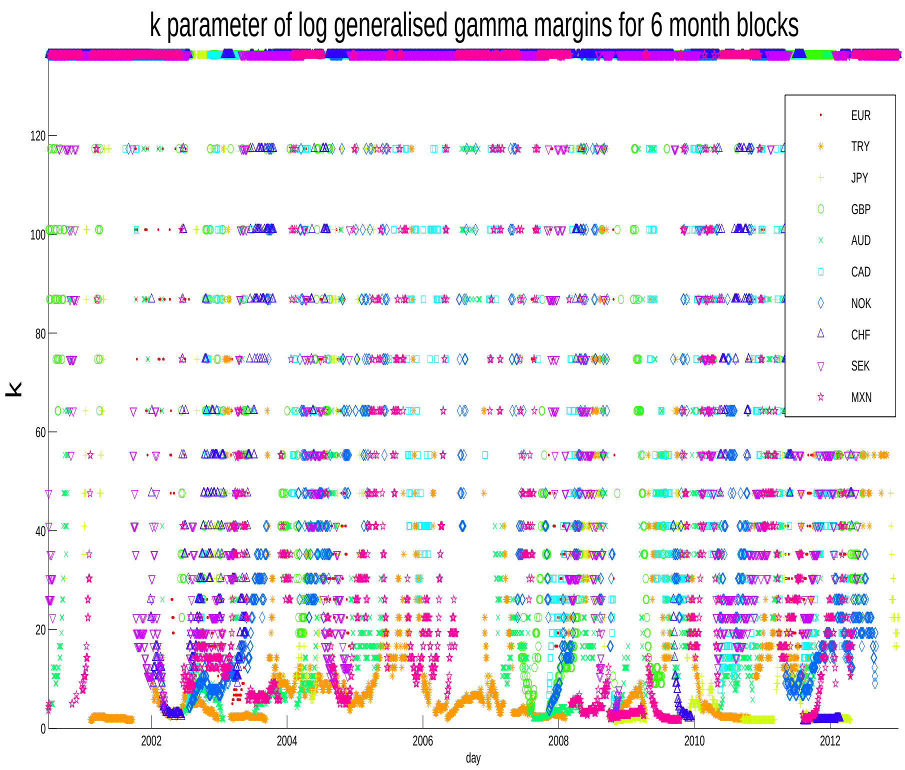


Figure 4.8: σ parameter of log generalised gamma margins using 6 month blocks



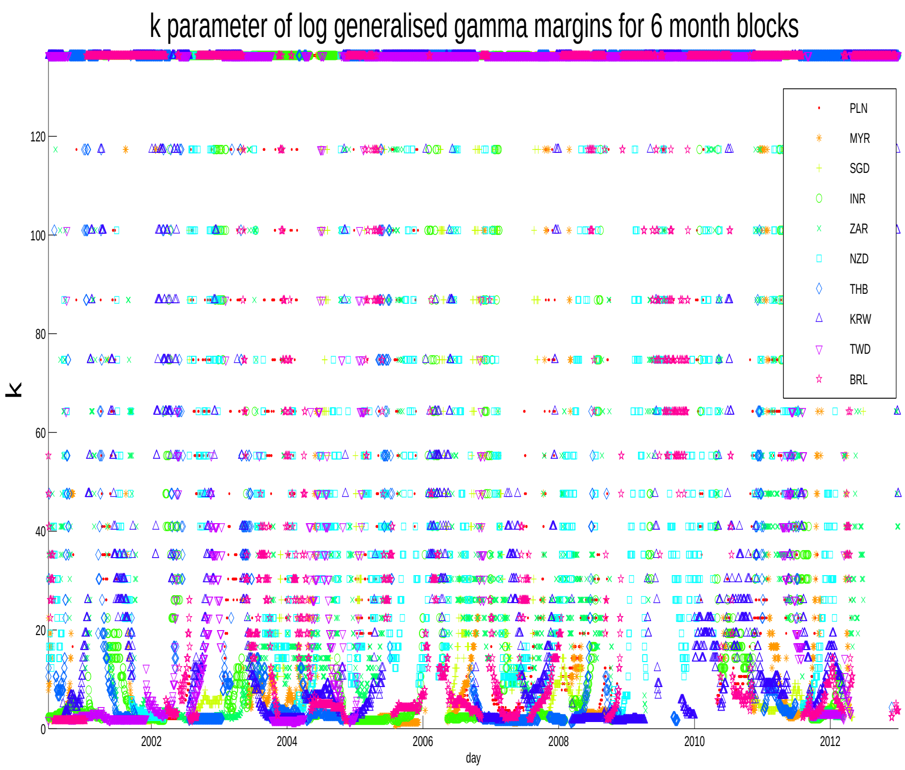


Figure 4.9: K parameter of log generalised gamma margins using 6 month blocks

4.3.2 Copula Modelling Results

I now utilised each of the l.g.g.d. marginal distribution fits for a given day's set of currencies in the high interest rate and low interest rate baskets to analyse the joint multivariate features. To achieve this for each of the currencies, the exchange rate log-return data was transformed via the l.g.g.d. marginal model's distribution function to uniform [0, 1] margins. Then the mixture Clayton-Frank-Gumbel copula (denoted C-F-G) and the outer-power versions were fitted each day to a sliding window of 6 months and one year log-returns data for both the high interest rate and low interest rate baskets. Below we will examine the time-varying parameters of the maximum likelihood fits of this mixture C-F-G copula model. Furthermore, the results for the outer-power transform cases did not demonstrate discernible differences from the base C-F-G model and so were excluded. This can be seen from the figures displaying the AIC for each of these models (Figures 4.5 and 4.6).

In this analysis there are several attributes to be considered for the mixture copula model, such as the relevant copula structures for the high and low interest rate baskets and how these copula dependence structures may change over time. In addition, there is the strength of the tail dependence in each currency basket and how this changes over time, especially in periods of heightened market volatility. The first of these attributes I will consider to be a structure analysis studying the relevant forms of dependence in the currency baskets and the second of these attributes that I shall study will be the strength of dependence present in the currency baskets, given the particular copula structures in the mixture.

Therefore I first consider the structural components of the multivariate copula model. To achieve this, I begin with a form of model selection in a mixture context, in which I consider the estimated relative contributions of each of the copula components (and their associated dependence features) to the joint relationship in the high and low interest rate currency baskets over time. This is reflected in the estimated mixture component weights, which can be seen in Figures 4.10 and 4.11 for the high interest rate basket and low interest rate basket respectively. The λ values show the relevance of each of the component copulae to the data. Thus a small λ value indicates the lack of a need for that particular copula component in order to model the associated 6 months or one year block of data. In contrast, for example a λ value for the Gumbel component very close to 1 indicates the block of data could be well modelled by a Gumbel copula alone. Hence, these plots convey the time varying significance of hypotheses about the presence of upper and lower tail dependence in each of the baskets over time. Examining these plots shows that in general the Clayton mixture weight tends to be lower when the Gumbel mixture weight is higher. We can also see that the Frank copula is systematically present in the mixture. In addition, we see that in the periods of high market volatility we observe differences in the relevant upper and lower tail dependence structural attributes when comparing the high versus low interest rate baskets. That is, there is an asymmetric tendency for the presence of particular copula components over time when comparing the high and low interest rate baskets. The implications of this will be discussed in further detail in the discussions.

In terms of the second attribute, the strength of the copula dependence, I analyse this in several ways. Firstly through an analysis of the estimation copula parameter components over time, then through an analysis of the transformation of these copula parameters to rank correlations and finally through an analysis of the multivariate strength of the mixture copula tail dependence over time.

The individual component copula parameters can be seen in Figures 4.12 and 4.13 for the high interest rate basket and low interest rate basket respectively. The strength of the copula parameters in the baskets shows a large degree of variance during the period 04/01/2000 to 02/01/2013. One interesting observation is the very large spikes in the Gumbel copula parameter observed for the high interest rate basket from 2006 to 2007 and again in 2009. This was significant as it also corresponds to periods in which the Gumbel copula mixture weight was non-trivial.

The measure of concordance as captured by Kendall's tau is decomposed in this analysis according to each of the mixture components, scaled by the mixture weights λ, and can be seen in Figure 4.14 for the high interest rate basket and Figure 4.15 for the low interest rate basket. These plots provide a more intuitive picture of the time-varying contributions of the individual copulae to the dependence structure present in each of the baskets. Interestingly, we see

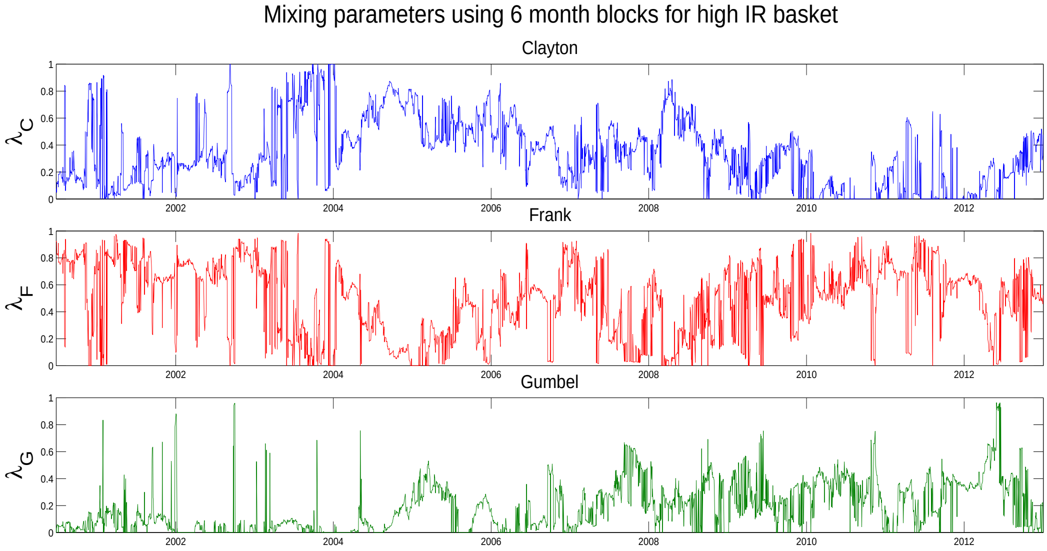


Figure 4.10: λ Mixing proportions of the respective Clayton, Frank and Gumbel copulae on the high interest rate basket, using 6 month blocks.

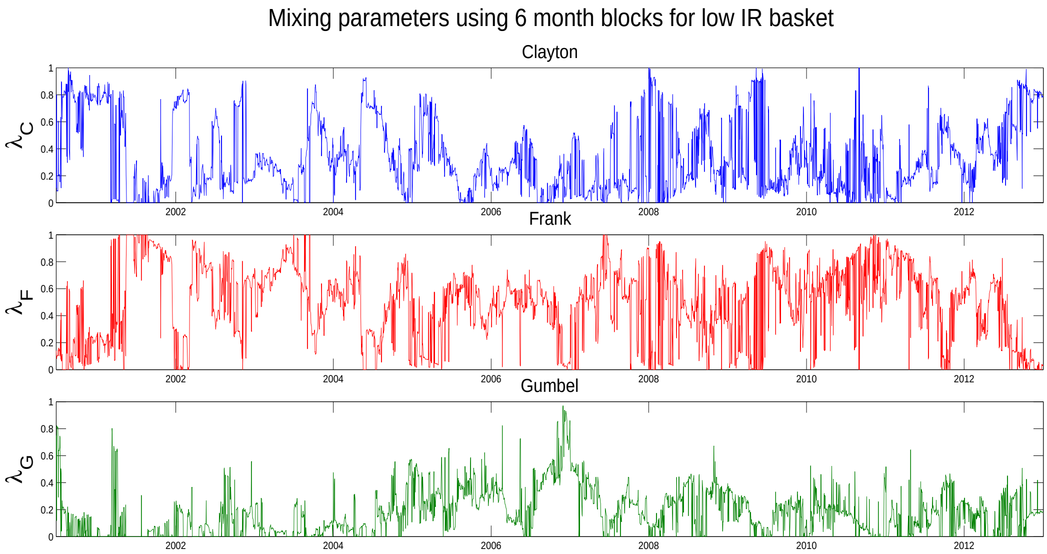


Figure 4.11: λ Mixing proportions of the respective Clayton, Frank and Gumbel copulae on the low interest rate basket, using 6 month blocks.

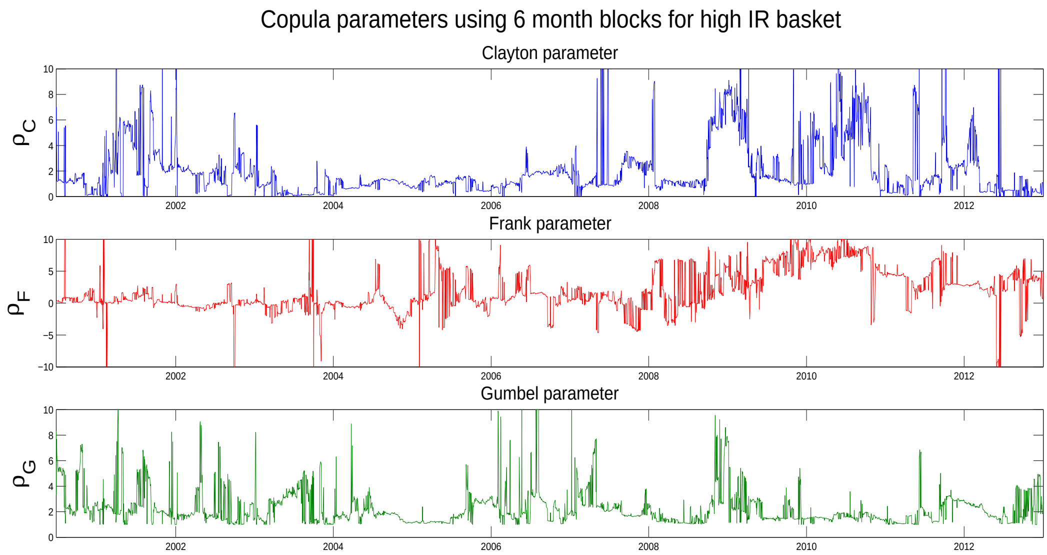


Figure 4.12: ρ Copula parameters for the Clayton, Frank and Gumbel copulae on the high interest rate basket, using 6 month blocks.

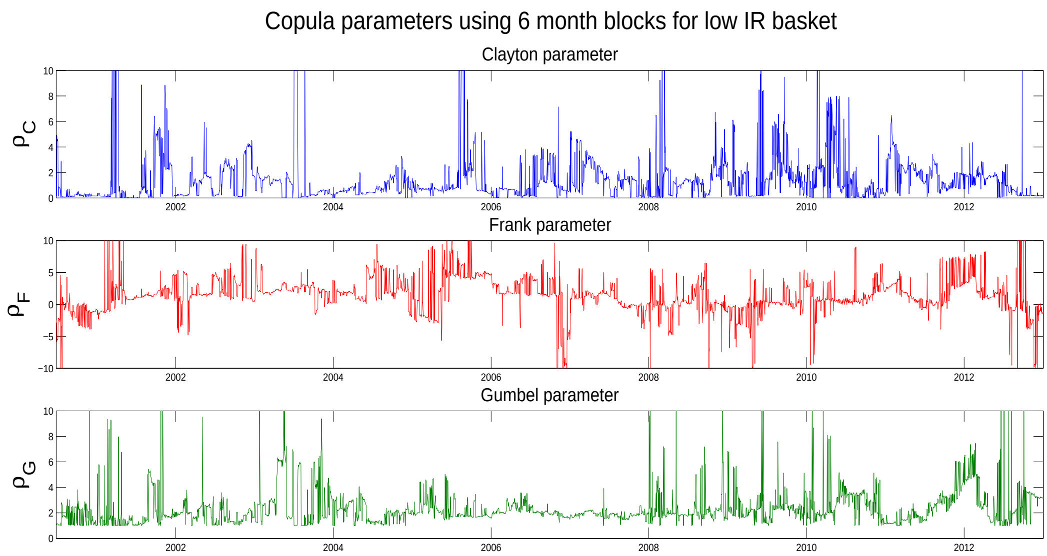


Figure 4.13: ρ Copula parameters for the Clayton, Frank and Gumbel copulae on the low interest rate basket, using 6 month blocks.

the rank correlation contribution from the Frank copula indicates the presence of negative as well as positive rank correlations. In addition, as discussed with the mixture weights, there is perhaps some asymmetry present between the high and low interest rate baskets over time.

Perhaps the most interesting and revealing representation of the tail dependence characteristics of the currency baskets can be seen in Figures 4.16 - 4.21. Here we can see that there are indeed periods of heightened upper and lower tail dependence in the high interest rate basket. There is a noticeable increase in upper tail dependence at times of global FX volatility. Specifically, during late 2007, i.e. the global financial crisis, there is a sharp peak in upper tail dependence. Preceding this, there is an extended period of heightened lower tail dependence from 2004 to 2007, which could tie in with the building of the leveraged carry trade portfolio positions.

In understanding this analysis we note that Figures 4.16 and 4.17 show the probability that one currency in the basket will have a move above/below a certain extreme threshold given that the other three currencies have had a move beyond this threshold. Then in Figures 4.18 and 4.19 I show the probability that two currencies in the basket will have a move above/below such an extreme threshold given that the other two currencies have had a move beyond this threshold. Finally, in Figures 4.20 and 4.21 I show the probability that three currencies in the basket will have a move above/below a certain threshold given that the remaining currency has had a move beyond this threshold.

To illustrate the relationship between heightened periods of significant upper and lower tail dependence features over time and to motivate the clear asymmetry present in the upper and lower tail dependence features between the high and low interest rate baskets over time I consider a further analysis. In particular I compare in Figures 4.22 and 4.23 the tail dependence plotted against the VIX volatility index for the high interest rate basket and the low interest rate basket respectively for the period under investigation. The VIX is a popular measure of the implied volatility of S&P 500 index options - often referred to as the *fear index*. As such it is one measure of the market's expectations of stock market volatility over the next 30 days. We can clearly see here that in the high interest rate basket there are upper tail dependence peaks at times when there is increased

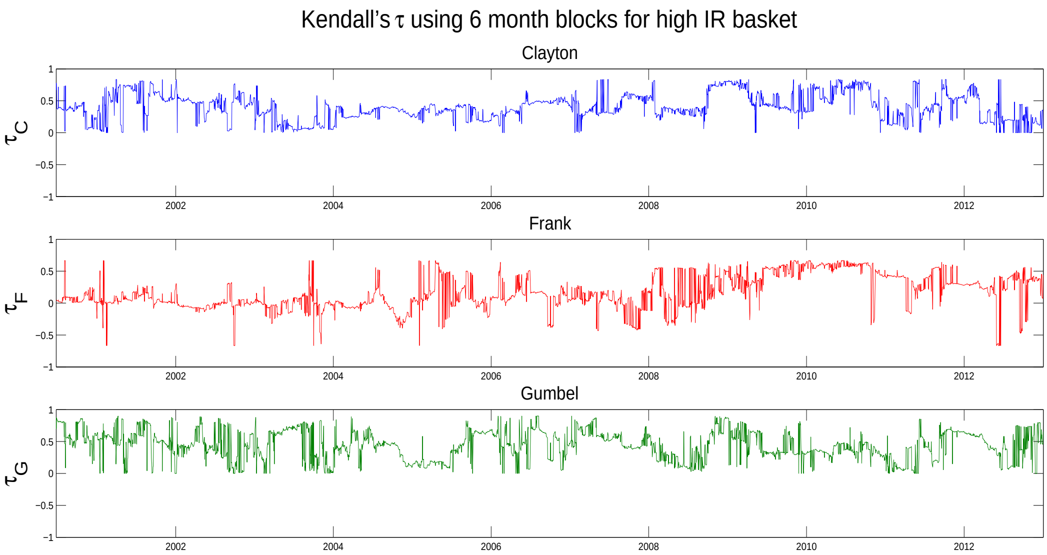


Figure 4.14: Kendall's τ for the Clayton, Frank and Gumbel copulae on the high interest rate basket, using 6 month blocks.

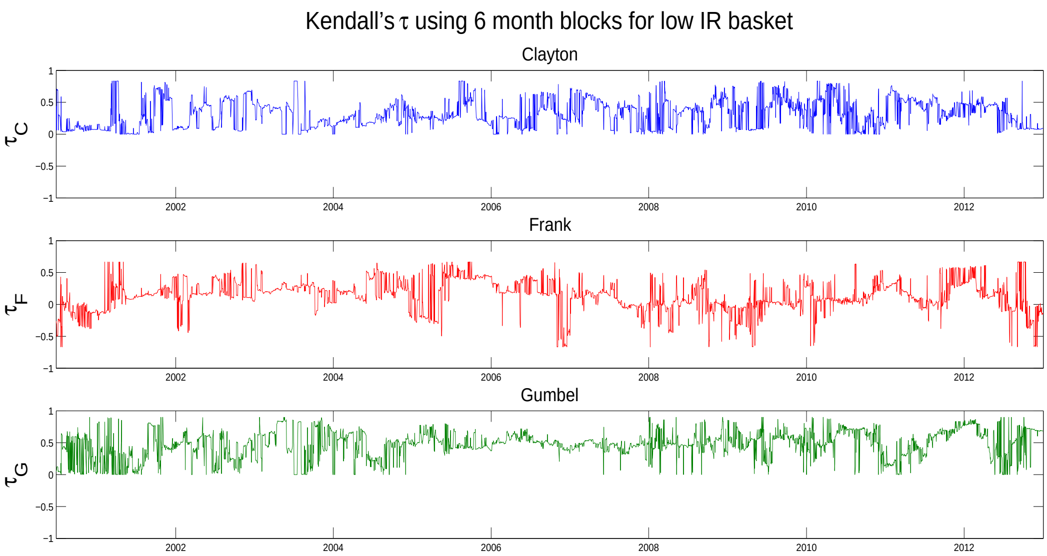


Figure 4.15: Kendall's τ for the Clayton, Frank and Gumbel copulae on the low interest rate basket, using 6 month blocks.

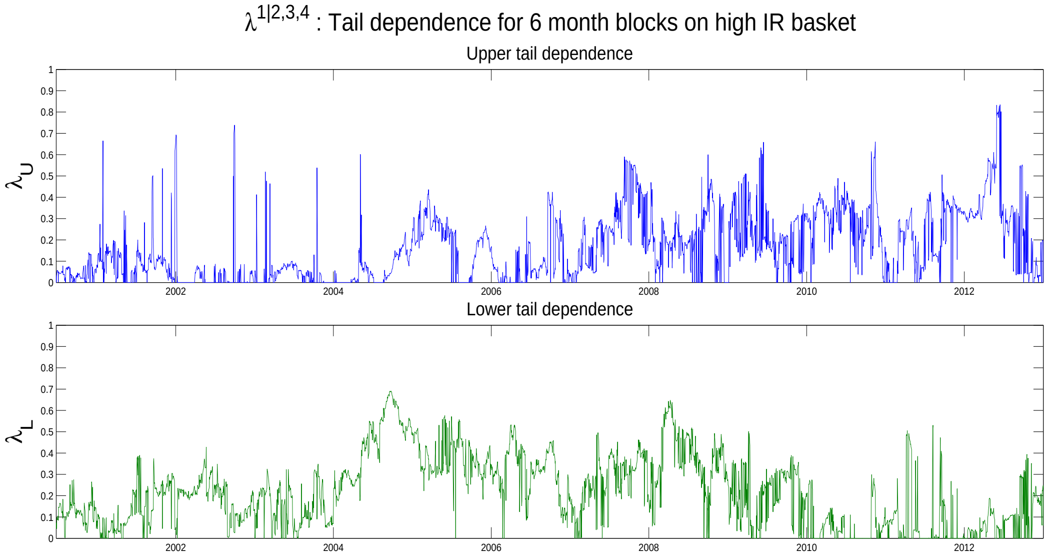


Figure 4.16: $\lambda^{1|234}$: 6 month blocks on high interest rate basket.

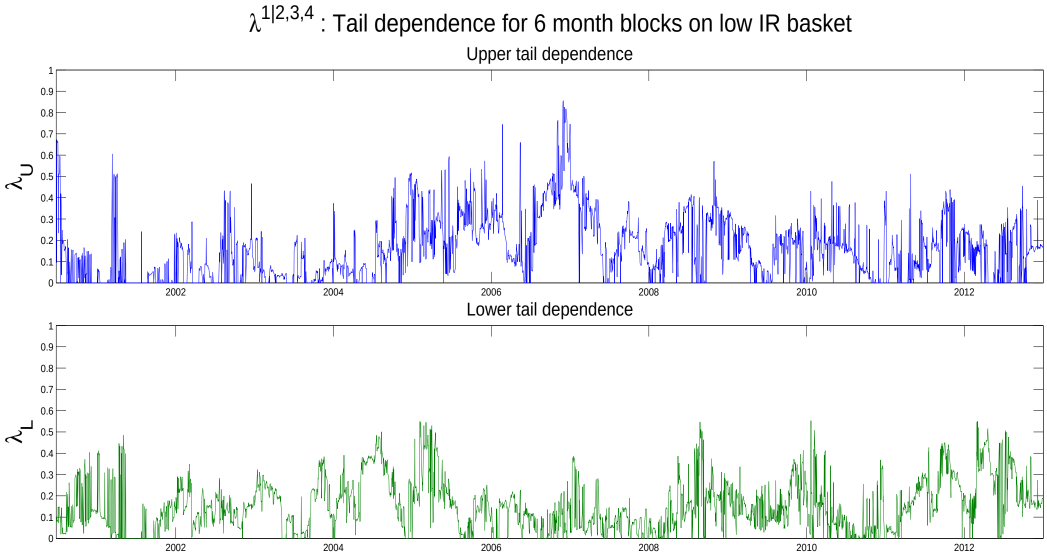


Figure 4.17: $\lambda^{1|234}$: 6 month blocks on low interest rate basket.

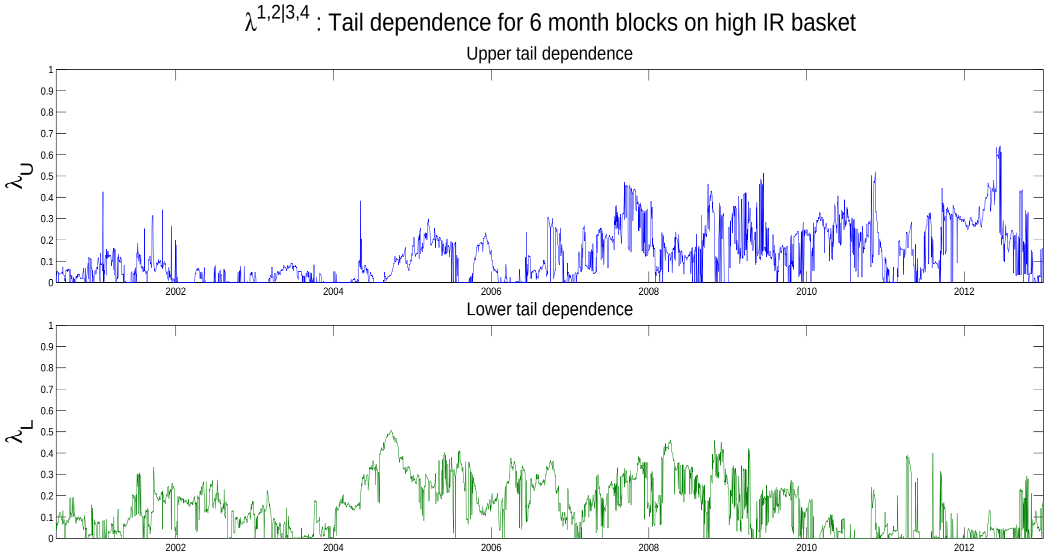


Figure 4.18: $\lambda^{12|34}$: 6 month blocks on high interest rate basket.

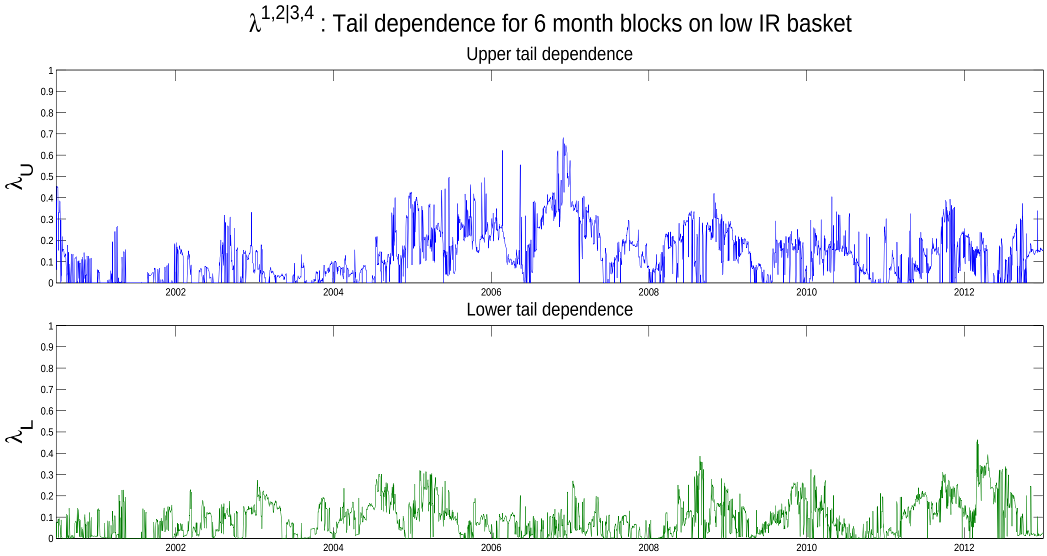


Figure 4.19: $\lambda^{12|34}$: 6 month blocks on low interest rate basket.

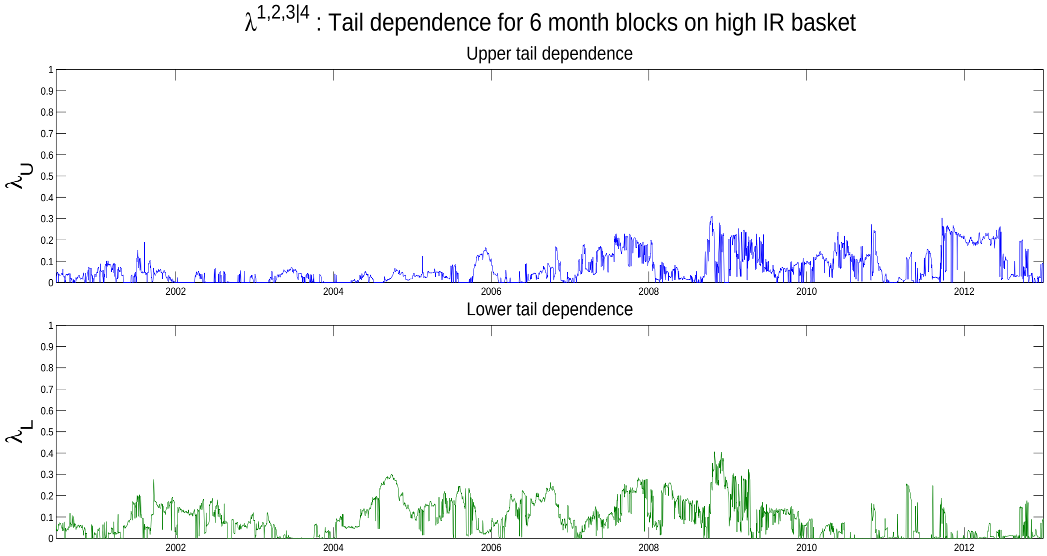


Figure 4.20: $\lambda^{123|4}$: 6 month blocks on high interest rate basket.

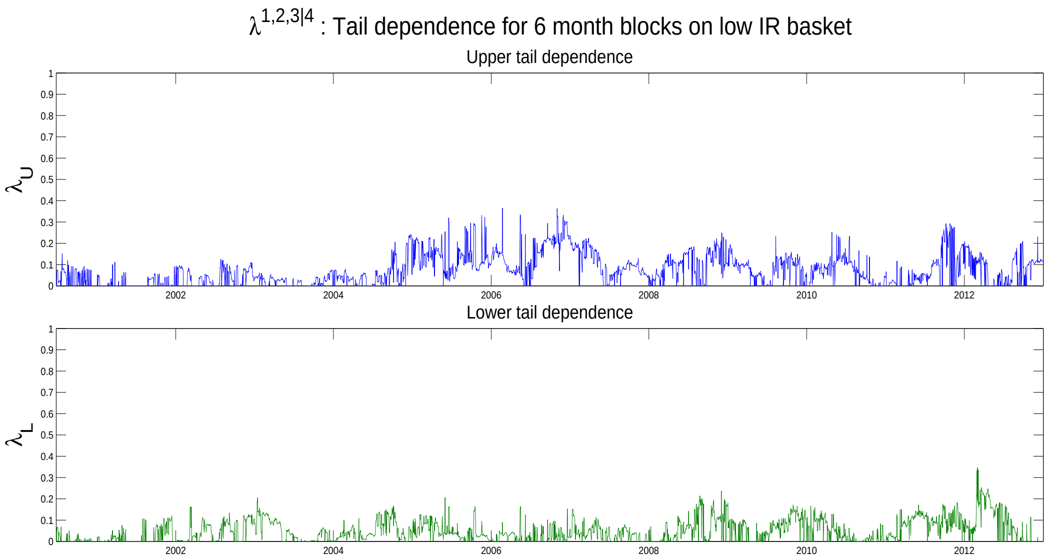


Figure 4.21: $\lambda^{123|4}$: 6 month blocks on low interest rate basket.

stock market volatility, particularly post-crisis. However, I would not expect the two to match exactly since the VIX is not a direct measure of global FX volatility. We can thus conclude that investors' risk aversion clearly plays an important role in the tail behaviour of high interest rate currencies and more importantly in their dependence structure. This statement can also be associated to the globalization of financial markets and the resulting increase of the contagion risk between countries. This conclusion corroborates some of the recent literature results with regards to the skewness and the kurtosis features characterizing the currency carry trade portfolios [Brunnermeier et al., 2008; Farhi and Gabaix, 2008; Menkhoff et al., 2012].

The black lines plotted in Figures 4.22 and 4.23 furthermore display the mean tail dependence before and after August 2007 (which corresponds to the beginning of the global financial crisis). The data shows a large increase in upper tail dependence in the high interest rate basket after the crisis, as well as a smaller decrease in lower tail dependence. Interestingly there is very little difference in the mean tail dependence before and after the crisis for the low interest rate basket. The carry trade portfolios were particularly impacted by the sub-prime crisis as most of these currency positions were implemented and held by financial institutions which faced sudden difficulties to finance the leverage of their positions. Furthermore, another interesting point we can make from the analysis of these two figures is the higher level of lower tail dependence before the financing crisis, especially between 2004 and 2007. If we put in parallel the fact that during this three year period the VIX index was noticeably and continuously going down we could imagine that this increase of the lower tail dependence results from lower risk aversion and the resulting tendency of investors to accordingly increase their leverage on risky positions such as currency carry trades.

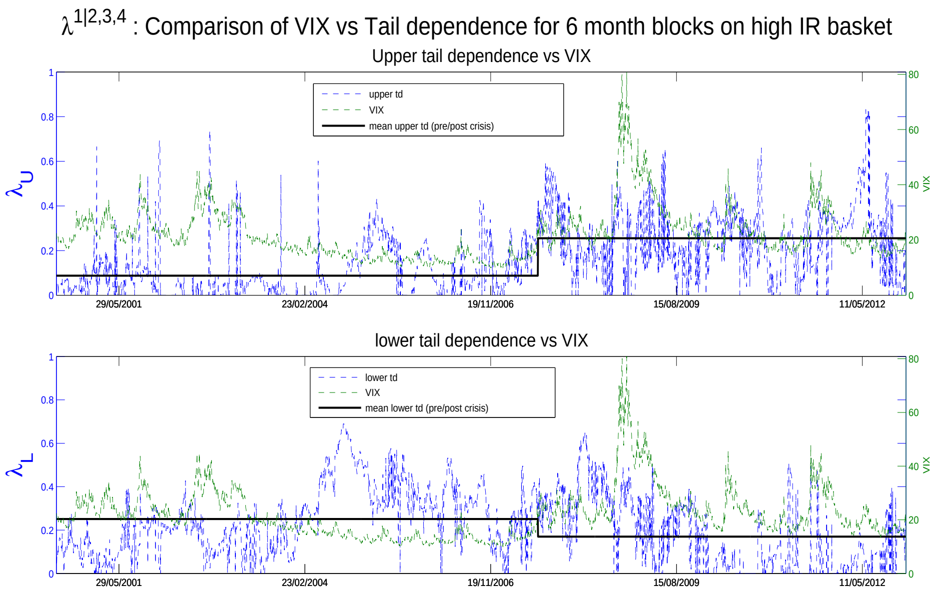


Figure 4.22: Comparison of Volatility Index (VIX) with upper and lower tail dependence of the high interest rate basket.

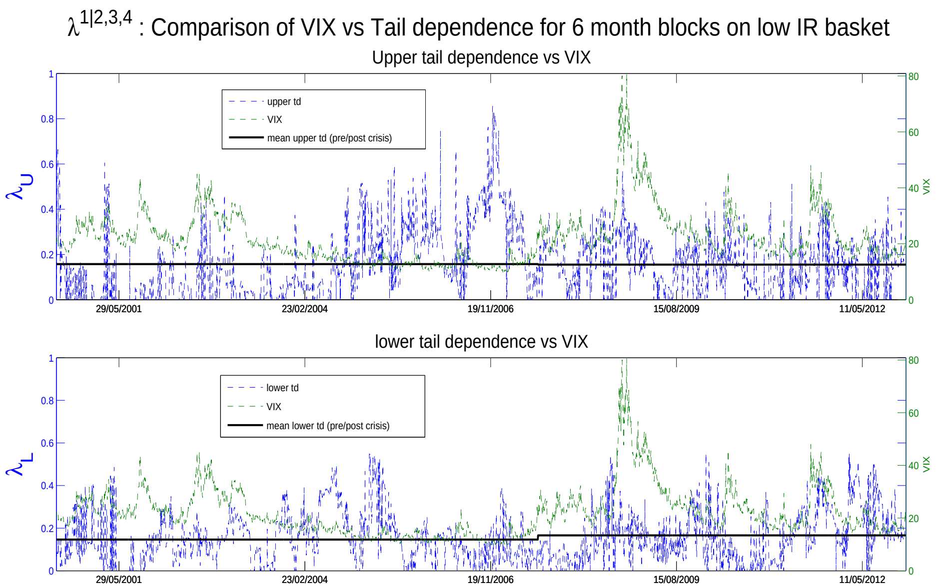


Figure 4.23: Comparison of Volatility Index (VIX) with upper and lower tail dependence of the low interest rate basket.

Chapter 5

Conclusions and Future Research

5.1 Conclusions

This dissertation has investigated one of the most robust puzzles in international finance, namely the currency carry trade. This market phenomenon is particularly interesting from a theoretical standpoint as well as for the understanding of financial market mechanisms. It has been demonstrated empirically that the currency markets were violating a fundamental relation in finance which connects the currency exchange rates and the interest rates associated with two different countries. The main contribution of this dissertation has been to propose a rigorous statistical modelling approach which captures the specific statistical features of both the individual currency log-return distributions as well as the joint features such as the dependence structures prevailing between all the exchange rates.

In achieving this goal, I first assessed the marginal statistical features of each of the 20 currencies on an assumed locally stationary sliding window of six months, over all the trading days in the period 04/01/2000 to 02/01/2013. I found that a simple log-normal marginal distribution would not produce a suitable statistical fit for some of the key currencies that are regularly present in the high interest rate basket throughout this period. As detailed in the results section this was notably the case in unstable economies such as developing countries (for instance Turkey, Brazil or South Africa) where political stability or default risk create sudden and violent adjustments to their currency exchange rates with other countries. We note that these currencies are still of direct significance to currency carry trade strategies since according to the modern portfolio theory this intrinsic risk borne by an investor in these currencies can be diversified and mitigated by adding to the considered portfolio other currencies which depend themselves on different sources of intrinsic risk. This would effectively establish a diversified portfolio of currencies violating the UIP hypothesis and would thus provide a very attractive average return for a very limited risk which has been the conclusion of several recent empirical studies in the finance literature.

The conclusion of this is that we cannot exclude these currencies from the high interest rate basket analysis, even though they may demonstrate attributes resulting primarily from significant changes in their countries political and financial structure. As a result I needed to obtain more flexible marginal models to capture the features of these currencies more adequately. Consequently I modelled each currency exchange rate return marginally via a flexible three parameter parametric model which offers a wide range of skew-kurtosis relationships as well as the possibility of light exponential tails and heavier sub-exponential tail behaviours such as the log-normal member. The parametric family of distributions I selected for this purpose was the log-generalized gamma distribution.

Having modelled the marginal attributes of the high and low interest rate currency baskets over time adequately, the main emphasis was then to assess the multivariate dependence features of the currency baskets. In particular how this may change over time within a given basket, where I was particularly interested in the effect of the composition of the basket over time, and the response of the multivariate dependence features of the modelled basket and how it may respond in periods of heightened market volatility versus more stable periods. In addition to this within basket temporal analysis, from the perspective of undertaking a currency carry trade strategy, we would need to consider the relative relationships between the temporal dependence features of the high interest rate and low interest rate currency baskets. I demonstrate several interesting features from the model fits relating to asymmetries between the high and low interest rate baskets over time, especially during periods of high volatility in global markets. One way I ascertained such periods was through a comparison of the VIX versus features of the multivariate dependence relationships I modelled. Importantly I found substantial evidence to support arguments for time varying behaviours in the structural dependence hypotheses posed about the currency baskets, as captured by the relevant contributing copula components to the multivariate mixture model. As well as substantial evidence for significant tail dependence features in both the high and low interest rate baskets, which again displayed interesting asymmetries between the high and low interest rate baskets over time.

The financial interpretation of the significance of these findings is related to the fact that it demonstrates that historically average rewards from a currency carry trade portfolio can be exposed to a significant risk of large losses arising from joint adverse movements in the currencies that would typically comprise the high and low interest rate baskets that an investor would go long and short on when trading. Hence, I conclude that our second contribution to the literature has been to rigorously demonstrate that such assertions relating to the profitability of the currency carry trade are failing to appropriately take into consideration an important component of the risk which characterizes these types of portfolios of currencies named carry trade portfolios.

I conclude that indeed the copula theory employed in this dissertation allows me to demonstrate statistically that beyond the intrinsic risk associated to high interest rate countries (which are generally paying higher interest rates to compensate for a higher risk) typically studied in the literature from a marginal perspective, another source of risk plays an important role. This second source of risk is related to the dependence structures linking these high interest rate currencies, more specifically the significant tail dependence features observed in this model analysis. I indeed proved through a mixture of Archimedean copulae the significant presence of tail dependence among high interest rate currencies which could have dramatic consequences on the carry trade portfolio's risk profile when accounted for appropriately in risk reward analysis. As a matter of fact, the tail dependence directly influences the diversity of the assets and thus reduces the appealing convergence property stated by the modern portfolio theory.

In other words, this copula based probabilistic modelling approach allows me to demonstrate that besides the intrinsic risk associated to each particular high interest rate currency, another factor constitutes a determining source of risk which turns out to be the level of risk aversion prevailing in the market. It was demonstrated in this analysis that both upper and lower tail dependence features displayed significant association and asymmetries with each other between the high and low interest rate baskets during periods of relative financial stability versus periods of heightened market volatility.

These tail dependence features in the high interest rate basket were significantly increasing during crisis periods leading to an increased amount of risk associated with utilising such currency baskets (which were no longer diversified due to the presence of significant tail dependence features) in a carry trade. That being said, a rational portfolio manager's natural risk aversion tells them that they should receive an additional remuneration in order to offset any additional sources of risk associated to an investment. Therefore, to properly assess the profitability of the currency carry trade, such tail dependence features should be incorporated into the analysis of such risk-rewards when developing a trading strategy. To conclude, this investigation rigorously tempers the too often claimed attractiveness of the currency carry trade and provides to investors a risk management tool in order to control and monitor the risk contained in such positions.

5.2 Future Research

The novel approach proposed in this dissertation paves the way for much further research in this and related areas. Most directly, I am currently investigating this puzzle using a much larger dataset (49 currencies) across a longer time period (1983 - 2013) with the expectation of similar results, reinforcing the findings here. Research ideas that are currently being explored or are intended to be explored in the near future include:

1. Analysing the returns from portfolios constructed not only using interest rate differentials, but also using stochastic ordering of individual currencies and multivariate Spearman's rank correlation.

2. Carrying out a regression analysis on the open interest of carry currencies with the multivariate tail dependence present in the funding and investment portfolios.

3. Modelling the carry portfolios using vine copulas to assess whether models built up of pairwise dependence blocks provides us with a better understanding of the tail characteristics.

4. Further exploring the outer power copula models and their relative advantages/disadvantages when compared to the mixture copula model proposed in this dissertation.

5. Creating a dynamic approach to the copula modelling framework in order to capture the time-varying nature of the dependence structure being considered.

Finally, the concept of joint tail exposure in portfolios that has been explored in this dissertation with regards to the currency carry trade, can also be applied to any asset class and indeed any portfolio of assets. This presents a more sophisticated approach to the challenges of risk management and optimal portfolio allocation.

Appendix A

Archimedean Copula Derivatives

A.1 Multivariate Clayton Copula

A.1.1 $C_\rho^C(\mathbf{u})$

$$C_\rho^C(\mathbf{u}) = \left(\sum_{i=1}^{d} u_i^{-\rho} - d + 1\right)^{-\frac{1}{\rho}} \quad , \quad \rho > 0 \tag{A.1}$$

A.1.2 $\psi_\rho^{(d)}$: d-th derivative of the Clayton generator

$$(-1)^d \psi_\rho^{(d)}(t) = \frac{\Gamma\left(d + \frac{1}{\rho}\right)}{\Gamma\left(\frac{1}{\rho}\right)} (1+t)^{-\left(d+\frac{1}{\rho}\right)} \tag{A.2}$$

A.1.3 Clayton Copula Density $\left(\frac{\partial^d C}{\partial u_1 \ldots \partial u_d}\right)$

$$c_\rho^C(\mathbf{u}) = \prod_{k=0}^{d-1}(\rho k + 1)\left(\prod_{i=1}^{d} u_i\right)^{-(1+\rho)} \left(1 + t_\rho^C(\mathbf{u})\right)^{\left(-d+\frac{1}{\rho}\right)} \tag{A.3}$$

where

$t_\rho^C(\mathbf{u}) = \sum_{i=1}^{d} \psi_C^{-1}(u_i)$

$\psi_C^{-1}(u_i) = (u_i^{-\rho} - 1)$

A.2 Multivariate Frank Copula

A.2.1 $C_\rho^F(\mathbf{u})$

$$C_\rho^F(\mathbf{u}) = -\frac{1}{\rho} \ln \left(1 + \frac{\prod_{i=1}^{d} (e^{-\rho u_i} - 1)}{(e^{-\rho} - 1)^{d-1}} \right) \quad , \quad \rho > 0 \tag{A.4}$$

A.2.2 $\psi_\rho^{(d)}$: d-th derivative of the Frank generator

$$(-1)^d \psi_\rho^{(d)}(t) = \frac{1}{\rho} Li_{-(d-1)} \left\{ (1 - e^{-\rho}) e^{-t} \right\} \quad , \quad t \in (0, \infty), \quad d \in \mathbb{N}_0 \tag{A.5}$$

where $Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$

A.2.3 Frank Copula Density $\left(\frac{\partial^d C}{\partial u_1 \ldots \partial u_d} \right)$

$$c_\rho^F(\mathbf{u}) = \left(\frac{\rho}{1 - e^{-\rho}} \right)^{d-1} Li_{-(d-1)} \left\{ h_\rho^F(\mathbf{u}) \right\} \frac{e^{\left(-\rho \sum_{j=1}^{d} u_j \right)}}{h_\rho^F(\mathbf{u})} \tag{A.6}$$

where

$h_\rho^F(\mathbf{u}) = (1 - e^{-\rho})^{1-d} \prod_{j=1}^{d} \{1 - e^{-\rho u_j}\}$

A.3 Multivariate Gumbel Copula

A.3.1 $C_\rho^G(\mathbf{u})$

$$C_\rho^G(\mathbf{u}) = e^{-\left(\sum\limits_{i=1}^{d}(-log\ u_i)^\rho\right)^{\frac{1}{\rho}}} \quad , \quad \rho \geq 1 \tag{A.7}$$

A.3.2 $\psi_\rho^{(d)}$: d-th derivative of the Gumbel generator

$$(-1)^d \psi_\rho^{(d)}(t) = \frac{\psi_\rho(t)}{t^d} P^G_{d,\frac{1}{\rho}}\left(t^{\frac{1}{\rho}}\right) \quad , \quad t \in (0, \infty),\ d \in \mathbb{N} \tag{A.8}$$

where

$P^G_{d,\frac{1}{\rho}}\left(t^{\frac{1}{\rho}}\right) = \sum\limits_{k=1}^{d} a^G_{dk}\left(\frac{1}{\rho}\right)(t^{\frac{1}{\rho}})^k$

$a^G_{dk}(\frac{1}{\rho}) = \dfrac{d!}{k!}\sum\limits_{i=1}^{k}\binom{k}{i}\binom{\frac{i}{\rho}}{d}(-1)^{d-i}\ , \quad k \in 1, ..., d$

A.3.3 Gumbel Copula Density $\left(\frac{\partial^d C}{\partial u_1 ... \partial u_d}\right)$

$$c_\rho^G(\mathbf{u}) = \rho^d e^{\left(-t_\rho(\mathbf{u})^{\frac{1}{\rho}}\right)} \frac{\prod\limits_{i=i}^{d}(-log\ u_i)^{\rho-1}}{t_\rho(\mathbf{u})^d \prod\limits_{i=1}^{d} u_i} P^G_{d,\frac{1}{\rho}}(t_\rho^G(\mathbf{u})^{\frac{1}{\rho}}) \tag{A.9}$$

where

$P^G_{d,\frac{1}{\rho}}(t^{\frac{1}{\rho}}) = \sum\limits_{k=1}^{d} a^G_{dk}(\frac{1}{\rho})(t^{\frac{1}{\rho}})^k$

$a^G_{dk}(\frac{1}{\rho}) = \dfrac{d!}{k!}\sum\limits_{i=1}^{k}\binom{k}{i}\binom{\frac{i}{\rho}}{d}(-1)^{d-i}\ , \quad k \in 1, ..., d$

$t_\rho^G(\mathbf{u}) = \sum\limits_{i=1}^{d} \psi_G^{-1}(u_i)$

$\psi_G^{-1}(u_i) = (-log\ u_i)^\rho$

A.4 Multivariate Clayton-Frank-Gumbel Mixture Copula

A.4.1 $C^{CFG}_{\rho_1,\rho_2,\rho_3}(\mathbf{u})$

$$
\begin{aligned}
C^{CFG}_{\rho_C,\rho_F,\rho_G}(\mathbf{u}) = & \lambda_C(C^C_{\rho_C}(\mathbf{u})) + \lambda_F(C^F_{\rho_F}(\mathbf{u})) + \lambda_G(C^G_{\rho_G}(\mathbf{u})) \\
= & \lambda_C \times \left(\sum_{i=1}^{d} u_i^{-\rho} - d + 1\right)^{-\frac{1}{\rho}} \\
& + \lambda_F \times -\frac{1}{\rho} \ln \left(1 + \frac{\prod_{i=1}^{d} (e^{-\rho u_i} - 1)}{(e^{-\rho} - 1)^{d-1}}\right) \\
& + \lambda_G \times e^{-\left(\sum_{i=1}^{d} (-log\ u_i)^{\rho}\right)^{\frac{1}{\rho}}}
\end{aligned}
\tag{A.10}
$$

A.4.2 Clayton-Frank-Gumbel Mixture Copula Density

$$
\begin{aligned}
c^{CFG}_{\rho_C,\rho_F,\rho_G}(\mathbf{u}) = & \lambda_C(c^C_{\rho_C}(\mathbf{u})) + \lambda_F(c^F_{\rho_F}(\mathbf{u})) + \lambda_G(c^G_{\rho_G}(\mathbf{u})) \\
= & \lambda_C \times \prod_{k=0}^{d-1}(\rho k + 1) \left(\prod_{i=1}^{d} u_i\right)^{-(1+\rho)} \left(1 + t^C_{\rho}(\mathbf{u})\right)^{\left(-d+\frac{1}{\rho}\right)} \\
& + \lambda_F \times \left(\frac{\rho}{1 - e^{-\rho}}\right)^{d-1} Li_{-(d-1)} \left\{h^F_{\rho}(\mathbf{u})\right\} \frac{e^{\left(-\rho \sum_{j=1}^{d} u_j\right)}}{h^F_{\rho}(\mathbf{u})} \\
& + \lambda_G \times \rho^d e^{\left(-t_{\rho}(\mathbf{u})^{\frac{1}{\rho}}\right)} \frac{\prod_{i=i}^{d}(-log\ u_i)^{\rho-1}}{t_{\rho}(\mathbf{u})^d \prod_{i=1}^{d} u_i} P^G_{d,\frac{1}{\rho}}(t^G_{\rho}(\mathbf{u})^{\frac{1}{\rho}})
\end{aligned}
\tag{A.11}
$$

where

$t_\rho^C(\mathbf{u}) = \sum_{i=1}^{d} \psi_C^{-1}(u_i)$

$\psi_C^{-1}(u_i) = (u_i^{-\rho} - 1)$

$$h_\rho^F(\mathbf{u}) = (1 - e^{-\rho})^{1-d} \prod_{j=1}^{d} \{1 - e^{-\rho u_j}\}$$

$$P_{d,\frac{1}{\rho}}^G(t^{\frac{1}{\rho}}) = \sum_{k=1}^{d} a_{dk}^G(\tfrac{1}{\rho})(t^{\frac{1}{\rho}})^k$$

$$a_{dk}^G(\tfrac{1}{\rho}) = \frac{d!}{k!} \sum_{i=1}^{k} \binom{k}{i} \binom{\frac{i}{\rho}}{d} (-1)^{d-i} \ , \quad k \in 1, ..., d$$

$t_\rho^G(\mathbf{u}) = \sum_{i=1}^{d} \psi_G^{-1}(u_i)$

$\psi_G^{-1}(u_i) = (-log\ u_i)^\rho$

Appendix B

Matlab Code

B.1 Script for Fitting Mixture Copulae to Rolling Daily Portfolios

```
%% General script for fitting a candidate copula to the basket of
% the 4 currencies that have highest interest rate differential
% (proxy) for that day: using data for previous block_length number
% of days, i.e. 6 months or 12 months.

% load data
load forwards;
load spots;

block_length = 125; % i.e. ~ 6 months

% calculate the IR_differential proxy for every day
calc_IR_proxy;
```

```
% calculate baskets
calc_baskets;

% Take log-Returns for full dataset
log_R_forwards = log_R(forwards);



%% pre-allocate storage
mu = zeros(size(log_R_forwards,1) - block_length + 1,...
 size(log_R_forwards,2));
sigma = zeros(size(log_R_forwards,1) - block_length + 1,...
 size(log_R_forwards,2));

d = zeros(size(log_R_forwards,1) - block_length + 1,...
 size(log_R_forwards,2));

lambda_c_hat = zeros(1, size(log_R_forwards,1) - block_length + 1);
lambda_f_hat = zeros(1, size(log_R_forwards,1) - block_length + 1);
lambda_g_hat = zeros(1, size(log_R_forwards,1) - block_length + 1);
rho_clayton_hat = zeros(1, size(log_R_forwards,1) - block_length + 1);
rho_frank_hat = zeros(1, size(log_R_forwards,1) - block_length + 1);
rho_gumbel_hat = zeros(1, size(log_R_forwards,1) - block_length + 1);
nll = zeros(1, size(log_R_forwards,1) - block_length + 1);

AIC = zeros(1, size(log_R_forwards,1) - block_length + 1);
BIC = zeros(1, size(log_R_forwards,1) - block_length + 1);
%% end storage

% for each day
```

```
for i = block_length:size(log_R_forwards,1)

    k = i - block_length + 1;

    % get data for previous block_length days
    datablock = log_R_forwards(i - block_length + 1:i, :);

    %% THIS CODE IS FOR THE LOGNORMAL FITS:
    % fit log-normal margins to dataset block
    % mu(k,:) = mean(datablock);
    % sigma(k,:) = std(datablock);

    %% U = the [0, 1] uniform marginals data
    % U = zeros(size(datablock));
    % for j = 1:size(datablock,2)
    %     U(:,j) = normcdf(datablock(:, j), mu(k,j), sigma(k,j));
    %
    %     % decision -> 0 = accept null (that is logNormal)
    %     d(k,j) = kstest((datablock(:,j) - mu(k,j)) ./ sigma(k,j));
    %
    %     %ecdf(log_R_forwards(k,j) - mu(k,j) ./ sigma(k,j))
    %     end
    %%

    % length of y vector => i.e. number of days
    n = size(datablock,1);

    % U = the [0, 1] uniform marginals data
    U = zeros(size(datablock, 1), 4);
```

```
j_index = 1;
for j = basket_high_IR(i, 1:4)

    [loglike, sigma_tilde, mu_tilde, k_grid] = ...
    generalised_gamma(datablock(:,j));

    % trim off at point when jumps to infinity
    Inf_cut_off = find(loglike == Inf,1);
    loglike = loglike(1:Inf_cut_off - 1);

    % find max log likelihood
    [MAX_loglike Ind] = max(loglike);



    % get mle parameters
    k_hat = k_grid(Ind);
    sigma_hat = sigma_tilde(Ind);
    mu_hat = mu_tilde(Ind);

    % TRANSFORM TO [0, 1] MARGINS
    [w P_cdf] = gengammacdf(k_hat);

    % get w
    w_evaluate = (datablock(:,j) - mu_hat) ./ sigma_hat;

    % loop through each datapoint to find
    % cdf value for w_evaluate
    for a = 1:n
        U_basket_H(a, j_index) =...
        P_cdf(find(w > w_evaluate(a), 1));
```

```
    end
    j_index = j_index + 1;
end

% truncate extremes to prevent numerical issues.
U_basket_H(U_basket_H > .9999) = .9999;
U_basket_H(U_basket_H < .0001) = .0001;



%% Fit 4-dimensional copula to this U datablock for
% highest 4 currencies (by IR differential proxy)

% mixture C-F-G copula
[lambda_c_hat(k), lambda_f_hat(k), lambda_g_hat(k), ...
    rho_clayton_hat(k), rho_frank_hat(k), rho_gumbel_hat(k)...
    nll(k)] = ...
    vec_copulafit_clayton_frank_gumbel(U_basket_H);

num_parameters = 6;

%% AIC & BIC
AIC(k) = (2*nll(k)) + (2*num_parameters);
BIC(k) = (2*nll(k)) + (k*log(size(block_length,1)));

% display progress every 1 percent
done = (100*k) / roundn((size(forwards,1) - block_length), 2);

if(mod(done,1) == 0)
    disp([num2str(done) '%']);
```

```
        % save workspace variables
        savefile = ['NEW_6_month_fit_high_IR_run_',...
        num2str(k), 'variables'];
        save(savefile, 'lambda_c_hat', 'lambda_f_hat',...
        'lambda_g_hat', 'rho_clayton_hat', 'rho_frank_hat',...
        'rho_gumbel_hat', 'nll', 'AIC', 'BIC', 'd');
    end
end % end script
```

B.2 Mixture Copula Fitting Function

```
%% Fits Clayton—Frank—Gumbel Mixture Copula to data
% that has been transformed to [0,1]margins.
% (i.e. after margins have been fitted).

% Input: (n x d) Data matrix U, with [0,1] margins.

% Outputs: Maximum likelihood estimates for:
% lambda_c_hat — clayton mixture component
% lambda_f_hat — frank mixture component
% lambda_g_hat — gumbel mixture component
% rho_clayton_hat — clayton copula parameter
% rho_frank_hat — clayton copula parameter
% rho_gumbel_hat — gumbel copula parameter

function [lambda_c_hat, lambda_f_hat, lambda_g_hat, ...
    rho_clayton_hat, rho_frank_hat, rho_gumbel_hat,...
    nll] = ...
    vec_copulafit_clayton_frank_gumbel(U)
```

```
%% Fit clayton-frank-gumbel mixture copula using fmincon

% initialisation of lambda, rho_clayton, rho_gumbel
x0 = [0.33; 0.33; 0.34; 2; 2; 2];

% bounds
lb = [0; 0; 0; eps; -10; 1];
ub = [1; 1; 1; 10; 10;10];

% Make sure three lambdas add up to 1 !!!
Aeq = [1 1 1 0 0 0];

beq = [1]; %1];

options = optimset('Display','off','Algorithm','interior-point');

% fmincon
[x, nll] = fmincon(@vec_negloglike_clayton_frank_gumbel_md,...
x0,[],[],Aeq,beq,lb,ub,[],options);
lambda_c_hat = x(1);
lambda_f_hat = x(2);
lambda_g_hat = x(3);
rho_clayton_hat = x(4);
rho_frank_hat = x(5);
rho_gumbel_hat = x(6);

disp(['fmincon estimates: ' num2str(x') ]);

%%-- AUXILIARY FUNCTION TO PASS TO OPTIMISATION ALGORITHMS --%%
```

```
% Multivariate Clayton-Frank-Gumbel copula
% negative log likelihood function
    function nll = vec_negloglike_clayton_frank_gumbel_md(X)

        lambda_nll_c = X(1);
        lambda_nll_f = X(2);
        lambda_nll_g = X(3);
        rho_c = X(4);
        rho_f = X(5);
        rho_g = X(6);

        %% check frank parameter is != 0
        if(rho_f < 0.01 && rho_f > -0.01)
            rho_f = 0.01;
        end

        nll = -sum( log( lambda_nll_c*(c_d) +...
            lambda_nll_f*(f_d)...
            + lambda_nll_g*(g_d) ) );
    end
end
```

B.3 Generalised Gamma Function

```
function [loglike, sigma_tilde, mu_tilde, k_grid] =...
 generalised_gamma(y)

% Input: y is log returns for 6 months
```

```
% length of y vector => i.e. number of days
n = size(y,1);

% storage counter for the values returned from grid of k values
index = 1;

%% Stage 1: Grid of k values
k_grid = logspace(-1,2.2);
for k = logspace(-1,2.2)

    %% Stage 2: SOLVE MLE ROOT SEARCH ON LAWLESS_EQUATION_7
    %      => TO FIND SIGMA_TILDE

    % set upper bound for sigma
    sigma0 = 100;

    % solve mle root search for sigma
    %!! Careful with setting lowerbound of interval !!%
    sigma_tilde(index) = fzero(@lawless_eqn_7,[0.1*std(y) sigma0]);

    %% 3: NOW SUBSTITUTE SIGMA_TILDE INTO LAWLESS_EQUATION_6

    mu_tilde(index) = lawless_eqn_6(sigma_tilde(index));

    %% 4: CALCULATE THE LOG-LIKELIHOOD WITH PARAMETERS
    %       SIGMA_TILDE, MU_TILDE, K

    loglike(index) =...
    log_likelihood_log_generalised_gamma(sigma_tilde(index),...
```

```
    mu_tilde(index), k);

    % move storage counter +1
    index = index + 1;

end

% Equation (7) from lawless paper on generalised gamma.
% Returns value of LHS of equation 7
    function value = lawless_eqn_7(sigma)

        part_1_num = sum(y .* exp(y ./ (sigma .* sqrt(k))), 1);
        part_1_denom = sum(exp(y ./ (sigma .* sqrt(k))), 1);
        part_2 = mean(y);
        part_3 = sigma / sqrt(k);

        % Return value of LHS of eqn_7
        value = (part_1_num / part_1_denom) - part_2 - part_3;
    end

% Equation (6) from lawless paper on generalised gamma.
% Returns mu_tilde, i.e. log of RHS.
    function mu_value = lawless_eqn_6(sigma_tilde)
        %NOTE:  changed to .*
        mu_value = log( ((1/n) .* sum( exp(y ./ (sigma_tilde .*...
        sqrt(k))), 1)) .^ (sigma_tilde .* sqrt(k)));
    end

% Returns log_likelihood of log_generalised_gamma
    function loglikelihood = ...
```

```
    log_likelihood_log_generalised_gamma(sigma_tilde, mu_tilde, k)
        % define w
        w = (y − mu_tilde) ./ sigma_tilde;

        % sum of log ( pdf of w evaluated at x_i 's)
        loglikelihood = sum( log((kˆ(k − (1/2)) / gamma(k)) .∗...
        exp((sqrt(k) .∗w) − (k .∗ exp(w ./ sqrt(k))))), 1);
    end
end
```

B.4 Generalised Gamma CDF

```
% Returns cdf value for w −> with pdf given by (4) in lawless
function [w P_cdf] = gengammacdf(k_hat)

delta = 0.001;   % step length along x−axis for cdf area calculation.
w = −1000:delta:1000;
P_eval =  (k_hatˆ(k_hat − (1/2)) / gamma(k_hat)) .∗ ...
exp((sqrt(k_hat) .∗w) − (k_hat .∗ exp(w ./ sqrt(k_hat))));

Z = sum(P_eval) ∗ delta;

P = P_eval .∗ delta;

P_cdf = cumsum(P ./ Z) ;
end
```

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