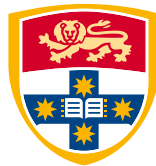


Generalised linear Gegenbauer long memory models for time series of counts with financial and insurance applications

HONGXUAN YAN

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School of Mathematics and Statistics
Faculty of Science
University of Sydney



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Declaration of authorship

I, Hongxuan Yan, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work. I have duly acknowledged all the sources of information which have been used in the thesis. This thesis has also not been submitted for any degree in any university previously.

Abstract

The contribution of this thesis is on developing new models and applying them to analyse data in finance and insurance. On model development, we propose to incorporate the family of Gegenbauer autoregressive moving average (GARMA) models and a special sub-family called the autoregressive fractionally integrated moving average (ARFIMA) models into the mean functions of the count distributions, including Poisson, negative binomial (NB), generalised Poisson (GP) and double Poisson (DP) within a generalised linear modelling framework. These count distributions display equi-, over- and under-dispersion to enhance model flexibility.

The proposed generalised linear GARMA models (GLGARMA) are defined for the error terms under two approaches, the parameter-driven (PD) and observation-driven (OD) approaches. Model properties including the oscillatory behaviour of the autocorrelation function (ACFs) in the time domain and spectral density function in the frequency domain are studied for Gegenbauer long memory model and compared to seasonal long memory model. Results show that seasonal long memory cannot replace Gegenbauer long memory. Moreover, the approximated spectral density function of the PD GLGARMA model is derived to obtain a deeper understanding of the oscillatory behaviour in the Gegenbauer polynomial and facilitate Whittle likelihood estimation. To estimate and forecast these models, we adopt a Bayesian approach implemented using the R package **Rstan**. Following this, various model selection criteria including deviance information criterion are evaluated to select some best-fitting models to undertake forecasting of future events.

For financial application, 136 open interest series from U.S. Commodity Futures Trading Commission (CFTC) cross classified by four fixed income securities from the U.S. federal government, market participant types and trading behaviour first tested for long memory using Hurst exponent and then analysed through the ACFs and periodograms for the types of long memory structures. Four types of structures, namely basic, oscillatory, combined and irregular, are identified and among them, basic and irregular long-memory structures are more prevalent in these highly liquid and important U.S.

treasury futures. All 136 CFTC data are fitted to 40 models with different distribution choices (Poisson, NB, GP, DP), model types (PD, OD), long memory types (ARFIMA, GARMA) and orders. Compared with other types of count models including a seasonal model, the GLGARMA models outperform these models in both in-sample fittings and out-of-sample forecasts for each type of long memory structures. Furthermore, the trade type spread with oscillatory long memory possibly due the seasonal rollover effect is still best modelled by Gegenbauer long memory but not seasonal long memory.

For insurance application, the prevalence of long memory in death count series from 16 selected countries from Human Mortality Database (HMD) cross classified by age groups and gender is demonstrated by adopting again the Hurst exponent. The different long memory patterns across genders, age groups, countries and time periods motivate us to incorporate the Lee-Carter (LC) model allowing for period effect with our GLGARMA model. For the univariate case of mortality models, we further extend this LC type GLGARMA (GLGARMA-LC) models to incorporate the graduation model resulting in three modelling components, namely, period effect, graduation effect and long memory and thereby eight submodels by inclusion of these components. These eight submodels are applied to fit the death count series and comparison is made between model 3 of basic LC model and model 8 with long memory and graduation effects. Results show that the long memory structures enhance the accuracy of in-sample fitting and out-of-sample forecast. By adopting model 8 (GLGARMA-LC model with graduation), the improvement in life expectancy calculation is clearly seen and it leads to more accurate forecast of life table 20 years ahead.

In order to capture cohort effects and avoid identification issues in LC models, a stationary long memory mortality model with long memory cohort effect (LMLM) model is proposed to analyse three selected countries' death count series. Compared with the multivariate extended LC cohort (MELCC) model, the in-sample fitting and out-of-sample forecast performances of our proposed LMLM model are much better. Consequently, life expectancies are also calculated to exhibit the enhancement by adopting LMLM model. Assuming constant interest rate, the annuity pricing is also studied to demonstrate the enhancement of adopting LMLM model. Attempt is further made to derive stochastic interest rate model using the popular Vasicek model but discretised to fit empirical data. Moreover we propose four dependency models, namely independence, linear correlated and rank correlated in a marginal approach and the jointly

distributed approach. With three mortality estimates from MELCC and LMLM models and from HMD, twelve formulations are applied to calculate pricing of annuity and generalised annuity options (GAO). Results again confirms the advantages of adopting LMLM models.

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Publications and presentations

Publications

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- b) Mortality models incorporating long memory improves life table: a comprehensive analysis HONGXUAN YAN, Gareth W. Peters and Jennifer S.K. Chan
- c) Multivariate long memory cohort mortality model HONGXUAN YAN, Gareth W. Peters and Jennifer S.K. Chan

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CHAPTER 1

Introduction

This chapter presents an overview of the thesis that includes the motivations for proposing a Generalised Linear Gegenbauer Autoregressive Moving Average (GLGARMA) model with discrete distribution and various applications in finance and insurance, the background information of previous related works, the contributions of this thesis and finally, an outline of the structure of the thesis.

1.1 Motivation

The principal motivation of this thesis is to develop and investigate a class of count processes in both univariate and multivariate setting. Specifically, we propose a GLGARMA model with discrete distribution to capture various long memory features. We pay particular attention to different regression structures, temporal structures and stochastic structures and derive time series models with state space formulation to capture their dynamics. A time series is said to have long range dependency or long memory if it has a slowly decaying autocorrelation function (ACF) which decays at a rate slower than an exponential decay. We also study different long memory processes including basic long memory, Gegenbauer long memory and seasonal long memory. Furthermore, the properties of these long memory processes in both time domain and frequency domain are compared. Different approaches to estimation, including both frequentist and Bayesian approaches, are explored. In the Bayesian approach, types of Markov chain Monte Carlo (MCMC) method which can speed up estimation and forecast procedures are investigated. Various model selection criteria are evaluated to identify models with

the best performance in in-sample fitting and out-sample forecast. Then we develop the GLGARMA model for different application scenarios in finance and insurance.

Based on these two key areas of application, this thesis focuses on modelling open interest in future markets for designing trading strategy as well as modelling mortality data for life table construction and insurance product formulation. From a financial perspective, the proposed models are applied to study future market liquidity dynamics which is a measure of the flow of money in and out of the futures market. To capture the most active futures contracts traded in financial markets, the U.S. Commodity Futures Trading Commission (CFTC) open interest time series data crossed classified with four fixed income securities provided by the U.S. federal government are analysed. Moreover, various market participant behaviours are investigated, and their differences in serial correlation and persistence in trading activities are identified. Consequently, the more profound understanding of future liquidity under different market participants and price volatility can be obtained. From an insurance perspective, this thesis proposes several mortality models incorporating long memory structures to improve the mortality rates modelling and forecasting and life table construction. The non-parametric measure called Hurst exponent is adopted to show the prevalence of long memory in the mortality data called Human Mortality Database (HMD) ([Vladimir et al., 2017](#)) which is publicly accessible. These mortality data are fitted to new mortality models by extending the Lee-Carter (LC) model to death counts and incorporating a long memory model structure. We begin with univariate long memory mortality models which model mortality rate of each single age group. We also demonstrate the ability of our multivariate extensions which model simultaneously mortality rates of all age groups to capture the cohort effect. We demonstrate that this multivariate model solves the identification issues caused by the structure of the LC model.

The *first key motivation* of this thesis is to propose flexible models to capture the long-term dependency between the present and all previous observations in a count process. The count process which defined as a non-negative integer-valued time series with various long memory features is crucial in explaining some natural events studied in geophysics, biology, finance, insurance and other areas. Autoregressive fractionally integrated moving average (ARFIMA) models with fractionally integrated values for the differencing parameter are extended from autoregressive integrated moving average

(ARIMA) model to model continuous time series with long memory. Gegenbauer ARMA (GARMA) model, as a generalised form of ARFIMA model, is adopted to model a slowly damping ACF with an oscillatory pattern. However, these long memory models are not applicable to time series of counts. On the other hand, current time series models for count data cannot properly capture Gegenbauer long memory feature and so, generalised linear ARFIMA (GLARFIMA) and GLGARMA models should be developed to seize both features of long memory and count measurement. Furthermore, these models should be extended to capture different dispersion properties of a count process by investigating and suitably choosing count distributions including Poisson, negative binomial (NB), generalised Poisson (GP) and double Poisson (DP) with equi-, over- and underdispersion.

Apart from model development, their properties including the long memory property should also be better understood. One way is via the ACF expressed alternatively through the frequency domain instead of time domain. This allows graphing of ACF along the frequency domain to provide a more vivid image of the oscillatory pattern of long memory. Other properties of long memory including the differences between seasonal long memory and Gegenbauer long memory should be distinctly identified so that suitable type of long memory model can be chosen to describe the long memory feature of a count process. This kind of study is currently missing for count processes. In model estimation, there are some unsolved issues in estimating the long memory parameters under the maximum likelihood (ML) approach as the likelihood function under the usual time domain can be complicated but such issue is much lessened when using the whittle likelihood function. In addition, Bayesian MCMC is also preferable for models with latent variables as it replaces the evaluation of likelihood function containing high dimensional integral with the more direct and efficient posterior simulation. Hence, Bayesian approach can also be considered and explored further. Lastly, the application of long memory count models is not commonly seen in finance and insurance. It will be a significant contribution if these estimation techniques can be applied to estimate the GLGARMA model to provide reliable in-sample fitting and out-of-sample forecast for stakeholders of relevance.

The *second key motivation* of this thesis is to study and investigate the dynamics of open interest which is the number of futures contracts that are not matured in financial

markets. Understanding these dynamics helps us to gain a clearer picture of the relationships between price discovery and price volatility in trading activities. To obtain such understanding and gain insight into liquidity dynamics in financial markets, one should model the dynamics of open interest through some time series models with long memory. The term liquidity provision signifies the monetary flows in and out of the futures markets as quantified by the open interest on futures contracts. Further studies of open interest by market participants reveal market participant behaviours and liquidity provision through market making and results are beneficial not only for providing more accurate open interest forecasts but also for fundamental trading strategy design. However, statistical models particularly developed to analyse open interest count series with different forms of long memory features are still limited. The study of open interest can be carried out through analysing the U.S. Commodity Futures Trading Commission (CFTC) time series of counts provided by U.S. federal government. This data contains 136 time series of open interest for futures-only and futures-and-options crossed classified with four fixed income securities: treasury notes at 2, 5 and 10-year terms and treasury bonds at 20 or 30-year terms and with four market participant types: namely, dealers, asset managers, leveraged funds and other reportables subdivided according to the categories of reportables and non-reportables and sub-categories of positions of their trading behaviours categorised by the buy and sell sides of long-call, short-put and spread being the difference between long-call and short-put. This open interest data represent some of the most active futures contracts traded in financial markets. The analysis developed provides the first detailed study of this kind undertaken by market participant type and is challenging to analyse because of its large scale and data versatility. However, in-depth studies of CFTC or other related data to identify any particular long-memory structures, forecastable trading behaviours and liquidities by market participant types are not yet reported or identified in the financial literature.

The *third key motivation* of this thesis is to examine and model mortality data. Mortality is studied in many areas such as demography, insurance and social policy. It is crucial for governments and insurance companies to obtain accurate forecasts of mortality rates to provide population projections for demographers and cash flow projections for actuaries. Over-estimated mortality rates increase premium costs for policyholders whereas under-estimated mortality rates cause longevity risk which is a high-risk profile for pension funds. For the past decades, there exists a significant increase in

life expectancy in many developed countries over the last century because of the improvements in medical technologies and scientific advances in disease control. Due to this longevity risk caused by the declining trends in mortality rates, governments and insurance companies need to modify expected payout levels to account for the longer life expectancy of policyholders. The existence of long memory and the dynamics of this feature across genders, age groups, countries and time periods before and after World War Two in mortality data have not been analysed before. Understanding and incorporating this stylised long memory features will provide more accurate modelling and reliable forecasting of mortality rates for life expectancy calculation in life table construction.

Lee-Carter (LC) model has been used extensively to model mortality rates for all age groups simultaneously. Because of its appealing model structure, it receives great popularity but it also suffers from parameter identification issue. Many attempts have been made to extend LC model. Firstly, modelling the death counts directly can also provide mortality rate estimates and allow dynamic volatility changes as the variance for most discrete count distributions can be expressed in terms of mean function and so it will change with the mean. Long memory and graduation models can be further included to provide more flexible and accurate modelling. LC model can also be applied to model a single age group and such model is called univariate model (as versus multivariate model for all age groups). Multivariate LC model has been extended to capture the cohort effect across age groups and attempts have been made to solve the identification issue by imposing parameter constraints. This multivariate LC model with cohort effect can be extended to multivariate extended LC cohort (MELCC) model for death counts and stationary multivariate long memory mortality model with long memory cohort effect (LMLM) for death counts which contains long memory model for both the mean and cohort functions. Comparing their model structures, LMLM model postulates to replace the period effect by a long memory model in the mean function. It is interesting to investigate the significance or non-significance of these model structures, including trend and long memory, for different functional components, namely, the mean, period and cohort. Studies can also be directed to how these structures, such as long memory, can avoid the identification issue of parameters faced by common LC models. Lastly, comparison can be made to the model performance and to confirm if in-sample fit and out-of-sample forecast are enhanced significantly by adopting the LMLM model with stationary long memory structures.

The *fourth key motivation* of this thesis is to derive some annuity and generalised annuity option (GAO) pricing procedures using the mortality rate estimates to provide more precise valuation. A life annuity provides an individual with the opportunity to convert a stock of wealth into an income until the end of life. GAO is a mortality-related guarantee to policyholder from the insurance company to convert an accumulated fund to a life annuity at a fixed rate when the policy matures. To enhance the accuracies of life insurance product design and pricing, there are three crucial stochastic factors to consider, namely, the prevailing interest rates at the time the annuity is purchased, the life expectancy of the annuitant and the causal relationship between mortality events and interest rate fluctuations. The annuity pricing procedures should adopt stochastic interest rates and reliable mortality rate estimates to reduce the impact of mis-pricing, which can be a significant risk while considering a large annuity portfolio. Secondly, classical approaches treat mortality rates as independent of interest rates. However, further experience shows a non-ignorable interplay between financial risk and mortality risk for insurance product pricing and such interplay should be suitably modelled based on different correlation assumptions. The annuity pricing procedures should adopt stochastic interest rates and reliable mortality rate estimates to reduce the impact of mis-pricing, which can be a significant risk while considering a large annuity portfolio. For the valuation of GAO, diffusion models were adopted by traditional researchers to model mortality rates. However, without long memory model, this type of mortality models may not provide adequate estimation and prediction of mortality rates as compared to LMLM models. Hence comparisons should be conducted to confirm this.

1.2 Literature review

This section reviews the literature development for long memory models, time series count models and mortality models. Technical details are presented in the corresponding Chapters later.

1.2.1 Development of long memory time series model for counts

The increasing prevalence of long memory phenomena in many fields motivates researchers to analyse the non-ignorable dependence between the present observation of

a time series and all previous observations (Graves et al., 2014b). The rate at such dependency decreases can often be slower than exponential decay, leading one to consider long-memory processes to explain and model such persistence features in a time series. Beran (1994) defined a condition for a stationary long memory process in terms of the divergence of the ACF, denoted by $\rho(j)$ for integers j , such that

$$\sum_{j=-\infty}^{\infty} \rho(j) = \infty. \quad (1.1)$$

The Hurst exponent H as a non-parametric measure is utilised to measure long memory in a time series. (Hurst, 1951) and Davidson and De Jong (2000) showed the relationship of H with the fractional differencing operator parameter d as $d = H - 0.5$. Rescaled range analysis (Hurst, 1951), Detrended Fluctuation Analysis (DFA) (Peters, 1994) and periodogram regression (Geweke and Porter-Hudak, 1983) are three frequently used methods to estimate H . To incorporate such long memory property into the autoregressive moving average (ARMA) model (see Equation (2.6)), Granger and Joyeux (1980) and Hosking (1981) extended the classical autoregressive integrated moving average (ARIMA) (see Equation (2.7)) model to ARFIMA model. This is achieved by allowing the differencing operator d to be fractional ($0 < d < 1/2$), such that the ARFIMA(0, d , 0) model describes a long memory stationary process with a hyperbolic decay of the ACF as compared to the geometric decay for an ARIMA model.

However, many financial time series exhibit strong persistence as well as clear oscillatory behaviour. The Seasonal ARFIMA (SARFIMA) model was applied to study cyclical phenomena exhibiting long-range dependence ((Porter-Hudak, 1990) and (Hassler, 1994)). Further extensions to this class of models were also made through the useful generalised ARFIMA model called GARMA model which was proposed by Hosking (1981) to describe data showing slow damping ACF with an oscillatory pattern. See Section 2.5 for a comparison between SARFIMA and GARMA models in terms of model structures and ACFs. Although, the GARMA model has been successfully applied to model, forecast and analyse time series on the real number domain, the data in many areas such as biology, finance and engineering are measured on the integer domain usually in the form of counts. It is challenging to apply these classical time series models to analyse discrete valued time series.

One approach to modelling discrete valued time series, such as count series, is via the binomial thinning operator (see Equation (2.1)). Then the common ARMA structure

can be incorporated into the model and the resultant model is called the integer-valued ARMA (INARMA) model (see (2.2)). The integer-valued AR(1) (INAR(1)) model as the first INARMA type model was proposed by McKenzie (1985) and Al-Osh and Alzaid (1987). Freeland (1998) studied some aspects of INAR(1) model including estimation, forecasting and testing for independence. Freeland and McCabe (2004) adopted new methods of assessing the adequacy of INAR(1) by providing a new score function and the information matrix. Weiß (2008) proposed the INMA(q) family of models and derived relevant regression properties. Enciso-Mora et al. (2009) extended INAR model by incorporating explanatory variables. Furthermore, an efficient MCMC method was proposed estimation of parameters and analysis of AR orders. Then Quoreshi (2006) extended the INARMA model to integer-valued ARFIMA(p, d, q) model (see 2.3). However, estimating such a family of models is challenging. For example, the likelihood function for this general INARFIMA(p, d, q) model is complicated and moreover, the unconditional variance is also intractable to obtain in a closed form. Consequently, Quoreshi (2006) proposed the conditional least squares and generalised least squares estimation methods in a non-parametric approach. To tackle these challenges, alternative approaches were developed.

Amongst these alternative approaches, one considers a generalised linear modelling framework to model directly the mean of a count distribution. Under this approach, Heinen (2003) proposed the autoregressive conditional Poisson (ACP) model adopting a GARCH-type structure to the mean of Poisson distribution (see Equation (2.4)). (Ferland et al., 2006) derived the condition for covariance stationarity as well as strict stationarity. Unlike the INARMA model that the unconditional variance cannot be derived, the ACP model can deal with over-dispersed count data since the unconditional variance is always greater than the mean. Groß-Klußmann and Hautsch (2013) provided an alternative representation of the model in Equation (2.6) which hinted further extension to adopt the ARIMA structure in Equation (2.7). Another model having a similar structure as ACP is the generalised autoregressive score (GAS) model which is based on a score function s_t (Creal et al., 2013) (see Equations (2.8) and (2.9)).

Apart from ACP model, Davis et al. (1999) proposed another type of generalised linear model, by adopting an ARMA-type structure to the log mean of a count distribution through the canonical log link function and the extended model is called generalised linear ARMA (GLARMA) model. Cox et al. (1981) categorised this generalised linear

model according to the ways to define the error terms into two classes: observation driven (OD) models and parameter driven (PD) models (see Equation (2.12)). For OD models, the error terms are functions of lagged dependent variables and lagged exogenous variables. Conditional on the past information, the parameters are predictable and the likelihood evaluation is simplified. Typical examples of these models include ACP (see Equation (2.4) with errors $\nu_t = y_t - \mu_t$) and GAS model (see Equation (2.8) based on the score function (Creal et al., 2013)). For PD models, the parameters have separate sources of errors. Conditional on past and concurrent information, the parameters are not completely predictable. Hence, parameter estimation is more difficult than OD models because computing the likelihood function requires evaluation of a high-dimensional integral and are often not available in closed form. Typical examples of PD models include the stochastic volatility (SV) model (Shephard, 2005) and the stochastic intensity models (Bauwens and Hautsch, 2006). For GLARMA model, it can be classified as either OD or PD depending on different representation of the error terms (see Equation (2.12)).

Besides, other literature considered the choice of count distributions. Davis et al. (1999) initially adopted the Poisson distribution with equi-dispersion. However, many count data sets display over-dispersion with variance greater than the mean. To fit these over-dispersed data, Davis et al. (1999) adopted also NB distribution. However, as NB distribution fails to capture under-dispersion, Consul (1989) introduced GP distribution, which captures both under- and over-dispersed count data and nests Poisson distribution as a special case. Another favourable distribution for both under- and over-dispersed count data is the DP distribution considered by Efron (1986). Currently, some flexible model structures incorporating all of Gegenbauer long memory model, generalised linear model and flexible count distribution are still lacking in the finance literature.

1.2.2 Development of model properties in frequency domain

To facilitate the understanding long memory model, the model properties in the frequency domain should be investigated. The Whittle likelihood function, as an approximation to the negative Gaussian likelihood function in the frequency domain, was applied to the univariate stationary processes with zero mean by Whittle (1953). The

Whittle likelihood approach has broader application in estimating the spectral density and other parameters in a stochastic process because this approach reduces the number of evaluation procedures from $O(n^3)$ to $O(n \log n)$ where O provides a useful approximation to the actual number of steps in the computation. For the properties of Whittle likelihood estimator, the asymptotic behaviour and boundary effect of estimators were studied. [Giraitis and Surgailis \(1990\)](#) derived a theorem to prove asymptotical normality of Whittle's estimate of the parameter of strongly dependent linear sequences. Under different assumptions, [Giraitis and Taqqu \(1999a\)](#) further described different asymptotic normality of Whittle's estimates. For implementation, [Pawitan and O'sullivan \(1994\)](#) further extended Whittle likelihood to non-parametric spectral density estimation using the negative logarithm of Whittle's likelihood to build a penalised likelihood. The authors developed an algorithm involving iterative least squares, which is initialised by the log-periodogram. The local polynomial techniques were utilised by [Fan and Kreutzberger \(1998\)](#) to fit Whittle's likelihood for smoothing the result of the spectral density estimation. Given a stationary time series, [Choudhuri et al. \(2004\)](#) adopted a Bayesian approach to estimate the spectral density with a nonparametric prior. Due to the complicated likelihood function, a pseudo-posterior distribution was derived by updating the prior using the Whittle likelihood. Additionally, a consistent result of the spectral density based on pseudo-posterior distribution was also considered. In application, by extending the central limit theorem, [Dunsmuir \(1979\)](#) adopted Whittle likelihood function to estimate the parameters of multivariate time series. [Robinson \(1995a\)](#) developed whittle estimation procedures for the long memory parameter d of a time series and further highlighted that the estimator with a simpler limiting normal distribution is consistent. [Phillips and Shimotsu \(2004\)](#) applied the local Whittle likelihood to estimate the long-range dependence parameters and studied the asymptotic properties and consistency in the non-stationary setting, including the unit root case. [Shimotsu and Phillips \(2005\)](#) proposed an exact form of the local Whittle likelihood for estimating d . In comparison with the previous approach, this method is a general-purpose estimation procedure that can be utilised in both stationary and non-stationary regions of d . With the development of non-Gaussian time series, [Naito et al. \(2006\)](#) studied the local Whittle likelihood estimator for spectral densities of non-Gaussian linear processes and the asymptotic behaviours. However most of these applications consider Gaussian model but for non-Gaussian model, the derivation of spectral density function is very challenging.

1.2.3 Development of Bayesian model estimation and forecast

In terms of model estimation, there are many advantages of adopting a Bayesian approach. The ML method is feasible for the OD model because the likelihood function is tractable and numerical procedures like Newton-Raphson can be used to optimise the likelihood function (Groß-KlußMann and Hautsch, 2013). However, for the PD model, it involves very high dimensional integrals. Consequently, several techniques based on asymptotic expansions, simulation methods and numerical integration (Davis et al., 1999) are needed to evaluate or approximate these integrals. Current estimation methods, which incorporate these techniques include generalised methods of moments, ML and composite likelihood methods (Davis and Dunsmuir, 2015). For the Bayesian approach, the problem of optimising these complicated likelihood functions can be replaced with posterior simulations. Barnett et al. (1996) presented a complete Bayesian approach for the estimation of autoregressive time series model, incorporating choices of autoregressive order, enforcing stationarity and tackling outliers in a more efficient way. Gerlach et al. (1999) provided a method to test the goodness of fit of a general state space model with latent variables using a Bayesian approach. Holan et al. (2009) proposed a hierarchical Bayesian model to estimate the memory parameter of a stationary long-memory Gaussian time series. Graves et al. (2014a) further studied the Bayesian approach for ARFIMA model and provided a new approximate likelihood for efficient parameter inference. The model structure may be complicated but the efficient Rstan package (Team, 2016) can be adopted to provide an easy alternative.

In terms of model forecast, the advantage of adopting a Bayesian simulation approach is its provision of the complete forecast predictive distributions which include all sources of uncertainties to provide the predictive values and predictive intervals for the forecasts ((Rydberg and Shephard, 2003) and (De Alba and Mendoza, 2007)). Harrison and Stevens (1976) studied a Bayesian approach to forecast using the dynamic linear model with Kalman filter recurrence relations. Fildes (1983) compared several types of time series forecasting approaches including the Bayesian by examining parameter sensitivity and contrasting empirical studies of Bayesian with other approaches. Pole et al. (1994) provided detailed practical Bayesian forecasting including the time series analysis and model formulation. West (1996) provided a broad introduction of principles, models and methods of Bayesian forecasting and time series analysis developments. However

none of these works on Bayesian approach considered long memory time series models for counts.

1.2.4 Development of mortality model

Long memory model has seen its role in modelling financial data. We now review the development of mortality models and identify the role of long memory model in modelling mortality data. To begin with, we point out that there are two classes of models: modelling single age group and modelling simultaneously all age group with two types: modelling mortality rates and modelling death counts. In the model development, there are also deterministic models and stochastic models. We begin with the historical deterministic models.

In traditional pension scheme actuaries tended to apply deterministic mortality models which generate only one estimate of mortality rate at a time. The first deterministic mortality model is the well known Gompertz's law of mortality proposed by [Gompertz \(1825\)](#) which states that the force of mortality at age x follows the function

$$\mu_x^* = ae^{bx}$$

in which the mortality rate is postulated to increase at exponential rate with age where a and b are model parameters. This law was later extended to Makeham's law by including a constant ([Makeham, 1860](#)). Other deterministic models such as logistic model ([Perks, 1932](#)) and Weibull model ([Weibull, 1951](#)) given by respectively

$$\mu_x^* = c + ae^{bx}/[1 + ae^{bx}] \quad \text{and} \quad \mu_x^* = ax^b$$

were also developed. However these deterministic mortality models first studied by [Gompertz \(1825\)](#) failed to provide reasonable estimates of mortality rates across all age group stratification especially for the older age group of 80+. An example was the single age group mortality model proposed by [Heligman and Pollard \(1980\)](#) and applied to post-war Australian national mortality data. However, since the predicted mortality rates according to these classic methods were inaccurate, [Stoto \(1983\)](#) provided a data-analytic approach to construct confidence interval for each estimate.

Alternative approaches to develop mortality models with regression structures have incorporated a stochastic component into the regression structure because deterministic

mortality models hardly quantify the uncertainty around future mortality rates and provide a full distribution of the liabilities associated with longevity risk. Longevity risk is an important risk profile for governments and insurance companies due to the increase in life expectancies. To capture mortality features and minimize longevity risk, stochastic mortality models are more appropriate to understand the uncertainty related to longevity risk and provide a completed mortality projection. [Juel \(1983\)](#) proposed a model illustrating the relationship between demographic factors and age-specific mortality rates. In the model, the number of deaths for a particular age group x was found to follow a Poisson distribution and the constant central mortality rate μ_x could be expressed by the function $\mu_x = ax^b$, where a and b are parameters to be estimated. [Macdonald \(1996\)](#) also confirmed that the Poisson distribution is an appropriate choice for the number of deaths with the mean being the product of total central exposure E_x and a constant mortality rate μ_x .

To improve the modelling of mortality features, the choices of mean function includes the graduation model of [Forfar et al. \(1988\)](#) who also studied the parameter estimation with confidence intervals and constructed a complete mortality table based on the graduated mortality rates. [Renshaw et al. \(1996\)](#) further developed a graduation model in which the log of mortality rate was adopted as the intensity function in an over-dispersed Poisson distribution. The mean function incorporating both age effects in Legendre polynomials and time trends is given by

$$\ln(\mu_{x,t}) = \beta_0 + \sum_{j=1}^s \beta_j L^{(j)}(x) + \sum_{i=1}^r \alpha_i t^i,$$

where the Legendre polynomials $L^{(n)}(\cdot)$ of degree n is generated by

$$L^{(0)}(x) = 1, \quad L^{(1)}(x) = x, \quad (n+1)L^{(n+1)}(x) = (2n+1)xL^{(n)}(x) - nL^{(n-1)}(x) \quad \text{for } n \geq 1.$$

This is an extension to the conventional parametric graduation models which incorporate unknown deterministic linear trend effects in mortality rates. [Sithole et al. \(2000\)](#) further improved the model of [Renshaw et al. \(1996\)](#) by introducing higher order terms in the deterministic time trend. Smoothing spline function is another choice for projecting mortality rates. [Currie et al. \(2004\)](#) considered univariate and bivariate Poisson regression models, with deterministic unknown parameters in the regression. In this work, the extension was to incorporate a bivariate tensor product P-spline for

the covariates of age and year. This approach using P-splines rather than B-splines introduces a penalty term for smoothness into the regression model. It has the advantage of creating a more parsimonious model, whilst balancing a bias/variance trade-off.

To further improve the modelling of mortality features, period effect were also included in the mean function. For instance, [Brouhns et al. \(2002\)](#) proposed a generalised linear model with Poisson random variation for the number of deaths allowing stochastic period effect κ_t on the logarithm of mortality rates. This type of structure in the mortality modelling literature is often referred to as a single age group (univariate) stochastic period effect model, which is the univariate equivalent of the famous multi-age group (multivariate) version of the widely used LC model. A widely used stochastic model

$$\ln \mu_{x,t} = \alpha_x + \beta_x \kappa_t + \varepsilon_{x,t}$$

called LC model proposed by [Lee and Carter \(1992\)](#) was applied to model mortality rates across all age groups simultaneously. The period effect is assumed an AR(1) process.

Alternatively, the period effect on the mortality rates can be modelled directly by a time series model. [Sarpong \(2013\)](#) studied an ARIMA model in modelling and forecasting maternal mortality ratios and found that the ARIMA (1,0,2) model provides reasonable forecasts. However these models still do not account for long memory. Another approach to account for the time effect is given by [Hobcraft et al. \(1985\)](#) and [Wilmoth \(1990\)](#) who claimed that changes of mortality rates across the dimensions of age, period and cohort need to be considered to provide a better projection of mortality rates. The age effect accounts for the increase in the probability of dying as individuals become older, the period effect describes effects of time specific events on the number of deaths whereas the cohort effect captures the impact of improvement in mortality rates over different birth cohorts distinguished by the year of birth. Hence extension of mortality models to model multiple age groups is natural as the cohort effect across age groups from the same birth cohort can be incorporated.

However, LC model did not yet consider cohort effect and ([Monetary and Department, 2012](#)) reported that the mortality rates estimated by LC models are consistently underestimated. Consequently, LC model has been extended to adopt different structures and to model various responses. Firstly, [Renshaw and Haberman \(2003\)](#) proposed to generalise the log mortality rate model of [Lee and Carter \(1992\)](#) to include multiple

period effects while [Renshaw and Haberman \(2006\)](#) instead extended the model to add age-specific cohort effect and [Currie \(2006\)](#) simplified the model by assuming the coefficient of period and cohort effects to be one. [Cairns et al. \(2006b\)](#) proposed to model alternatively the logit of the probability of death $q_{x,t} = 1 - \exp(-\mu_{x,t})$ with period effect and [Cairns et al. \(2009\)](#) extended it to include cohort effect. [Plat \(2009\)](#) further improved the model by introducing an additional term $(\bar{x} - x)^+ := \max(\bar{x} - x; 0)$ to capture the feature of young age mortality. A summary of these works can be found in Table 8.1 of Section 8.1.

Recently, further extensions of LC model includes [Toczydlowska et al. \(2017\)](#) who developed the family of LC models by including exogenous observable demographic features to improve model fit and forecast. However, the main weakness of LC models when modelling the mortality rates is that the errors are assumed to be homoscedastic. [Brouhns et al. \(2002\)](#) improved this by assuming the number of deaths follows a Poisson distribution which is intrinsically heteroscedastic. [Renshaw and Haberman \(2006\)](#) also adopted the Poisson distribution in modelling the number of deaths. Furthermore, the binomial distribution was used in the LC model of [Cossette et al. \(2007\)](#). In order to capture the over-dispersion in mortality data, [Delwarde et al. \(2007b\)](#) extended LC model with a NB distribution. LC model was also considered in state space modelling. [Fung et al. \(2017a\)](#) introduced stochastic period effect and cohort effect into the LC model and [Fung et al. \(2017b\)](#) further extended the model structure to include non-Gaussian features for modelling as well death counts using Poisson distribution.

On model implementation, [Fung et al. \(2017b\)](#) provided clear formulations to estimate and forecast mortality rates under a Bayesian state-space modelling framework. Bayesian approach is then developed to provide quantification of parameter uncertainty and improve mortality forecasts. In fact, the Bayesian LC model has been widely applied to study mortality rates in various literature. [Pedroza \(2006\)](#) gave a fully Bayesian specification of LC model using MCMC methods and applied the models to forecast mortality rates of U.S. males. [Giroi and King \(2008\)](#) improved the Bayesian models with an issue of incorrectly formulating prior knowledge for explanatory variables and proposed guidelines for choosing suitable priors in the presence of partial prior ignorance. [Kogure and Kurachi \(2010\)](#) adopted a Bayesian approach to price longevity risk based on Japanese mortality rates using LC model. [Fung et al. \(2015\)](#) provided some general assumptions for the priors of the LC model under the Bayesian

approach. Furthermore, they suggested a new form of identification constraints for estimating the model under the state-space modelling framework. They also derived posterior distributions of the parameters and latent processes for LC models via Gibbs sampling. Furthermore, the impact of longevity risk on the pricing of annuities can be captured by exploiting these posterior distributions. [Fung et al. \(2017b\)](#) and [Fung et al. \(2017a\)](#) further developed a cohort-based state space model formulation. The LC models including the many extensions also employed other estimation methods including the singular value decomposition (SVD) and ordinary least squares (OLS) which extrapolate the common trend with all age-specific parameters in forecasting.

From an application perspective, [Richards and Currie \(2009\)](#) studied the financial impact projected by different extensions of LC models and found that model risk causes significant uncertainty for actuarial pricing and reserving for pensions and annuities. [Rossa \(2011\)](#) applied standard LC models that a new recursive approach to provide more accurate forecast of mortality rates and life expectancies. [DAmato et al. \(2011\)](#) extended LC model basing on simulation procedures using bootstrap methodologies to obtain more reliable and accurate mortality projections. Assuming stochastic interest rates, the survival uncertainty for life annuity portfolios and guaranteed annuity option (GAO) were also investigated. [Fung et al. \(2015\)](#) studied the impact of longevity risk on the accurate pricing of annuities by adopting the predictive distributions of mortality rates. LC models have been widely applied to study mortality data in various countries ([Koissi and Shapiro, 2008](#)), including Australia ([Booth et al., 2002](#)), Austria ([Carter and Prskawetz, 2001](#)), Belgium ([Brouhns et al., 2002](#)), Canada ([Lee and Nault, 1993](#)), Chile ([Lee and Rofman, 1994](#)), China ([Jiang, 1995](#)), England and Wales ([Cairns et al., 2011](#)), Finland ([Alho, 1998](#)), Norway ([Keilman et al., 2002](#)), Spain ([Felipe et al., 2002](#)), Sweden ([Lundström and Qvist, 2004](#)), the U.K. ([Renshaw and Haberman, 2003](#)), United State (([Lee and Carter, 1992](#)) and ([Cairns et al., 2011](#)), the Nordic countries ([Koissi et al., 2006](#)) and the seven most economically developed nations (G7) ([Tuljapurkar et al., 2000](#)).

1.2.5 Development of annuity and GAO pricing model

The development of mortality models to provide accurate mortality rate estimates and forecasts in facilitating pension fund planing is of vital importance. Recently, traditional life insurance and annuity products provide many mortality-linked securities and mortality-related guarantees in the capital market, such as GAO. Under the GAO, an insurance company guarantees to convert a policyholder's accumulated funds to a life annuity at a fixed rate when the policy matures (Boyle and Hardy, 2003). London (1972) reported that the origin of GAOs dates back to 1839. However, these instruments became prevalence in UK in the era of 1970-1990. With the increase in life expectancy, accurate pricing of GAOs has drawn great attention as the underpricing of such guarantees has already caused serious solvency problems to insurers, for example in the U.K., as an after effect of encashment of too many GAOs, the world's oldest life insurer - Equitable Life had to close its new business in 2000 (Bahl and Sabanis, 2017).

Subsequently, the pricing of GAO and other mortality-linked insurance products was directed to linking mortality rates in both independent and dependent framework with stochastic interest rates. In terms of the choice of stochastic factor model, there are various approaches to model the term structure of interest rates in continuous time. For example, one factor short-term interest rates can be assumed to follow a univariate diffusion process. Under this assumption, Vasicek (1977) proposed one factor interest rate model to describe interest rate movements as driven by only one source of market risk. This model is the first one to capture mean reversion feature which is an essential characteristic of interest rate.

Deterministic mortality models were usually adopted by traditional pension scheme actuaries. However, the uncertainty of mortality rates cannot be captured by deterministic mortality models. In order to quantify the uncertainty related to longevity risk and provide a more accurate mortality projection, stochastic mortality models were proposed. Similar to interest rates, diffusion models were also adopted to model stochastic mortality rates by traditional researchers. Furthermore, mortality rates were assumed to be independent from interest rates. However, Norberg (2010) pointed out that there exist many fundamental differences between interest rates and mortality rates. Hence, mortality models need to be carefully defined. Moreover, Dhaene et al. (2013) stated that the real-world independence between interest risk and mortality risk

cannot be applied to pricing insurance products such as annuity and GAO. The pricing frameworks need to allow dependence between mortality rates and interest rates because extreme mortality events such as catastrophes and pandemics show a stronger effect in short term interest rates and improving life expectancy affects the financial market in a gradual manner. [Boyle and Hardy \(2003\)](#) pointed out that there exists a interplay between financial risk and mortality risk. This interplay is essential for the GAO pricing process. Furthermore, [Liu et al. \(2013\)](#) valued a GAO under a generalised modelling setup in which both interest rate and mortality risk are stochastic and correlated. [Liu et al. \(2014\)](#) further proposed a generalised pricing framework in which the dependence between the two risks can be explicitly modelled and derived a simplified formula for this GAO price under the generalised framework. [Gao et al. \(2015\)](#) proposed a Markov-modulated affine framework for dependent risk factors to value a GAO assuming both diffusion components of the interest rate and mortality rate are driven by a finite-state continuous time Markov chain. Under the assumption that both interest rate and mortality rate are stochastic and correlated, [Bahl and Sabanis \(2017\)](#) adopted doubly stochastic stopping times to incorporate the randomness of the time of death and modified a measure to evaluate the survival benefit. Despite these works, there are research gaps in deriving stochastic mortality models with long memory and developing dynamic models to describe the interplay between mortality rate and interest rate in a more coherent fashion in order to provide more accurate pricing of annuity and GAO.

1.3 Thesis Contributions

Model development, model properties and parameter estimation

The main contribution of this thesis is on proposing the GLGARMA model that combine the generalised linear model with GARMA model to provide efficient models for long memory time series of counts. This contribution includes some detailed studies on the model properties including long memory using ACFs and spectral density and

the differences between Gegenbauer long memory versus seasonal long memory. Different estimation methods are explored and Bayesian method is adopted. Techniques of posterior simulation for model parameters as well as posterior predictive distribution generation are provided to facilitate posterior inference. From a modelling perspective, the focus of this thesis is to propose new models that combine the GLARMA model with the long memory time series models, namely the ARFIMA and GARMA models to capture the long memory and oscillatory long memory properties in a count series. This can be done by replacing the data Y_t in Equation (2.7) with the mean function μ_t of a count data distribution. These extended models are called GLARFIMA and GLGARMA models respectively. We consider four choices of count data distribution, namely, Poisson, negative binomial (NB), generalised Poisson (GP) and double Poisson (DP). Our proposed GLARFIMA and GLGARMA models with either ARFIMA and GARMA mean functions along with DP and GP data distributions are new in the literature and their properties are vividly documented in this thesis.

To understand these models, we provide a precise explanation of different model structures, features and attributes for these long memory count processes including, firstly, the characterisation of population moment properties, stationarity and invertibility conditions for these processes; secondly, the time and frequency domain characterisation of the long memory features of our models to describe these processes; and lastly, a clear distinction between the two processes in seasonal ARFIMA (SARGIMA) and GARMA models. To explore the features of our proposed model in the frequency domain, we review properties of spectral density and Whittle likelihood function for Gaussian and non-Gaussian models and further derive a modified spectral density function which can be utilised to obtain Whittle estimator for the long memory parameter d in the GLGARMA parameter driven model by analysing the limiting behavior of Gegenbauer coefficients and approximating the exponential function in the generalised linear model with higher order Taylor series expansion.

Regarding the distinction between seasonality and Gegenbauer long memory processes, we demonstrate that Gegenbauer long memory and seasonality are distinct features and so we clarify that there is a clear distinction between periodic oscillation and the oscillation that comes from a Gegenbauer long memory process in their respective ACFs. Moreover, to illustrate this point, we undertake a range of simulation studies and empirical model fittings. In the simulation study, we show that deseasonalisation

cannot remove the oscillating behaviour in the ACF of the Gegenbauer long memory time series. We also compare the model performance of our proposed GLGARMA model with various seasonal time series models for the CFTC spread data which possess future rolling periodicity as well as for the long and short data which do not possess such periodicity. Results show that our proposed GLGARMA model consistently gives better model performance regardless of the possession of periodicity.

For modelling estimation and prediction, the ML method is not feasible for parameter driven model since the likelihood function involves high dimensional integrals, which are non-trivial to evaluate. We demonstrate the advantages of adopting a Bayesian approach via the user-friendly **Rstan** package for the estimation of our proposed models. The **Rstan** package applies the efficient Hamiltonian Monte Carlo (HMC) and No-U-Turn samplers for the posterior simulation. Dependence and convergence of posterior samples can be monitored via the number of effective samples and $\hat{R}$, a convergence index. Rstan programs for all model implementations and forecasts are available for practitioners.

Financial application to study open interest using CFTC data

Our second contribution is to study the behaviour of open interest on a range of important U.S. treasury futures contracts. This study and its findings of long memory patterns in traders behaviours, cross-classified by market participant type, is the first of its kind and has important ramifications to understand market liquidity and to forecast such market dynamics. We believe that the findings in futures market can have important applications in price discovery studies, market making practice, regulatory analysis of market participant trade behaviour under trading regulations such as Dodd Frank, MiFid II and EMIR II as well as trading strategy design. We are particularly interested in forecasting accuracy for net monetary flows in and out of the commodity markets over time by different large key market participants, this is widely believed to be captured best by the observation of open interest of futures contracts. Hence, this thesis analyses 136 individual U.S. CFTC time series of open interest for futures contracts. The contribution of our analyses to understanding of commodity market dynamics pertains specifically to studying open interest of distinct types, with different maturities and from different market participants obtained from the CFTC and analysing their stochastic structures across these factors. This then has practical implications for developing models to forecast the dynamics of such data and study how

the dynamics are related to market behaviour and market participants interactions in such commodity markets. We aim to highlight in this study the fact that there are statistical stylised long memory features present in U.S. treasury futures relating to different market participant behaviours. This consistently observed long-memory feature indicates forecastable patterns in trading behaviour and market liquidity. These forecasts are highly relevant to financial regulators who are concerned about promoting market stability in the highly liquid future markets and to the market participants who aim to set up trading strategies that are robust and consistent. We demonstrate that this finding significantly aids in the ability to forecast future monetary flow dynamics in futures markets by market participant types.

Evidence of long memory is first provided by the analysis of Hurst exponent, ACF plot and spectral density plot which reveal four types of long-memory features presented consistently in different open interest series. This consistently observed long-memory feature in open interest which indicates forecastable patterns in trading behaviour and market liquidity. These forecasts are highly relevant to financial regulators who are concerned about promoting market stability in the highly liquid future markets and to the market participants who aim to set up trading strategies that are robust and consistent. Through further fitting of each series to forty models with different orders, distribution choices, error types and memory types, the clear distinctions between different market participant trading characteristics and their association with four different long memory features, namely the basic, oscillatory, composite and irregular, are revealed. The **Rstan** package offers a powerful tool to undertake this large-scale data analysis. This is the first time such a detailed financial study has been undertaken with regards to this market data for open interests. The fact that we identify systematic long-memory patterns in the count processes modelled for market participant trading behaviours when taking long, short and spread positions on the US treasury futures, as we demonstrate it significantly elevates our ability to forecast such trading dynamics utilising the models we propose. Comparing the forecasts using the ACP, GAS, GLARMA and seasonal models with only short memory, the supremacy of our proposed GLGARMA model with long memory is demonstrated.

Insurance application to study mortality rate using HMD data

Our third contribution is to study the behaviour of mortality rates using death counts from sixteen selected countries cross-classified by gender and age groups which are

downloaded from HMD. A non-parametric measure called Hurst exponent H is applied to detect the presence of the long memory in this data. Then the features of long memory across genders, age groups, countries and periods before and after World War two are analysed. To model the mortality data, we first extend in the LC model to incorporate our GLGARMA model as well as the graduation and period effect models to provide more flexible model structures for modelling mortality rate of each age group.

With different choices of model features including types of long memory, graduation and period effect, we consider seven sub-models nested within the full model and they are fitted to five specially chosen countries, namely Australia, Iceland, Japan, U.K. and U.S. and Japan with different population sizes and mortality patterns to demonstrate these univariate models. Moreover, Poisson distribution has been routinely applied to model death counts. However, as this distribution fails to accommodate both over- and under-dispersed data, we adopt a GP distribution which includes the standard Poisson distribution as a special case. The in-sample fit and out-of-sample forecast performance are evaluated through simulation study and real data analyses. Life expectancies are calculated based on the mortality rate forecasts and life tables for all sixteen countries are forecasted for 20 year starting from 2015.

To capture the cohort effect, the univariate models are further extended to model death counts for all age groups simultaneously. The proposed LMLM model incorporates long memory structure in both mean function and cohort effect function but it excludes the period effect which is present in the MELCC model as an extension of LC model to death counts. Both LMLM and MELCC models are fitted to Australia, U.K. and U.S. death counts to obtain in-sample fits and out-of-sample forecasts of mortality rate. In summary, forecasts using both univariate and multivariate models without long memory structures provide inaccurate mortality rate forecasts leading to improper life expectancy forecasts in life tables.

Calculation of annuities and GAO for insurance product pricing

Our fourth contribution is directed towards evaluating annuity and GAO pricing by adopting the mortality rates provided from HMD and estimated from MELCC and LMLM models. In the calculation of annuity pricing, we consider two cases: interest

rate is fixed or varied with stochasticity. We demonstrate that LMLM model outperforms MELCC model in annuity pricing when compared the mortality rates from HMD. We highlight that even a small improvement in annuity pricing can reduce a non-ignorable amount of shortfall when considering a large annuity portfolio.

We further derive model to calculate annuity as well as GAO pricing when the interest rate is assumed to be stochastic. Vasicek model is used to model interest rate. After discretising the model, it is applied to model both mortality rate and interest rate and the two models are expressed marginally as well as jointly. To calibrate the Vasicek models, we apply three mortality rate data obtained from HMD or forecasted using MELCC and LMLM models whereas the interest rate data obtained from LIBOR. With the calibrated Vasicek models, we then re-estimate the mortality and interest rates. Furthermore, there are four formulations for the dependency structure between interest rate and mortality rate: independence, linearly correlated and rank correlated in the marginal modelling approach and jointly correlated in the joint modelling approach. Together with the three types of mortality rate estimates, we derive totally twelve models for annuity and GAO pricing. Comparing the results with the bench mark model using mortality rates from HMD and assuming joint distribution, our proposed formulation using mortality rates forecasted from LMLM model and joint distribution assumption provides the best performance.

1.4 Thesis Structure

This thesis is structured as follows:

- Chapter 2 introduces a class of GLGARMA models with four count distributions. The model structure is presented with two types of error terms in the PD and OD approaches. The model properties including moment properties, stationarity, invertibility and long memory conditions in the time domain are studied. Furthermore, the oscillatory properties of GARMA model are compared with various seasonal ARFIMA models to show the fundamental differences between the Gagenbauer fractional difference and seasonal fractional difference operators.
- Chapter 3 investigates model properties in the frequency domain. In both linear and non-linear settings, Whittle likelihood estimations with Gaussian and non-Gaussian

innovations are introduced respectively to simplify the estimation procedures of the long memory parameter d . Furthermore, the properties of Gegenbauer polynomial including the limiting behaviour for the product of Gegenbauer coefficients and the approximation of exponential function in generalised linear model using higher order Taylor series expansion are studied. A modified spectral density function for the proposed model is derived.

- Chapter 4 introduces the CFTC open interest data and discusses the importance of studying the dynamics of open interest in futures contracts and characterising market participant behaviours. Hurst exponent H as a non-parametric measure to detect the long memory strength in a time series is described. Three estimators of H including rescaled range analysis, detrended fluctuation analysis and periodogram regression are presented. Furthermore, the relationship between H and d is discussed. Then the rescaled range estimator of H is applied to study the long memory features of CFTC data across terms of open interest futures contracts.
- Chapter 5 further identifies four types of long memory features by inspecting the ACFs for each time series of CFTC data. The Bayesian approach is adopted for models parameter estimation and forecast. The Bayesian R package **Rstan** which utilises the **stan** program is introduced. Then the deviance information criterion and other model selection criteria are evaluated to assess model performance. Furthermore, to capture these long memory features, the proposed GLGARMA models including various higher order models and count distributions are adopted to fit CFTC data. GLGARMA model is compared to various popular time series count models including seasonal models according to in-sample fit and out-of-sample forecast and results confirm the superiority of GLGARMA model relative to other seasonal and count models.
- Chapter 6 introduces the development of mortality models and demonstrates the prevalence of long memory features in mortality data of HMD by adopting the Hurst exponent. The dynamics of long memory in mortality data across genders, age groups, countries and time periods are analysed and discussed.
- Chapter 7 introduces a univariate Gegenbauer long memory LC model for death counts with seven sub models. Bayesian inference is applied to estimate the model parameters. Mortality data studies along with in-sample fitting and out-of-sample

forecast are performed. Lastly, life expectancies are also calculated using the mortality rate forecasts and life tables are then constructed.

- Chapter 8 introduces the multivariate extended LC cohort model (MELCC) with discrete distribution and our proposed stationary long memory mortality model with long memory cohort effect (LMLM) model. These multivariate models can capture cohort effect across different age groups. Moreover, we demonstrate and explain that identification issue can be resolved in our proposed LMLM model as compared to MELCC model which has such issue. Then in-sample fitting and out-of-sample forecast for mortality rate and life expectancy are presented.
- Chapter 9 introduces the background of longevity risk in insurance and some practical mechanisms of risk transfer. Given fixed interest rate, various annuity prices with credible intervals based on mortality rates from MELCC and LMLM models are evaluated to show the advantages of adopting LMLM model. However, the developments of mortality rate and interest rate models have highlighted the significance of adopting stochasticity in both mortality rate and interest rate data. Furthermore, the importance of the interplay between financial risks and mortality risk is explained. Hence, the annuity pricing is further extended to incorporate stochastic interest rate and stochastic mortality rate. Four types of dependency structure between interest rates and mortality rates under the discretised Vasicek model and three types of mortality rates using directly the mortality rates from HMD or estimates from MELCC and LMLM models, result in twelve formulations for the pricing of annuity, pure endowment and GAO. Comparing the shortfall calculated from various formulations, results confirm the advantage of LMLM model in the pricing of annuities as well as GAO.
- Lastly, chapter 10 concludes the thesis and provides future research directions.

CHAPTER 2

Proposed long memory models for time series of counts

2.1 Background

This chapter incorporates the family of Gegenbauer autoregressive moving average (GARMA) models and a special sub family called the autoregressive fractionally integrated moving average (ARFIMA) models into the mean functions of the count distributions, including Poisson, negative binomial (NB), generalised Poisson (GP) and double Poisson (DP) within a generalised linear modelling framework. These distributions display equi-, over- and under-dispersion and together with the additional long memory structure in the mean functions have enhanced modelling flexibility. Furthermore, the error terms of the generalised linear GARMA (GLGARMA) model are defined under two different approaches, the parameter-driven (PD) and observation-driven (OD) approaches. Model properties including stationarity, invertibility, long memory, autocorrelation function (ACF) and spectral density function for mean function are presented. Furthermore, the differences between Gegenbauer long memory and seasonal long memory are precisely studied.

2.2 Development of time series model for counts

Literature review for the development of time series count model can be found in Section 1.2.1. This section focuses on the development from a technical perspective.

One approach to modelling discrete value time series, such as count series, is via the binomial thinning operator construction McKenzie (1985) as follows. Consider

$$\iota \circ \delta = \sum_{i=1}^{\delta} w_i, \quad (2.1)$$

introduced by Steutel and Van Harn (1979), where the binary w_i are i.i.d. random variables independent of the integer-valued δ with $P(w_i = 1) = \iota = 1 - P(w_i = 0)$. This operator is incorporated to describe a count process $y_1, \dots, y_T$ with an ARMA time series structure where the resultant model is given by

$$y_t - \alpha_1 \circ y_{t-1} - \dots - \alpha_p \circ y_{t-p} = u_t + \theta_1 \circ u_{t-1} + \dots + \theta_q \circ u_{t-q}. \quad (2.2)$$

This is known as an integer-valued ARMA (INARMA) model, where u_t are i.i.d. non-negative discrete valued random variables and α_i for all $i \in \{1, \dots, p\}$ and θ_j for all $j \in \{1, \dots, q\}$ take values in the interval $[0,1]$. Then Quoreshi (2006) further extended the INARMA model to integer-valued ARFIMA(p, d, q) model defined as

$$y_t - \alpha_1 \circ y_{t-1} - \dots - \alpha_p \circ y_{t-p} = (1 - B^\circ)^{-d} (u_t + \theta_1 \circ u_{t-1} + \dots + \theta_q \circ u_{t-q}), \quad (2.3)$$

where

$$\begin{aligned} (1 - B^\circ)^{-d} &:= 1 + \varphi_1 \circ B + \varphi_2 \circ B^2 + \varphi_3 \circ B^3 + \dots, \\ \varphi_j &= \frac{\Gamma(j+d)}{\Gamma(d)j!}. \end{aligned}$$

To tackle model estimation challenges, Heinen (2003) proposed the autoregressive conditional Poisson (ACP) model. Conditional on the model parameters, we assume that the count data $Y_{t \geq 0}$ form an independent Markov process. The ACP model for $\mathbf{Y}_{t \in \{0,1,2,\dots\}}$

is defined as

$$\begin{aligned}
 Y_t | \mathcal{F}_{1:t-1} &\sim \text{Pois}(\mu_t), \\
 \mu_t &= \alpha_0 + \sum_{j=1}^p \alpha_j Y_{t-j} + \sum_{j=1}^q \theta_j \mu_{t-j} \\
 &= \alpha_0 + \alpha^*(B) Y_t + \theta^*(B) \mu_t,
 \end{aligned} \tag{2.4}$$

where $\alpha^*(B) = \alpha_1 B + \dots + \alpha_p B^p$ and $\theta^*(B) = \theta_1 B + \dots + \theta_q B^q$, $\alpha_j, \theta_j > 0$, and the natural sigma algebra for the observed data filtration is given by

$$\mathcal{F}_{1:t-1} = \sigma(Y_1, Y_2, \dots, Y_{t-1}). \tag{2.5}$$

The condition for covariance stationarity as well as strict stationarity is given by the constraint $\sum_{i=1}^p \alpha_i + \sum_{j=1}^q \theta_j < 1$ (Ferland et al., 2006). This model can deal with over-dispersed count data because the unconditional variance is always greater than the mean. This is an advantage over the INARMA model in which the unconditional variance cannot be derived. Subsequently, Groß-KlußMann and Hautsch (2013) rearranged the form of the model in Equation (2.4) into the following representation

$$\phi(B) (Y_t - c) = \theta(B) \nu_t, \tag{2.6}$$

which corresponds to the ARMA($\max\{p, q\}, p$) specification with $c = \alpha_0 / (1 - \phi(1))$, $\phi(B) = 1 - \alpha^*(B)$, $\theta(B) = 1 - \theta^*(B)$ and $\nu_t = Y_t - \mu_t$. Under such a representation, Groß-KlußMann and Hautsch (2013) considered the DP distribution and extended Equation (2.6) to the long-memory ACP model given by

$$(1 - B)^d \phi(B) (Y_t - c) = \theta(B) \nu_t. \tag{2.7}$$

Other model specifications have been proposed such as the generalised autoregressive score (GAS) model which can be applied to time series with different levels of measurement. The univariate GAS(1,1) model with Poisson distribution is defined as

$$\begin{aligned}
 Y_t | \mathcal{F}_{1:t-1} &\sim \text{Pois}(\mu_t), \\
 \mu_t &= \alpha_0 + \alpha_1 s_{t-1} + \theta_1 \mu_{t-1},
 \end{aligned} \tag{2.8}$$

$$s_t = \left[\mathbb{E}_{t|t-1} \left(\frac{\partial^2 \log f_P(Y_t; \mu_t)}{\partial \mu_t^2} \right) \right]^{-\gamma} \frac{\partial \log f_P(Y_t; \mu_t)}{\partial \mu_t}, \tag{2.9}$$

where $f_P(\cdot)$ denotes the probability mass function (pmf) for a Poisson distribution, s_t is comprised of two components, the scaled score vector and the information matrix, which accounts for the variance of μ_t and $\mathbb{E}_{t|t-1}$ denotes an expectation with respect to $f_p(y_t|\mu_{t-1}, \mathcal{F}_{t-1}, \boldsymbol{\theta})$. The index parameter γ can be set to be $\{0, 1/2, 1\}$, where the zero index corresponds to an identity matrix. An advantage of the GAS model formulation is that it may avoid some difficulties of the ML estimation by using the conditional score function ((Creal et al., 2013) and (Harvey, 2013)).

Another type of the generalised linear model, extended by Davis et al. (1999), is the generalised linear ARMA (GLARMA) model which incorporates the ARMA-type mean function within the generalised linear model structure. According to the different representation of error terms in the mean function, this model is further classified into two types, PD and OD respectively, and the canonical log link is adopted to link this function to the mean of a count distribution, to ensure the intensity is strictly positive. We note that the OD type of GLARMA model is an extension of the models defined in Equation (2.6).

2.3 Proposed long memory models for time series of counts

We propose the GLGARMA model which combines the generalised linear model and the long memory GARMA time series models. The general model for $\mathbf{Y}_{t \in \{0,1,2,\dots\}}$ of order (p, d, q) is defined as

$$Y_t | \mathcal{F}_{1:t-1} \sim D(\mu_t, \nu), \quad (2.10)$$

where the mean function μ_t satisfies

$$(1 - 2uB + B^2)^d \phi(B)(\ln \mu_t - c) = \theta(B)\varepsilon_t, \quad (2.11)$$

$$\varepsilon_t = \begin{cases} z_t \stackrel{\text{i.i.d}}{\sim} N(0, \sigma^2) & \text{for PD,} \\ e_t = \frac{Y_{t-1} - \mu_{t-1}}{\sqrt{\mu_{t-1}}} & \text{for OD.} \end{cases} \quad (2.12)$$

$D(\cdot, \cdot)$ represents the Poisson, NB, GP or DP distributions. The dispersion parameter is defined as $\nu \in (-1, 1)$ for GP distribution and $\nu > 0$ for NB and DP distributions. The

Gegenbauer parameter $|u| < 1$ controls the oscillation and the long memory parameter $0 < d < 1/2$ determines the strength of long memory. B is the backshift operator and $\phi(B) = 1 - \phi_1 B - \dots - \phi_p B^p$ and $\theta(B) = 1 + \theta_1 B + \dots + \theta_q B^q$ are the autoregressive and moving-average operators, respectively with no common roots.

For the data distributions, the Poisson and NB are well-known. GP distribution has the pmf, mean and variance given by

$$\begin{aligned} f(y_t; \mu_t, \nu) &= \mu_t(1 - \nu)[\mu_t(1 - \nu) + \nu y_t]^{y_t-1} e^{-\mu_t(1-\nu) - \nu y_t} / y_t!, \\ \mathbb{E}(Y_t) &= \mu_t \quad \text{and} \quad \mathbb{V}\text{ar}(Y_t) = \mu_t(1 - \nu)^{-2}, \quad \mu_t > 0, \quad -1 \leq \nu < 1, \end{aligned} \quad (2.13)$$

respectively. Furthermore, the GP distribution is over-, under- and equi-dispersed when ν is greater than, less than and equal to 0 respectively. On the other hand, the pmf for the DP distribution is given by

$$\tilde{f}(y_t; \mu_t, \nu) = c(\nu, \mu_t) f(y_t; \mu_t, \nu), \quad \mu_t > 0, \quad \nu > 0, \quad (2.14)$$

where the normalizing term $c(\nu, \mu_t)$, given by [Efron \(1986\)](#), is

$$\frac{1}{c(\nu, \mu_t)} = \sum_{y=0}^{\infty} f(y_t; \mu_t, \nu) \approx 1 + \frac{1 - \nu}{12\mu_t\nu} \left(1 + \frac{1}{\mu_t\nu} \right).$$

Because of the complicated structure of $c(\nu, \mu_t)$, some properties of the DP distribution are difficult to derive. The unnormalised pmf, mean and variance are given by

$$\begin{aligned} f(y_t; \mu_t, \nu) &= (\nu^{1/2} e^{-\nu\mu_t}) \left(\frac{e^{-y_t} y_t^{y_t}}{y_t!} \right) \left(\frac{e\mu_t}{y_t} \right)^{\nu y_t}, \\ \mathbb{E}(Y_t) &\approx \mu_t \quad \text{and} \quad \mathbb{V}\text{ar}(Y_t) \approx \frac{\mu_t}{\nu}, \end{aligned}$$

respectively. The DP distribution is over-, under- and equi-dispersed when ν is less than, greater than and equal to 1 respectively.

For the error terms in the mean function μ_t , there are two definitions under the PD and OD modelling approaches and they are defined in Equation (2.12). These state equations combined with the structural equation in Equation (2.10) form the structure of a class of count valued state space models. The errors of PD model follow a normal distribution which together with the data distributions, Poisson, NB, GP and DP, constitute two sources of randomness whereas the errors of OD model are defined in terms of observations Y_t and mean functions μ_t .

2.4 Mean functions

Based on the definitions of PD GLGARMA model of order $(0, d, 0)$ (written as GLGARMA-PD $(0, d, 0)$) in Equation (2.11) and Equation (2.12), the mean function with $c = 0$ is given by a random variable μ_t according to:

$$\mu_t = \exp \left(\sum_{j=0}^{\infty} \psi_j z_{t-j} \right) \sim \text{LN} \left(0, \sigma^2 \sum_{j=0}^{\infty} \psi_j^2 \right), \quad (2.15)$$

where $\text{LN}(\cdot, \cdot)$ denotes a lognormal distribution. One can then easily derive the following results

$$\mathbb{E}(\mu_t) = \exp \left(\frac{\sigma^2}{2} \sum_{i=1}^{\infty} \psi_i^2 \right) \quad \text{and} \quad \mathbb{V}\text{ar}(\mu_t) = \exp \left(\sigma^2 \sum_{i=1}^{\infty} \psi_i^2 \right) \left[\exp \left(\sigma^2 \sum_{i=1}^{\infty} \psi_i^2 \right) - 1 \right].$$

For the OD model, the error term e_t is a martingale difference sequence with $\mathbb{E}(e_t | \mathcal{F}_{t-1}) = 0$. Hence,

$$\mathbb{E}(\ln \mu_t) = 0 \quad \text{and} \quad \mathbb{V}\text{ar}(\ln \mu_t) = \sum_{i=1}^{\infty} \psi_i^2,$$

since $\mathbb{V}\text{ar}(e_t) = \mathbb{E}(e_t^2) = \mathbb{E}(\mathbb{E}(e_t^2 | \mu_t)) = \frac{\mu_t}{\mu_t} = 1$ (Davis et al., 2003). The mean in Equation (2.11) for the GLGARMA (p, d, q) model can be written as

$$\phi(B)(\ln \mu_t - c) = \theta(B)(1 - 2uB + B^2)^{-d} \varepsilon_t \equiv \theta(B) \left(\sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} \right), \quad (2.16)$$

where ψ_j denote the coefficients of the generating function for the Gegenbauer polynomials $(1 - 2uB + B^2)^{-d}$ (Stein and Weiss, 1971). These coefficients are formulated as

$$\psi_j = \sum_{q=0}^{[j/2]} \frac{(-1)^q (2u)^{j-2q} \Gamma(d - q + j)}{q! (j - 2q)! \Gamma(d)}, \quad (2.17)$$

where $[j/2]$ represents the integral part of $j/2$. It is clear that these coefficients are functionally dependent on d and u (Rainville, 1960). Furthermore, the coefficients ψ_j can be easily computed using the recursive formula:

$$\psi_j = 2u \left(\frac{d-1}{j} + 1 \right) \psi_{j-1} - \left(2 \frac{d-1}{j} + 1 \right) \psi_{j-2}, \quad (2.18)$$

where the first three terms are $\psi_0 = 1$, $\psi_1 = 2du$ and $\psi_2 = -d + 2d(1+d)u^2$. For any $d > 0$ and $u \in [-1, 1]$, Rainville (1960) shows that Equation (2.17) can be expressed as

$$\begin{aligned}
\psi_j &= \sum_{q=0}^{[j/2]} \frac{(2d)_j u^{j-2q} (u^2 - 1)^q}{2^{2q} q! (d + \frac{1}{2})_q (j - 2q)!} \\
&= (2d)_j \sum_{q=0}^{[j/2]} \frac{(\frac{1}{2})_q u^{j-2q} (u^2 - 1)^q}{(2q)! (d + \frac{1}{2})_q (j - 2q)!} \\
&= \frac{(2d)_j \Gamma(d + \frac{1}{2})}{j! \Gamma(\frac{1}{2}) \Gamma(d)} \int_0^\pi (u + \sqrt{u^2 - 1} \cos \beta)^j \sin^{2d-1} \beta d\beta, \tag{2.19}
\end{aligned}$$

where the operator $(\cdot)_m$ is the Pochhammer's symbol defined as $(x)_m = \frac{\Gamma(x+m)}{\Gamma(x)}$ and it is the forward factorial function.

The ARFIMA model is a special case of the GARMA model when $u = 1$ ([Granger and Joyeux, 1980](#)) such that the factor $(1 - 2uB + B^2)^{-d}$ in Equation (2.16) can be written as

$$(1 - B)^{-d_a} = \sum_{j=0}^{\infty} \frac{\Gamma(j + d_a)}{\Gamma(j + 1) \Gamma(d_a)} B^j = \sum_{j=0}^{\infty} \varphi_j B^j \quad \text{with } d_a = 2d. \tag{2.20}$$

Hence the mean function μ_t for the GLARFIMA(p, d, q) model satisfies

$$\phi(B)(\ln \mu_t - c) = \theta(B) \left(\sum_{j=0}^{\infty} \varphi_j \varepsilon_{t-j} \right). \tag{2.21}$$

Practically, the infinite sums in Equation (2.16) and Equation (2.20) need to be truncated and the asymptotic behaviour of such truncations needs to be investigated. According to Equation (2.19), the bounds for the coefficients in the Gegenbauer polynomials are:

$$|\psi_j| \leq \frac{(2d)_j}{j!} = \varphi_j,$$

which are demonstrated in Figure 2.1. Furthermore, [Kokoszka and Taqqu \(1995\)](#) found that the right hand side of Equation (2.20) converges almost surely, if and only if $\sum_{j=0}^{\infty} |\varphi_j|^\alpha < \infty$ given $0 < \alpha \leq 2$ and $\alpha(d_a - 1) < -1$. In our case, $\alpha = 2$ and by Stirling's formula:

$$\varphi_j = \frac{j^{d_a-1}}{\Gamma(d_a)} + O(j^{-1}) \quad \text{for large } j,$$

where $f(n) = O(g(n))$ means that for positive constants a and b , such that $0 \leq f(n) \leq bg(n)$ for all $n \geq a$. So Equation (2.20) converges almost surely.

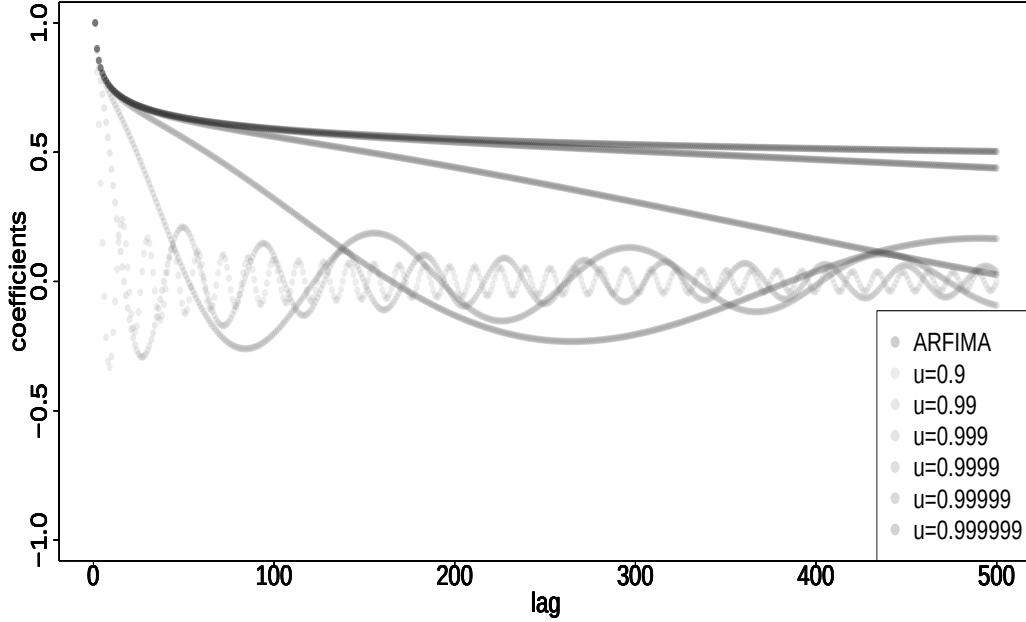


Figure 2.1. Plot of coefficients ψ_j for the Gegenbauer polynomial across values of u when $d = 0.45$. The case for ARFIMA model corresponds to $u = 1$.

2.4.1 Stationarity, invertibility and long memory for mean function

The stationarity, invertibility and long memory conditions of GARMA $(0, d, 0)$ and ARFIMA $(0, d, 0)$ were investigated by Gray et al. (1989). The long memory parameter d and Gegenbauer parameter u control the long memory strength and oscillatory behaviours of the model respectively. For different values of d and u , the autocorrelation function (ACF) plots of long memory time series show different patterns. Figures 2.2 and 2.3 shows that the long memory strength is controlled by the value of d and the value of u determines the cycle of long memory process. It further confirms that the ARFIMA model is a special case of the GARMA model because as u approaches 1, the oscillation of ACF approaches infinity which corresponds to the ARFIMA-type ACF. Hence, u can be approximately estimated from the ACF plot as illustrated by Figure 2.3. It is obvious that the oscillation and long memory phenomena of the ACF ceases as d approaches 0. Since it is known from Equation (3.10) that $u = \cos(\lambda_0) = \cos\left(\frac{2\pi}{\omega}\right)$,

the estimated oscillation when $u = 0.3$ is $\varpi \simeq \frac{2\pi}{\cos^{-1}(0.3)} = 4.9$ which agrees with the oscillation observed in Figures 2.2 and 2.3. If $u = 1$, we have the ARFIMA model with $\lambda_0 = 0$ indicating no oscillatory phenomenon and this agrees with Figure 2.2.

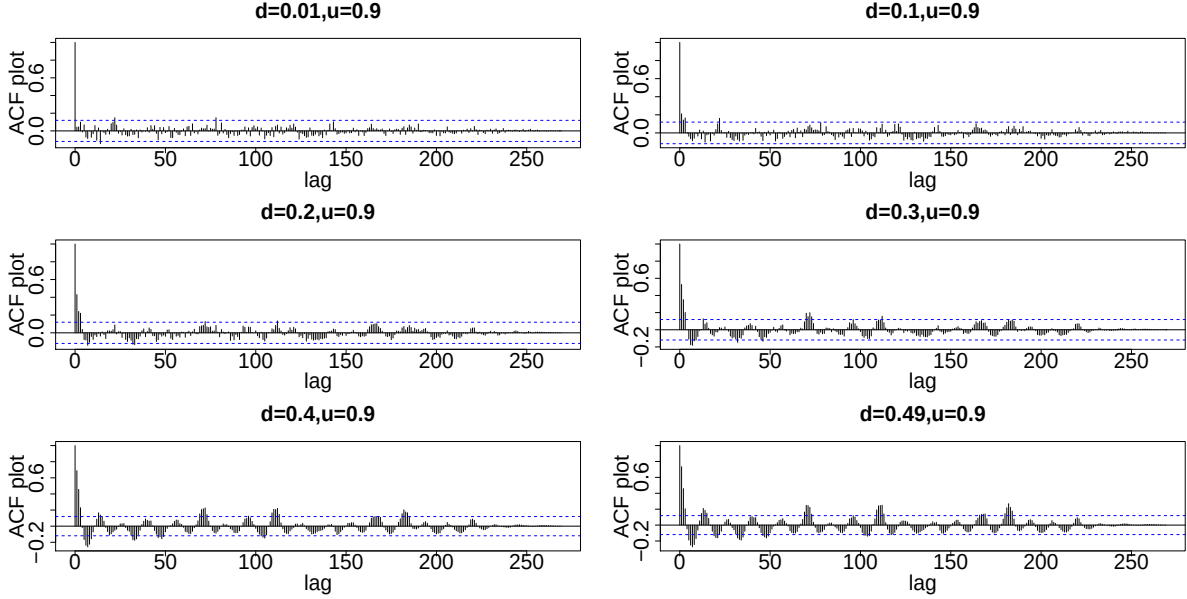


Figure 2.2. The ACF plot for data simulated from GARMA model with different values of d and fixed value $u = 0.9$

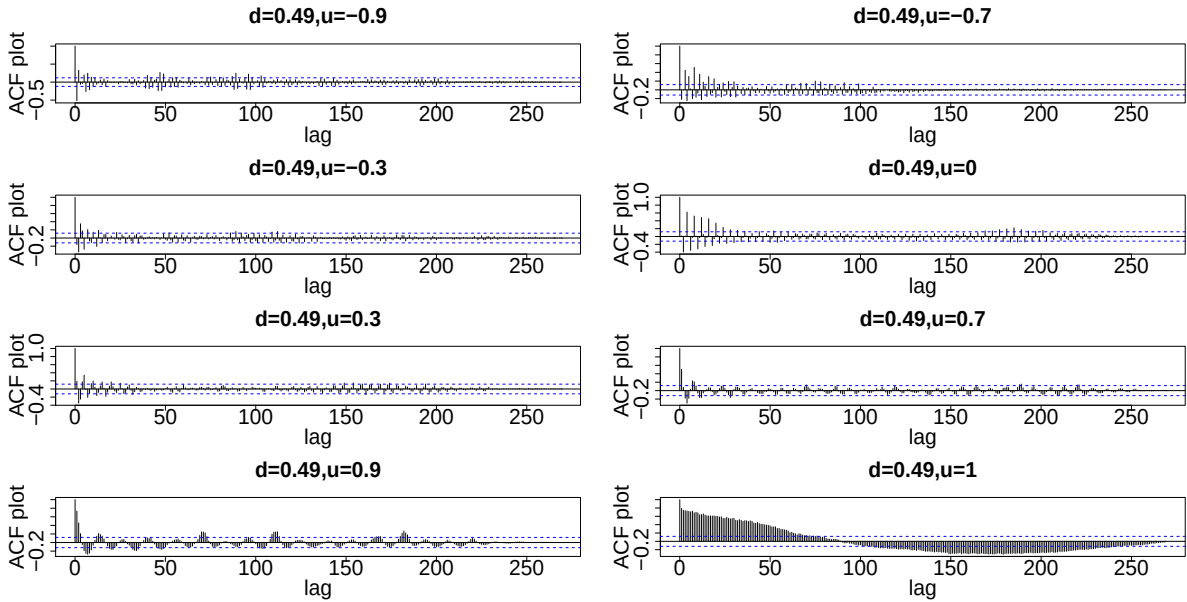


Figure 2.3. The ACF plot for data simulated from GARMA model with different values of u and fixed value $d = 0.49$

Given the constraint $|u| < 1$ for the GARMA(0, d , 0) model, it is stationary if $d < 1/2$, invertible if $d > -1/2$ and has long memory if $0 < d < 1/2$. For the constraint $|u| = 1$ which corresponds to the ARFIMA(0, d_a , 0) model ($d_a = 2d$), the stationarity and invertibility conditions are again $d_a < 1/2$ and $d_a > -1/2$ respectively. Furthermore, the model has long memory if $0 < d_a < 1/2$.

Given a stationary process $\ln \mu_t$, the ACF was worked out by [Beran \(1994\)](#). In particular, when $u = -1$ and $0 < d_a < 1/2$, the ACF can be written as

$$\rho(j) = (-1)^j \frac{\Gamma(1 - 2d)\Gamma(j + 2d)}{\Gamma(2d)\Gamma(j - 2d + 1)} \sim \text{constant} \cdot (-1)^j j^{4d-1}, \text{ as } j \rightarrow \infty.$$

For ARFIMA(0, d , 0) with $u = 1$ and $0 < d < 1/4$, it is given asymptotically by,

$$\rho(j) = \frac{\Gamma(1 - 2d)\Gamma(j + 2d)}{\Gamma(2d)\Gamma(j - 2d + 1)} \sim \text{constant} \cdot j^{4d-1}, \text{ as } j \rightarrow \infty. \quad (2.22)$$

In addition, we note that the close form ACF for the GARMA(0, d , 0) model with the constraint given by $|u| < 1$, $0 < d < 1/2$ is unavailable ([Woodward et al., 1998](#)) but is asymptotically given by

$$\rho(j) \sim \text{constant} \cdot j^{2d-1} \sin(\pi d - j\lambda_0), \text{ as } j \rightarrow \infty, \quad (2.23)$$

where $\cos(\lambda_0) = u$ and the upper bound ([Woodward et al., 1998](#)) is

$$|\rho(j)| \sim \text{constant} \cdot j^{4d-1} \rightarrow 0, \text{ as } j \rightarrow \infty.$$

Moreover, when the lag increases, the sign and size of each lagged correlation changes oscillatorily, as shown in [Figure 2.2](#).

Under the assumption that there are no common zeros for $\phi(B)$ and $\theta(B)$ and GARMA(0, d , 0) and ARFIMA(0, d_a , 0) are well-defined stationary processes, [Palma \(2007\)](#) demonstrated that both GARMA(p , d , q) and ARFIMA(p , d_a , q) models (for $\ln \mu_t$ in our proposed models) are stationary if the zeros of $\phi(B)$ lie outside the unit disk. In addition, if the zeros of $\theta(B)$ lie outside the closed unit disk, the process $\ln \mu_t$ is invertible.

2.4.2 Spectral density for mean function

The spectral measure is absolutely continuous with respect to the Lebesgue measure on $[-\pi, \pi]$ ([Palma, 2007](#)). It is obvious that the spectral density $f(\lambda)$ for ARFIMA

model will be infinity if we plug $\lambda = 0$ in the equation, hence $f(\lambda)$ is unbounded and discontinuous at $\lambda = 0$. The peak of GARMA model is unbounded when $\lambda = \lambda_0$. In this case, the $f(\lambda)$ is no longer properly specified over range $[-\pi, \pi]$, and such misspecification will cause inconsistent estimation at high frequency (Robinson, 1995a). As a result, the semi-parametric estimation methods are applied in this situation because this method only bases on the shape of the spectral density rather than the specification of a parametric model for the time series. Figures 2.4 and Figures 2.5 present the spectral density plots when $d = 0.09$ and $d = 0.49$ respectively for different values of u . The peak at λ_0 indicates that a substantial source of variability is contributed by this frequency band around the peaks. Hence, the value of u can be easily determined from the spectral density plot. For instance, when $u = 1$, $\lambda_0 = \cos^{-1}(1) = 0$ and hence the oscillation $\varpi = \frac{2\pi}{\lambda_0} = \infty$ as shown in Figure 2.1. It is also obvious that the height around the peak tends to increase with d . In summary, the combined features of oscillation and height around the peak in the spectral density plot help to identify the type of GARMA model that could be used to characterise a given observed count time series that contain oscillation and long memory features.

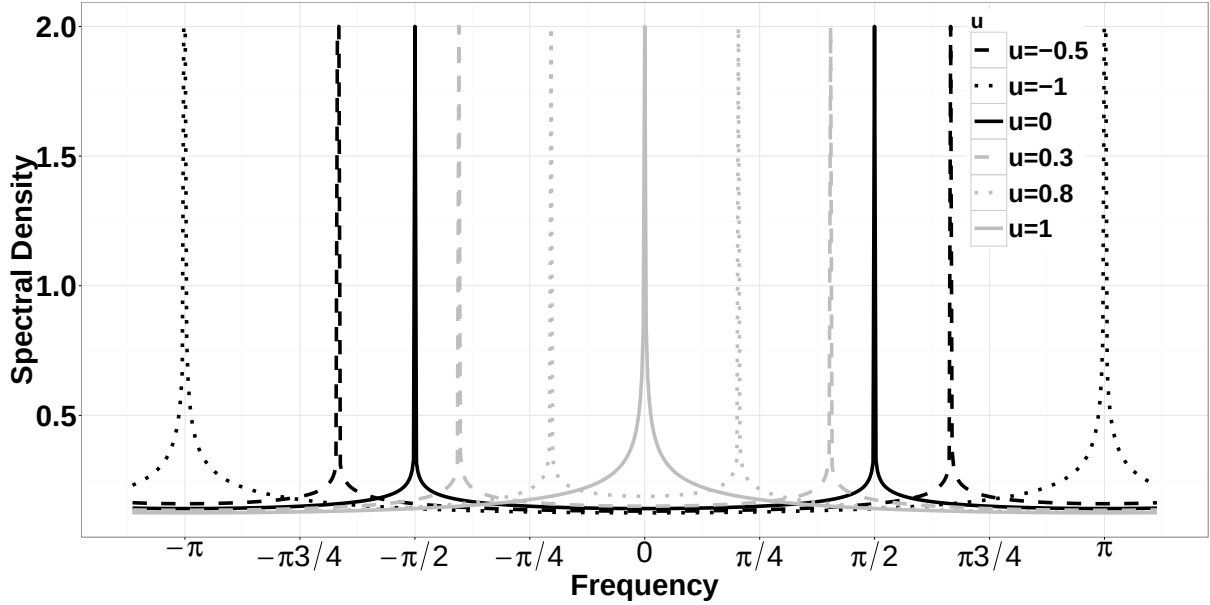


Figure 2.4. Spectral density plots for GARMA model for different values of u when $d = 0.09$.

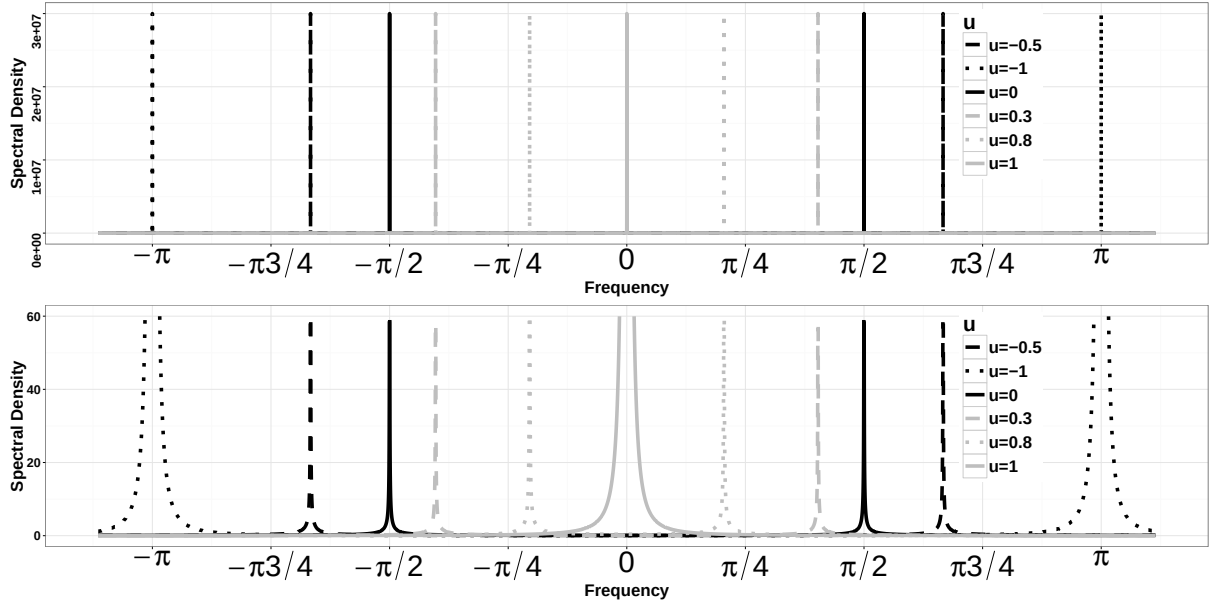


Figure 2.5. Spectral density plots for GARMA model for different values of u when $d = 0.49$ (above) and its enlarged version (below).

2.5 Compared with integer and fractional seasonal difference SARFIMA models.

In addition to the model properties and spectral density specification, this section considers other fractional derivatives, namely, the seasonal fractional difference. ACVF oscillation in Gagenbauer fractional differences will be compared with integer and fractional seasonal difference SARFIMA models. To begin with, we define the standard integer seasonal difference operator and the fractional seasonal difference operator as $(1 - B^S)^{-1}$ and $(1 - B^S)^{-d_S}$, respectively, where the integer S is the seasonal period which defines the frequency of the seasonal pattern. We further explain the difference between three types of process: the seasonal ARIMA known as the SARIMA model; the integer seasonal difference model with ARFIMA structure which we will denote as ISARFIMA models; and the combination of two fractional differences, a fractional seasonal difference combined with an ARFIMA model that we will denote by the

SARFIMA class of models. Then the SARIMA model is given by

$$\phi_S(B^S)(Y_t - c) = (1 - B^S)^{-1} \theta_S(B^S) z_t, \quad (2.24)$$

where $\phi_S(B) = 1 - \phi_{S,1}B - \dots - \phi_{S,p_S}B^{p_S}$ and $\theta_S(B) = 1 - \theta_{S,1}B - \dots - \theta_{S,q_S}B^{q_S}$. Extending the model to incorporate ARFIMA structure gives the ISARFIMA model

$$\phi_S(B^S)(Y_t - c) = (1 - B)^{-d_a} (1 - B^S)^{-1} \theta_S(B^S) z_t, \quad (2.25)$$

whereas replacing the integer seasonal difference in SARIMA model in Equation (2.24) by the fractional seasonal difference gives the basic SARFIMA model

$$\phi_S(B^S)(Y_t - c) = (1 - B^S)^{-d_s} \theta_S(B^S) z_t. \quad (2.26)$$

Lastly, combining the two fractional differences gives the most general SARFIMA(p, d_a, q) $\times$ (p_S, d_S, q_S) $_S$ model

$$\phi(B)\phi_S(B^S)(Y_t - c) = (1 - B)^{-d_a} (1 - B^S)^{-d_s} \theta(B)\theta_S(B^S) z_t, \quad (2.27)$$

which nests all of the three models as special cases. Furthermore, we can further extend SARFIMA to SGARMA by replacing $(1 - B)^{-d_a}$ with $(1 - 2uB + B^2)^{-d}$

$$\phi(B)\phi_S(B^S)(Y_t - c) = (1 - 2uB + B^2)^{-d} (1 - B^S)^{-d_s} \theta(B)\theta_S(B^S) z_t. \quad (2.28)$$

Brietzke et al. (2005) derived the spectral density function for the general SARFIMA model as

$$f_S(\lambda) = \frac{\sigma^2}{2\pi} \left(2 \sin \frac{\lambda S}{2} \right)^{-2d_s} \left(2 \sin \frac{\lambda}{2} \right)^{-2d_a} \frac{|\theta_S(e^{-iS\lambda})|^2}{|\phi_S(e^{-iS\lambda})|^2} \frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2}. \quad (2.29)$$

Figure 2.6 compares four models, namely the (0,0) order ARFIMA, ISARFIMA (Equation (2.25)), SARFIMA (Equation (2.26)) and GARMA models through their ACFs and spectral density functions. It shows that the ACFs of ARFIMA and ISARFIMA models are very similar. Moreover the ACFs of ARFIMA and SARFIMA models are also similar if discounting the gaps, since the ACF of SARFIMA model (Brietzke et al., 2005) is given by

$$\rho_S(j, S + k) = \begin{cases} \frac{\Gamma(1-d_S)\Gamma(j+d_S)}{\Gamma(d_S)\Gamma(j-d_S+1)} = \rho(j), & \text{if } k = 0, \\ 0, & \text{if } k \in \{1, \dots, S-1\}. \end{cases}$$

which is the same as the ACF for ARFIMA model in Equation (2.22) when $d_S = 2d$. Regarding spectral densities, they are very similar for ISARFIMA and SARFIMA models as can be explained from Equation (2.29). In particular, the peaks in the

spectral density of ISARFIMA and SARFIMA models are periodically repeated at frequencies $\frac{2\pi\nu}{S}$ for $\nu = 1, \dots, [S/2]$. The only difference is the height around the peaks which indicates different strength of long memory. On the other hand, there is only one peak at a given frequency $\lambda_0 = \cos^{-1}(u) \in [0, \pi]$ for the GARMA model which is similar to the ARFIMA model but the peak occurs at zero frequency. These features of spectral densities show that within two seasonal frequencies, a SARFIMA process behaves similar to an ARFIMA process but they have different long memory structures to a GARMA process. Hence, the seasonality feature in SARFIMA model cannot describe the oscillatory feature of the GARMA model.

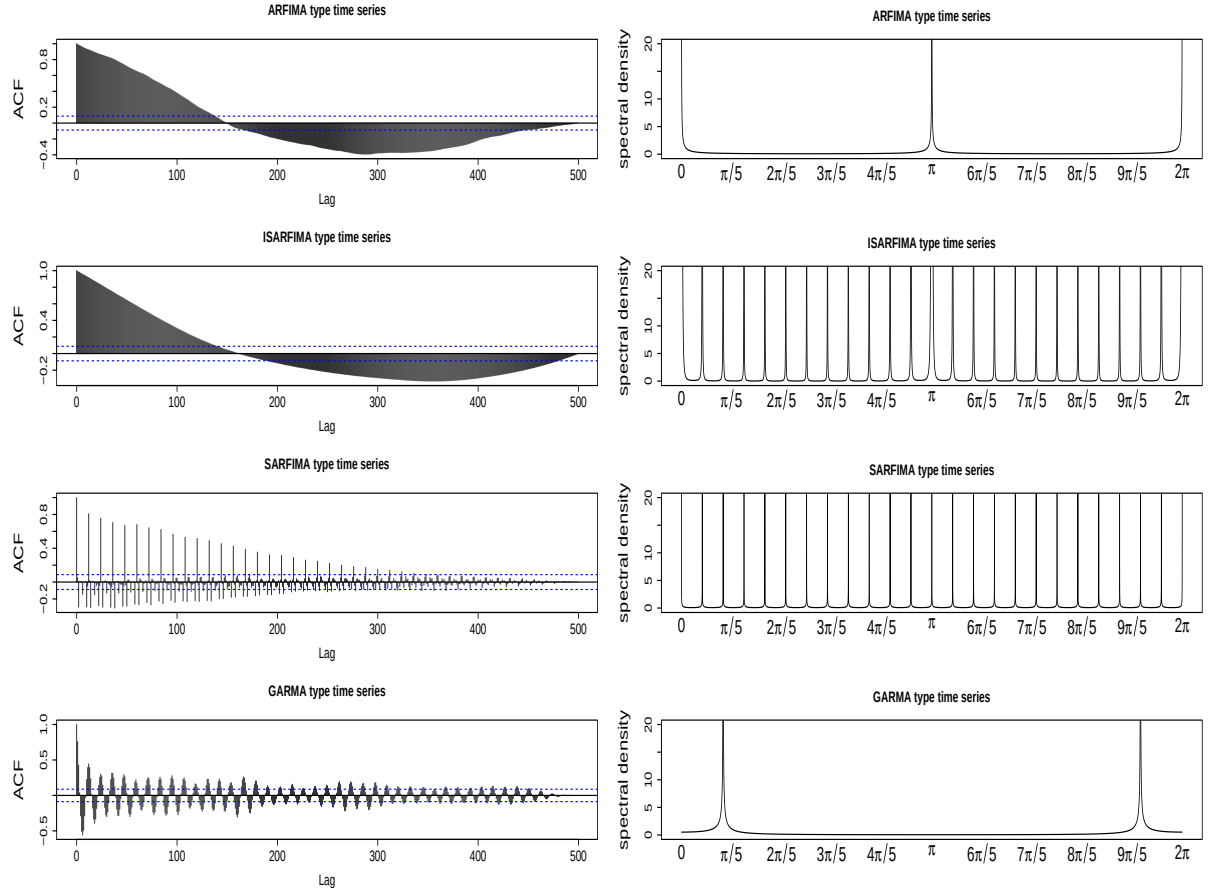


Figure 2.6. Plot of ACF (left) and spectral density (right) for ARFIMA(0, 0.45, 0) (row 1), ISARFIMA(0, 0.45, 0) $\times$ (0, 1, 0)₁₂ (row 2), SARFIMA(0, 0, 0) $\times$ (0, 0.45, 0)₁₂ (row 3), and GARMA(0, 0.45, 0) ($u = 0.866$ for $s = 12$) (row 4) models.

2.6 Conclusion

This chapter introduces GLGARMA model with four different distributions. Moment properties, stationary, invertibility and long memory conditions are explained. Furthermore, the differences between the Gegenbauer long memory component and seasonality component are demonstrated through spectral density function and its plot together with ACFs. Furthermore, the model properties will be further studied in the frequency domain in the next chapter to obtain a better understanding of model structure particularly the oscillatory behaviour of the long memory process.

CHAPTER 3

Proposed long memory time series count model in frequency domain

This chapters review the properties of generalised linear Gegenbauer autoregressive moving average (GLGARMA) model in the frequency domain. We first study spectral density function that reveals important long memory properties, namely the strength and oscillatory pattern of a long memory process and introduce periodogram which is a numerical realisation of spectral density function. We note further that Whittle likelihood involving both spectral density function and periodogram is useful to estimate the long memory parameter d and we also introduce some estimators of d using local log-periodogram and whittle likelihood. Then we study the Whittle likelihood with a Gaussian distribution and its extension to incorporate a long memory component and provide some asymptotic results for Whittle likelihood estimator. For the non-Gaussian case particularly for our proposed GLGARMA-PD model, we derive an approximated spectral density function which can be used to estimate d .

3.1 Whittle likelihood model with Gaussian innovations

3.1.1 Whittle likelihood function

In this section, we consider a univariate time series which admits Wold representation. The Wold representation introduced by Wold (1938) states that

Theorem 3.1.1. *Any stationary process $\mathbf{Y}_{t \in \{1,2,3,\dots,T\}}$ is the sum of a regular process and a singular process; these two processes are orthogonal and the decomposition is unique.*

According to the Wold representation theorem, a zero-mean nondeterministic covariance-stationary process may be expressed as

$$Y_t = c_t + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} = \Psi(B)\varepsilon_t + c_t, \quad \varepsilon_t \sim \text{WN}(0, \sigma^2), \quad (3.1)$$

where ε_t and ψ_j are uniquely defined and satisfy $\psi_0 = 1$, $\sum_{j=0}^{\infty} \psi_j^2 < \infty$, the coefficients $\{c_t; t \in \mathbb{Z}\}$ is a deterministic term with $\mathbb{E}(c_t, \varepsilon_s) = 0, \forall t, s$, WN stands for white noise. In the form of the Wold representation, the spectral density of $Y_t = \Psi(B)\varepsilon_t$ can be written as

$$f_s(\lambda) = \frac{\sigma^2}{2\pi} |\Psi(e^{-i\lambda})|^2. \quad (3.2)$$

There exists a relationship between spectral density and variance given by the Szego-Kolmogorov formula:

Proposition 3.1.1 (The relationship between variance and spectral density). *Given an stationary process $\mathbf{Y}_{t \in \{1,2,3,\dots,T\}}$ with error terms $\varepsilon_t \sim \text{WN}(0, \sigma^2)$, the variance can be represented by a spectral density function as*

$$\sigma^2 = \exp \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log(2\pi f_s(\lambda)) d\lambda \right).$$

Suppose a linear time series $\mathbf{Y}_{t \in \{1,2,3,\dots,T\}}$ is normally distributed with zero mean and variance $\mathbf{\Gamma}_\theta$. Then the log-likelihood function can be written as

$$L(\theta) = -\frac{T}{2} \log(2\pi) - \frac{1}{2} \log \det \mathbf{\Gamma}_\theta - \frac{1}{2} \mathbf{Y}' \mathbf{\Gamma}_\theta^{-1} \mathbf{Y}. \quad (3.3)$$

The log-likelihood function in Equation (3.3) can be approximated by a Whittle likelihood function (Palma, 2007) which is a function of Equation (3.3) given by the following definition.

Definition 3.1.1 (Whittle likelihood). *Consider a linear time series $\mathbf{Y}_{t \in \{1,2,3,\dots,T\}}$ with Gaussian additive innovations, a Whittle likelihood function for unknown parameters θ can be derived by taking the Fourier transform of log-likelihood function (3.3)*

$$L_W(\theta) = -\frac{1}{4\pi} \left[\int_{-\pi}^{\pi} \log f_s(\lambda) d\lambda + \int_{-\pi}^{\pi} \frac{I_T(\lambda)}{f_s(\lambda)} d\lambda \right]. \quad (3.4)$$

According to Proposition (3.1.1), the spectral density $f_s(\lambda)$ can be derived by taking the Fourier transform of autocovariance functions $\gamma_\theta(l-j) = (\mathbf{\Gamma}_\theta)_{lj}$ and $I_T(\lambda_k)$ can be derived by taking Discrete Fourier Transformation ($\mathfrak{D}(\cdot)$) of $\mathbf{Y}_{t \in \{1,2,3,\dots,T\}}$ where $\lambda_k = \frac{2\pi k}{T}$ for $k = 1, \dots, \left[\frac{T}{2}\right]$, $[\cdot]$ represents the integer part and only half of frequencies are enough to demonstrate the features because of symmetry. The spectral density is given by

$$f_s(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma_\theta(k) \exp(-i\lambda k) dk, \quad -\pi < \lambda < \pi \quad \text{and} \quad k = l - j.$$

The periodogram is given by

$$\begin{aligned} I_T(\lambda_k) &= \frac{1}{2\pi T} |\mathfrak{D}(\mathbf{Y}_{t \in \{1,2,\dots,T\}})|^2 \\ &= \frac{1}{2\pi T} \left| \sum_{j=1}^T Y_j e^{-ij\lambda_k} \right|^2 \\ &= Y_c^2(\lambda_k) + Y_s^2(\lambda_k), \\ Y_c(\lambda_k) &= \frac{1}{\sqrt{2\pi T}} \sum_{j=1}^T \cos(j\lambda_k) Y_j, \\ Y_s(\lambda_k) &= \frac{1}{\sqrt{2\pi T}} \sum_{j=1}^T \sin(j\lambda_k) Y_j. \end{aligned}$$

For computational convenience, the integrals in Equation (3.4) can be approximated by Riemann sums

$$\begin{aligned}\hat{L}_W(\theta) &= -\frac{1}{2T} \left[\sum_{k=1}^T \log f_s(\lambda_k) + \sum_{k=1}^T \frac{I_T(\lambda_k)}{f_s(\lambda_k)} \right], \\ f_s(\lambda_k) &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_{\theta}(k) \exp(-i\lambda_k), \\ \int_{-\pi}^{\pi} \log f_s(\lambda) d\lambda &\approx \frac{2\pi}{T} \sum_{k=1}^T \log f_s(\lambda_k), \\ \int_{-\pi}^{\pi} \frac{I_T(\lambda)}{f_s(\lambda)} d\lambda &\approx \frac{2\pi}{T} \sum_{k=1}^T \frac{I_T(\lambda_k)}{f_s(\lambda_k)}.\end{aligned}$$

Remark 3.1.1. Under the condition $\mathbb{V}ar(Y_t) = \sum_{j=0}^{\infty} \psi_j^2 \sigma^2 = 1$, which is equivalent to one of the normalization conditions of spectral density $f_s(\lambda)$, such that

$$\int_{-\pi}^{\pi} \log f_s(\lambda) d\lambda = 0,$$

the Equation (3.4) can be transformed into following expression (Palma, 2007)

$$L_W(\theta) = -\frac{1}{4\pi} \left[\int_{-\pi}^{\pi} \frac{I_T(\lambda)}{f_s(\lambda)} d\lambda \right], \quad (3.5)$$

which can also be replaced in practice by a discrete approximation

$$\hat{L}_W(\theta) = \sum_{k=1}^{\left[\frac{T}{2}\right]} \frac{I_T(\lambda_k)}{f_s(\lambda_k)}.$$

3.1.2 Whittle likelihood function for long memory model

Given a stationary time series process $\mathbf{Y}_{t \in \{1,2,3,\dots,T\}}$, Beran (1994) defined a condition for a long memory stationary process in terms of the divergence of the autocorrelation function (ACF) for Y_t and Y_{t+j} at lag j , such that

$$\lim_{T \rightarrow \infty} \sum_{j=-T}^T |\rho(j)| \rightarrow \infty \quad \text{where} \quad \rho(j) = \frac{\text{Cov}(Y_t, Y_{t+j})}{\sqrt{\mathbb{V}ar(Y_t) \mathbb{V}ar(Y_{t+j})}}.$$

In order to extend the ARMA model to capture such a long memory feature, the ARFIMA (p, d, q) model with mean μ was proposed by Granger and Joyeux (1980)

and Hosking (1981) to describe a long memory stationary process with integrated order $d \in (0, 1/2)$. Robinson (1995a) developed a Whittle estimation procedure for the long memory parameter d for a stochastic time series and further showed that the estimator with limiting normal distribution is consistent. Phillips and Shimotsu (2004) applied the local Whittle likelihood to estimate the long memory parameter d and studied the asymptotic properties and consistency in the non-stationary setting, including the unit root case. Shimotsu and Phillips (2005) proposed an exact form of the local Whittle likelihood for estimating d . Compared with the previous approach, this method is a general-purpose estimation procedure that can be used when d lies in both stationary and non-stationary regions. For the ARFIMA(p, d, q) model

$$\phi(B)(y_t - c) = \theta(B)(1 - B)^{-da} \varepsilon_t \equiv \theta(B) \left(\sum_{j=0}^{\infty} \varphi_j \varepsilon_{t-j} \right), \quad (3.6)$$

the spectral density function can be expressed as

$$\begin{aligned} f_s(\lambda) &= \frac{\sigma^2}{2\pi} |1 - e^{-i\lambda}|^{-4d} \frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2} \\ &= \frac{\sigma^2}{2\pi} |1 - \cos \lambda + i \sin \lambda|^{-4d} \frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2} \\ &= \frac{\sigma^2}{2\pi} \left(2 \sin \frac{\lambda}{2} \right)^{-4d} \frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2}. \end{aligned}$$

Furthermore, Hosking (1981) showed that as $\lambda \rightarrow 0$,

$$f_s(\lambda) = \frac{\sigma^2}{2\pi} \left(2 \sin \frac{\lambda}{2} \right)^{-4d} \frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2} \sim \frac{\sigma^2}{2\pi} \frac{|\theta(1)|^2}{|\phi(1)|^2} \lambda^{-4d} \sim \sigma^2 \lambda^{-4d}, \quad (3.7)$$

for $0 < \lambda \leq \pi$. Consequently, $\lim_{\lambda \rightarrow 0} \lambda^{4d} f_s(\lambda)$ exists and is finite. For the special case of $\phi(B) = \theta(B) = 1$, we have

$$f_s(\lambda) = \frac{\sigma^2}{2\pi} |1 - e^{-i\lambda}|^{-4d} = \frac{\sigma^2}{2\pi} \left(2 \sin \frac{\lambda}{2} \right)^{-4d}.$$

A useful generalised ARFIMA model called Gegenbauer ARMA (GARMA)(p, d, q) model (see Equation (3.8)) proposed by Hosking (1981) describes data showing slowly damping ACF with a cyclic pattern. The GARMA is given by

$$\phi(B)(y_t - c) = \theta(B)(1 - 2uB + B^2)^{-d} \varepsilon_t \equiv \theta(B) \left(\sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} \right). \quad (3.8)$$

For the GARMA(p, d, q) process with long memory, the spectral density function was derived in [Gray et al. \(1989\)](#) as follow:

$$\begin{aligned}
f_s(\lambda) &= \frac{\sigma^2}{2\pi} |1 - 2u \exp(-i\lambda) + \exp(-2i\lambda)|^{-2d} \frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2} \\
&= \frac{\sigma^2}{2\pi} |1 - 2u(\cos \lambda - i \sin \lambda) + \cos(2\lambda) - i \sin(2\lambda)|^{-2d} \frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2} \\
&= \frac{\sigma^2}{2\pi} (2|\cos(\lambda) - \cos(\lambda_0)|)^{-2d} \frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2} \quad \text{where } |e^{-i\lambda}| = 1 \\
&= \frac{\sigma^2}{2\pi} \left(4 \left| \sin \left(\frac{\lambda + \lambda_0}{2} \right) \sin \left(\frac{\lambda - \lambda_0}{2} \right) \right| \right)^{-2d} \frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2},
\end{aligned}$$

where $0 < d < 1/2$. Furthermore, it was demonstrated in [Gray et al. \(1989\)](#) that the limiting behaviour of the spectral density function for the GARMA(p, d, q) model as $\lambda \rightarrow \lambda_0$ is

$$\begin{aligned}
f_s(\lambda) &= \frac{\sigma^2}{2\pi} \frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2} \left(4 \left| \sin \left(\frac{\lambda + \lambda_0}{2} \right) \sin \left(\frac{\lambda - \lambda_0}{2} \right) \right| \right)^{-2d} \\
&\sim \frac{\sigma^2}{2\pi} \frac{|\theta(\lambda_0)|^2}{|\phi(\lambda_0)|^2} (4|\sin(\lambda_0)|)^{-2d} (\lambda - \lambda_0)^{-2d} \\
&\sim \sigma^2 (\lambda - \lambda_0)^{-2d}.
\end{aligned} \tag{3.9}$$

Hence, $\lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0)^{2d} f_s(\lambda)$ exists and is finite. For the special case $\phi(B) = \theta(B) = 1$, we have

$$f_s(\lambda) = \frac{\sigma^2}{2\pi} [4(\cos(\lambda) - \cos(\lambda_0))^2]^{-d} = \frac{\sigma^2}{2\pi} [4(\cos(\lambda) - u)^2]^{-d}. \tag{3.10}$$

The spectral measure is absolutely continuous with respect to the Lebesgue measure on $[-\pi, \pi]$ (see [Palma, 2007](#)). The limits in Equations (3.7) and (3.9) give the spectral density functions around the peak and are useful in Whittle likelihood estimation. It is obvious that the spectral density $f_s(\lambda)$ for ARFIMA model will approach infinity if we plug $\lambda = 0$ into Equation (3.7). Hence, $f_s(\lambda)$ is unbounded and discontinuous at $\lambda = 0$. Similarly, the peak of GARMA model is unbounded when $\lambda = \lambda_0$. In this case, the $f_s(\lambda)$ is no longer properly defined over the range $[-\pi, \pi]$, and such issue will cause inconsistent estimation estimation of Hurst exponent H defined in Section 4.3.1 at high frequencies ([Robinson, 1995a](#)). As a result, the semi-parametric estimation methods are applied in this situation because this method bases only on the locations of unbound points of the spectral density rather than the specification of a parametric model for the time series.

3.1.3 Estimation of d using log-periodogram and Whittle likelihood

The log-periodogram (LP) regression studied by Geweke and Porter-Hudak (1983) and Robinson (1995b) and local Whittle (LW) likelihood estimation studied by Kuensch (1987) and (Robinson, 1995a) are two main types of semi-parametric methods. The long memory parameter d is estimated by log-periodogram regression and local Whittle likelihood methods and the estimates are denoted by $\hat{d}_{LP}$ and $\hat{d}_{LW}$ and defined in Definition 3.1.2 and Definition 3.1.3 respectively. The approach of estimating d from log-periodogram regression is very straight forward (Robinson, 1995b). For a stochastic process $\mathbf{Y}_{t \in \{1,2,3,\dots,T\}}$ with the periodogram $I_T(\lambda_k)$, the estimator of d can be regarded as a least square estimator in a linear regression model

$$\begin{aligned} \log I_T(\lambda_k) &= c + da_k + u_k, \quad k = l+1, \dots, m, \quad 0 \leq l < m < T \\ a_k &= -\log \left(4 \sin^2 \left(\frac{\lambda_k}{2} \right) \right), \\ u_k &= \log \left(\frac{4 \sin^2 \left(\frac{\lambda_k}{2} \right)^d I_T(\lambda_k)}{C} \right) + \eta, \\ c &= \log C - \eta, \end{aligned}$$

where η is the Euler's constant and C is a constant. Making use of the result in linear regression, we have the LP estimator of d as below.

Definition 3.1.2 (Estimator $\hat{d}_{LP}$ by adopting log-periodogram). *The estimator $\hat{d}_{LP}$ developed by adopting log-periodogram approach can be shown to be*

$$\begin{aligned} \hat{d}_{LP}(l) &= \frac{\sum_{k=l+1}^m (a_k - \bar{a}) \log I_T(\lambda_k)}{S_l}, \\ \bar{a} &= (m-l)^{-1} \sum_{k=l+1}^m a_k, \\ S_l &= \sum_{k=l+1}^m (a_k - \bar{a})^2. \end{aligned}$$

For the range $|d| < 0.5$ in the stationary case, estimator of d using LW likelihood is more efficient than estimator using LP (Shimotsu and Phillips, 2005). Given a ARFIMA

model defended in Equation 3.6, the estimators of d and Λ using LW likelihood is given in Definition 3.1.3.

Definition 3.1.3 (Estimator $(\hat{\Lambda}, \hat{d}_{LW})$ by adopting local Whittle likelihood). *The parameters (Λ, d) can be estimated by minimizing the objective function*

$$Q_m(\Lambda, d) = \frac{1}{m} \sum_{k=1}^m \left(\log(\Lambda \lambda^{-2d}) + \frac{\lambda_k^{2d}}{\Lambda} I_T(\lambda_k) \right), \quad m < T.$$

Hence, we have

$$(\hat{\Lambda}, \hat{d}_{LW}) = \arg \min_{\Lambda \in (0, \infty), d \in [\Delta_1, \Delta_2]} Q_m(d, \Lambda), \quad -\frac{1}{2} < \Delta_1 < \Delta_2 < \infty,$$

where Δ_1 and Δ_2 are the lower and upper bounds of the admissible values of d such that $-\infty < \Delta_1 < \Delta_2 < \infty$. For example, for the LW estimator is discontinuous at $d = 3/4$ and again at $d = 1$.

Given the estimate $\hat{\Lambda}, \hat{d}_{LW}$ can be obtained by the following local Whittle likelihood estimator.

Definition 3.1.4 (Estimator $\hat{d}_{LW}$ by adopting local Whittle likelihood given $Q_m(\Lambda, d)$). *The estimator $\hat{d}_{LW}$ developed by adopting LW approach given $Q_m(\Lambda, d)$ can be shown as*

$$\begin{aligned} \hat{d}_{LW} &= \arg \min_{d \in [\Delta_1, \Delta_2]} R(d), \\ R(d) &= \log \left[\frac{1}{m} \sum_{k=1}^m \lambda_k^{2d} I_T(\lambda_k) \right] - 2d \frac{1}{m} \sum_{k=1}^m \log \lambda_k. \end{aligned}$$

The consistency of estimator $\hat{d}_{LW}$ is subjected to the following assumptions (Robinson, 1995a).

Assumption 3.1.1. *This assumption introduces the range of Λ and d .*

$$f_s(\lambda) \sim \Lambda_0 \lambda^{1-2d}, \quad \text{as } \lambda \rightarrow 0_+,$$

where $\Lambda_0 \in (0, \infty)$ denote the true value of Λ which is σ^2 in our case and $d \in [\Delta_1, \Delta_2]$.

Assumption 3.1.2. *This assumption provides a regularity condition for the estimator $\hat{d}_{LW}$. In a neighborhood $(0, \delta)$ of the origin, $f_s(\lambda)$ is differentiable with respect to λ and*

$$\frac{d}{d\lambda} f_s(\lambda) = O(\lambda^{-1}), \quad \text{as } \lambda \rightarrow 0_+.$$

Assumption 3.1.3. *This assumption provides a mild homogeneity restriction.*

$$u_t = (B)(B)\varepsilon_t = \sum_{j=0}^{\infty} b_j \varepsilon_{t-j}, \quad \sum_{j=0}^{\infty} b_j^2 < \infty,$$

with

$$\mathbb{E}(\varepsilon_t | \mathcal{F}_{t-1}) = 0, \quad \mathbb{E}(\varepsilon_t^2 | \mathcal{F}_{t-1}) = 1, \quad t = 0, \pm 1, \dots$$

There exist ε_t with $\mathbb{E}(\varepsilon_t^2) < \infty$. For all η and some $K > 0$, $P(|\varepsilon_t| > \eta) \leq KP(|\varepsilon_t| > \eta)$.

Assumption 3.1.4. *For the consistency of estimator $\hat{d}_{LW}$, m must tend to infinity but slower than n . Hence, this assumption is introduced to satisfy the behaviour of m*

$$\frac{1}{m} + \frac{m}{T} \rightarrow 0, \quad \text{as } T \rightarrow \infty,$$

Remark 3.1.2. *The estimator $\hat{d}_{LW}$ is consistent and asymptotically normally distributed for any true value $d_0 \in (\Delta_1, \Delta_2)$. The range of d was extended by [Velasco \(1999\)](#) and the estimator is consistent for $d \in (\frac{1}{2}, 1)$ and asymptotically normally distributed for $d \in (\frac{1}{2}, \frac{3}{4})$. The asymptotic properties of the LW estimator in the non-stationary case for $d > \frac{1}{2}$ were studied by [Phillips and Shimotsu \(2004\)](#).*

Remark 3.1.3. *Similar to the ARFIMA model, the estimator of d in GARMA model (see Equation 3.8) can also be obtained. [Fox and Taqqu \(1987\)](#) proved the Central Limit Theorems with asymptotical normality for whittle likelihood estimates from Gaussian models with long-range dependence.*

3.1.4 Relationship between periodogram and spectral density

The spectral density is a critical component in Whittle likelihood estimation. However, for much more complicated hierarchy model, we need to study the behaviour of the spectral density function to facilitate better understanding of the model structure. For a Gaussian white noise process $\mathbf{Y}_{t \in \{1, 2, 3, \dots, T\}}$, the covariance

$$\text{Cov}(I_T(\lambda_i), I_T(\lambda_j)) = 0,$$

of periodograms $I_T(\lambda_k)$ also holds for finite T and $i \neq j$ (Percival and Walden, 1993). Furthermore, the following properties are derived to demonstrate the relationship between the periodogram $I_T(\lambda_k)$ and the spectral density $f_s(\lambda)$. The expected value, variance and covariance of the two components of periodogram $Y_c(\lambda_k)$ and $Y_s(\lambda_k)$ are given by

$$\begin{aligned} \mathbb{E}(Y_c(\lambda_k)) &= \mathbb{E}(Y_s(\lambda_k)) = 0, \\ \mathbb{V}\text{ar}(Y_c(\lambda_k)) &= \sigma^2 \sum_{j=1}^T \cos^2(j\lambda_k) = \frac{T\sigma^2}{2}, \\ \mathbb{V}\text{ar}(Y_s(\lambda_k)) &= \sigma^2 \sum_{j=1}^T \sin^2(j\lambda_k) = \frac{T\sigma^2}{2}, \\ \mathbb{C}\text{ov}(Y_c(\lambda_k), Y_s(\lambda_k)) &= \mathbb{E} \left[\sum_{i=1}^T \sum_{j=1}^T Y_i Y_j \cos(i\lambda_k) \sin(j\lambda_k) \right] \\ &= \sigma^2 \sum_{j=1}^T \cos(j\lambda_k) \sin(j\lambda_k) = 0. \end{aligned}$$

Hence, $Y_c(\lambda_k)\sqrt{\frac{T\sigma^2}{2}}$ and $Y_s(\lambda_k)\sqrt{\frac{T\sigma^2}{2}}$ are independent Normal random variables. In particular, the distribution, expected value and variance of the periodogram $I_T(\lambda_k)$ are given by

$$\begin{aligned} I_T(\lambda) &\stackrel{\text{i.i.d}}{\sim} \frac{1}{2}\chi_2^2\sigma^2, \quad 0 < \lambda < \pi, \\ \mathbb{E}(I_T(\lambda)) &= \sigma^2, \\ \mathbb{V}\text{ar}(I_T(\lambda)) &= \sigma^4. \end{aligned}$$

Hence, $I_T(\lambda)$ is unbiased but inconsistent estimator of $f_s(\lambda_k)$ because the spectrum $f_s(\lambda_k) = \sigma^2$ for a Gaussian white noise process $\mathbf{Y}_{t \in \{1,2,3,\dots,T\}}$.

Proposition 3.1.2 (The relationship between periodogram and spectral density). *Given a stationary and Gaussian process $\mathbf{Y}_{t \in \{1,2,3,\dots,T\}}$ with spectral density $f_s(\lambda_k)$, (Koopmans, 1995) stated that the relationship between periodogram and the spectral density as below:*

$$\begin{aligned} I_T(\lambda_k) = \frac{T}{4\sigma^2}(Y_c^2(\lambda_k) + Y_s^2(\lambda_k)) &\stackrel{\text{i.i.d}}{\sim} \frac{1}{2}\chi_2^2 f_s(\lambda_k), \quad 0 < \lambda_k < \pi, \\ I_T(\lambda_k) &\stackrel{\text{i.i.d}}{\sim} \chi_1^2 f_s(\lambda_k), \quad \lambda_k \in \{0, \pi\}. \end{aligned}$$

Hence, the power spectral density can be approximated by the periodogram

$$\begin{aligned}
& \mathbb{E} \left(\frac{1}{2\pi T} |\mathfrak{F}(Y_t)|^2 \right) \\
&= \frac{1}{T} \mathbb{E} \left(\int_0^{2\pi T} Y_t e^{-it\lambda} dt \cdot \int_0^{2\pi T} Y_s e^{is\lambda} ds \right) \\
&= \frac{1}{2\pi T} \int_0^{2\pi T} \int_0^{2\pi T} \mathbb{E}(Y_t Y_s) e^{-i\lambda(t-s)} dt ds \\
&= \frac{1}{2\pi T} \int_0^{2\pi T} \int_0^{2\pi T} \gamma(t-s) e^{-i\lambda(t-s)} dt ds \\
&= \frac{1}{2\pi T} \int_0^{2\pi T} \left[\int_{t-2\pi T}^t \gamma(\tau) e^{-i\lambda\tau} d\tau \right] dt \\
&= \frac{1}{2\pi T} \int_{-2\pi T}^0 \left[\int_0^{2\pi T+\tau} \gamma(\tau) e^{-i\lambda\tau} dt \right] d\tau + \frac{1}{2\pi T} \int_0^{2\pi T} \left[\int_{\tau}^{2\pi T} \gamma(\tau) e^{-i\lambda\tau} dt \right] d\tau \\
&= \int_{-2\pi T}^{2\pi T} \left(1 - \frac{|\tau|}{2\pi n} \right) \gamma(\tau) e^{-i\lambda\tau} d\tau,
\end{aligned}$$

where $\mathfrak{F}(\cdot)$ is the Fourier transform function, $\tau = t - s$, $t \in (0, 2\pi T)$ and $s \in (0, 2\pi T)$ implies $\tau \in (-2\pi T, 2\pi T)$. It is clear that the periodogram is an unbiased estimator as

$$\lim_{T \rightarrow \infty} \mathbb{E} \left(\frac{1}{2\pi T} |\mathfrak{F}(Y_t)|^2 \right) = f_s(\lambda).$$

However, the periodogram is inconsistent, which means the variance does not decrease when we take more observations. Let us define $\hat{f}_s(\lambda) = \mathbb{E} \left(\frac{1}{2\pi T} |\mathfrak{F}(Y_t)|^2 \right)$ given T , then we have

Proposition 3.1.3 (Inconsistent estimator). *The periodogram is an inconsistent estimator of spectral density function.*

The proposition 3.1.3 can be proved by following equations. The variance of spectral density function is given by

$$\mathbb{V}\text{ar} \left(\hat{f}_s(\lambda) \right) = \mathbb{E} \left(\hat{f}_s(\lambda)^2 \right) - \mathbb{E} \left(\hat{f}_s(\lambda) \right)^2,$$

where

$$\begin{aligned}
\mathbb{E} \left(\hat{f}_s(\lambda)^2 \right) &= \frac{1}{(2\pi T)^2} \mathbb{E} \left(\int_0^{2\pi T} \int_0^{2\pi T} \int_0^{2\pi T} \int_0^{2\pi T} Y_t Y_s Y_u Y_v e^{-\lambda(t-s+u-v)} dt ds du dv \right) \\
&= \frac{1}{(2\pi T)^2} \left(\int_0^{2\pi T} \int_0^{2\pi T} \gamma(t-s) e^{-\lambda(t-s)} dt ds \cdot \int_0^{2\pi T} \int_0^{2\pi T} \gamma(u-v) e^{-\lambda(u-v)} du dv \right) \\
&\quad + \frac{1}{(2\pi T)^2} \left(\int_0^{2\pi T} \int_0^{2\pi T} \gamma(t-u) e^{-\lambda(t+u)} dt du \cdot \int_0^{2\pi T} \int_0^{2\pi T} \gamma(s-v) e^{-\lambda(s+v)} ds dv \right) \\
&\quad + \frac{1}{(2\pi T)^2} \left(\int_0^{2\pi T} \int_0^{2\pi T} \gamma(t-v) e^{-\lambda(t-v)} dt dv \cdot \int_0^{2\pi T} \int_0^{2\pi T} \gamma(s-u) e^{-\lambda(s-u)} ds du \right) \\
&= 2\mathbb{E} \left(\hat{f}_s(\lambda) \right)^2 + \frac{1}{(2\pi n)^2} \left| \int_0^{2\pi T} \int_0^{2\pi T} \gamma(t-u) e^{-\lambda(t+u)} dt du \right|^2.
\end{aligned}$$

Hence, we have

$$\text{Var} \left(\hat{f}_s(\lambda) \right) = \mathbb{E} \left(\hat{f}_s(\lambda) \right)^2 + \frac{1}{(2\pi T)^2} \left| \int_0^{2\pi T} \int_0^{2\pi n} \gamma(t-u) e^{-i\lambda(t+u)} dt du \right|^2 \geq \mathbb{E} \left(\hat{f}_s(\lambda) \right)^2,$$

and for $T \rightarrow \infty$, we have

$$\begin{aligned}
\lim_{T \rightarrow \infty} \text{Var} \left(\hat{f}_s(\lambda) \right) &= f_s(\lambda)^2 + O(1/T) \\
&\geq f_s(\lambda)^2.
\end{aligned}$$

Hence, the $\text{Var} \left(\hat{f}_s(\lambda) \right)$ does not decrease when we take more observations. The periodogram $I_T(\lambda_k)$ is inconsistent estimator of spectral density function $f_s(\lambda)$.

The periodogram and spectral density function in Figure 3.1 are generated from GARMA model using $d = 0.45$ and $u = 0.9$ respectively. The figure shows that the periodogram is a sample estimate of spectral density. Because of the symmetry, only half of the range $(0, \pi)$ is enough to demonstrate the relationship between periodogram and spectral density. The position of the peak in the periodogram and spectral density plots represent u which follows $u = \cos(\lambda)$ so that $\lambda = \cos^{-1}(0.9) = 72/500\pi$ and the curvature of the spectral density indicates the magnitude of d . If the periodogram can closely approximate the shape of the spectral density function, the estimates of u and d using Whittle likelihood will be more precise and accurate.

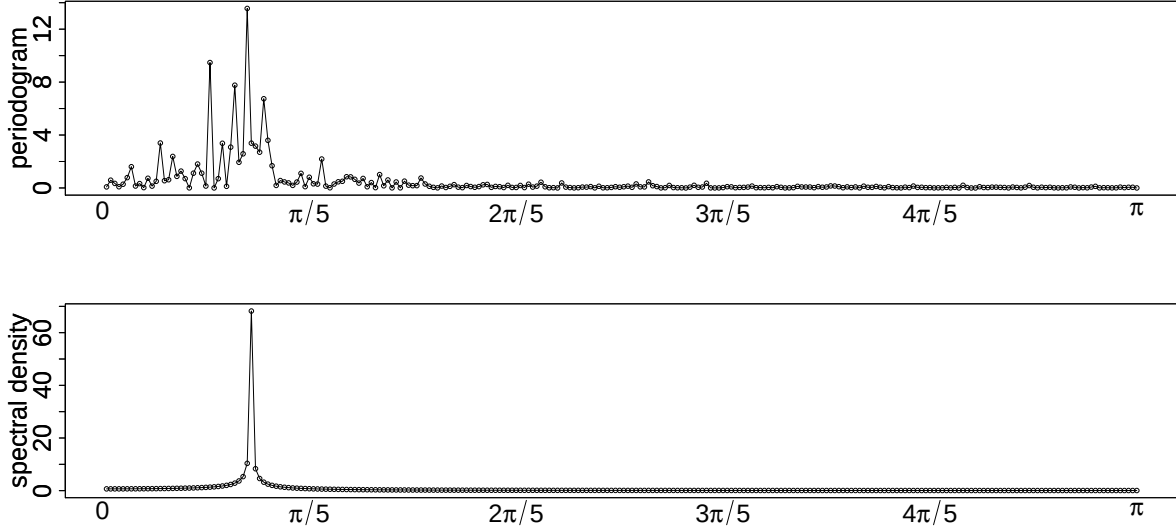


Figure 3.1. Periodogram (above) and spectral density (below) for $n = 500$ data simulated from GARMA model with $d=0.45$ and $u=0.9$.

3.2 Whittle likelihood model with non-Gaussian innovations

In this section, we study the Whittle likelihood with non-Gaussian innovations and the extension that incorporates a non-linear process.

3.2.1 Whittle likelihood for linear process with Gaussian distribution

Giraitis and Surgailis (1990) proved the asymptotical normality of Whittle likelihood estimate for the parameter in a dependent linear sequences. Fox and Taqqu (1987) further generalised the central limit theorem to the non-Gaussian process if the process $\mathbf{Y}_{t \in \{1,2,3,\dots,T\}}$ can be written as Wold representation. Let us introduce the following set of conditions and four assumptions. There exists $0 < \alpha(\theta) < 1$ and $C = C(\theta) < \infty$ such that for all $\theta \in \Theta$ and $\lambda \in \Pi$

Assumption 3.2.1. $\int_{\Pi} \log f_s(\lambda) d\lambda (\equiv 0)$ is twice differentiable in $\theta \in \Theta$ under the sign of integral.

Assumption 3.2.2. $f_s(\lambda)$ is continuous at all $(\lambda; \theta)$, $\lambda \neq 0$, $f_s^{-1}(\lambda)$ is continuous at all $(\lambda; \theta)$ and

$$f_s(\lambda) \leq C|\lambda|^{-\alpha(\theta)}.$$

Assumption 3.2.3. $\frac{\partial}{\partial \theta_i} f_s(\lambda)^{-1}$ and $\frac{\partial^2}{\partial \theta_i \partial \theta_j} f_s(\lambda)^{-1}$ are continuous at all $(\lambda; \theta)$

$$\left| \frac{\partial^2}{\partial \theta_i \partial \theta_j} f_s(\lambda)^{-1} \right| \leq C|\lambda|^{\alpha(\theta)}, \quad 1 \leq j \leq q,$$

where q in this section denote number of unknown parameters.

Assumption 3.2.4.

$$\left| \frac{\partial^2}{\partial \theta_i \partial \lambda} f_s(\lambda)^{-1} \right| \leq C|\lambda|^{\alpha(\theta)-1}, \quad 1 \leq j \leq q.$$

Under above conditions and four assumptions. Giraitis and Surgailis (1990) proved following theorem.

Theorem 3.2.1. Let $\hat{\theta}_T$ be the estimates that maximize the log Whittle likelihood function 3.5. $\hat{\theta}_T$ is consistent and

$$\sqrt{T}(\hat{\theta}_T - \theta_0) \rightarrow N(0, \mathbf{\Gamma}_{\theta_0}^{-1}), \quad \text{as } T \rightarrow \infty,$$

where $\mathbf{\Gamma}_{\theta} = (\mathbf{\Gamma}_{\theta})_{ij}$ is the matrix defined by

$$(\mathbf{\Gamma}_{\theta})_{ij} = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\frac{\partial \log f_s(\lambda)}{\partial \theta_i} \right) \left(\frac{\partial \log f_s(\lambda)}{\partial \theta_j} \right) d\lambda.$$

Given a non-Gaussian linear processes, Naito et al. (2006) proposed a local Whittle likelihood based on the spectral density $f_s(\lambda)$ and a local Whittle estimator $\hat{f}_s(\lambda)$ around each frequency $\lambda \in [-\pi, \pi]$. In addition, the asymptotic behaviours of $f_s(\lambda)$ and $\hat{f}_s(\lambda)$ were studied. The results can also be extended to stationary nonlinear processes, such as GARCH processes. To test whether the spectral density of a class of non-Gaussian linear processes belongs to a parametric family, a local Whittle likelihood ratio test statistic was applied to the local Whittle likelihood estimator, and the test will converge asymptotically in distribution to a normal distribution.

For a non-Gaussian process, the empirical likelihood is another option because the data distribution does not need to be assumed. Under the assumption of independent data, [Owen \(2001\)](#) showed that the empirical likelihood is asymptotically chi-square distributed. [Monti \(1997\)](#) further proved that this asymptotical behaviour still holds for dependent data. Given non-Gaussian and dependent process, [Ogata \(2005\)](#) provided a proof for asymptotic normality of the empirical Whittle likelihood ratio.

For non-Gaussian time series, [Giraitis and Taqqu \(1999b\)](#) showed that consistency of the Whittle estimator does not always hold and that the limit is not necessarily Gaussian. [Contreras-Cristan et al. \(2006\)](#) measured the loss of efficiency of Whittles method in nonstandard settings. For small and highly correlated samples, Whittles likelihood is not efficient in the Gaussian case. For moderate sample sizes, Whittles likelihood can still produce unreliable estimates in the non-Gaussian case.

3.2.2 Whittle likelihood for non-linear process with non-Gaussian distribution

In order to model discrete time series with long memory features, [Yan et al. \(2017\)](#) developed a GLGARMA model which combines the generalised linear model and the GARMA time series models. In the non-linear non-Gaussian models, the solution in this form can be intractable. For non-linear non-Gaussian model in Equation (2.11), there are two main differences from GARMA model. Firstly, the observations Y_t follow a discrete distribution, such as Poisson. The spectral density function is different from Equation (3.2) which is derived based on Gaussian distribution. Hence, the Whittle likelihood of the model in Equation (2.11) needs to be modified. Secondly, the mean function μ_t adopts a log link to ensure the values of μ_t are all positive. This non-linear transformation needs to be taken into the consideration when estimating the spectral density function in Whittle likelihood estimation.

To find the Whittle likelihood function of the model in Equation (2.11), we need to find the spectral density and the periodogram of this model. The periodogram can be easily calculated from the observations, but the derivation of the spectral density of this model is complicated. According to the relationship between variance and spectral

density function in Proposition 3.1.1, this spectral density function can be derived from autocovariance function by taking a Fourier transform.

Proposition 3.2.1 (Autocovariance function of GLGARMA). *Given a GLGARMA process $\mathbf{Y}_t$ (see Section 2.3), the autocovariance function can be represented as*

$$\gamma(h) = \exp\left(\sigma^2 \sum_{j=0}^{\infty} \psi_j^2\right) \exp\left(\sigma^2 \sum_{j=0}^{\infty} \psi_{h+j} \psi_j\right) - \exp\left(\sigma^2 \sum_{j=0}^{\infty} \psi_j^2\right). \quad (3.11)$$

The proposition 3.2.1 can be proved by following derivations

$$\begin{aligned} \gamma(h) &= \text{Cov}(Y_t, Y_{t+h}) \\ &= \exp\left(\frac{\sigma^2}{2} \left\{ \sum_{j=0}^{h-1} \psi_j^2 + \sum_{j=0}^{\infty} [\psi_{h+j} + \psi_j]^2 \right\}\right) - \exp\left(\sigma^2 \sum_{j=0}^{\infty} \psi_j^2\right) \\ &= \exp\left(\sigma^2 \sum_{j=0}^{\infty} \psi_j^2\right) \exp\left(\sigma^2 \sum_{j=0}^{\infty} \psi_{h+j} \psi_j\right) - \exp\left(\sigma^2 \sum_{j=0}^{\infty} \psi_j^2\right). \end{aligned}$$

The mean function for GLGARMA $(0, d, 0)$ with Poisson distribution and PD model is

$$\mu_t = \exp\left(\sum_{j=0}^{\infty} \psi_j Z_{t-j}\right) \quad \text{and} \quad Z_t \stackrel{\text{i.i.d.}}{\sim} N(0, \sigma^2),$$

then we have

$$\begin{aligned} \mu_t &\stackrel{\text{i.i.d.}}{\sim} \text{LN}\left(0, \sigma^2 \sum_{j=0}^{\infty} \psi_j^2\right), \\ \mathbb{E}(\mu_t) &= \exp\left(\frac{\sum_{j=0}^{\infty} \psi_j^2 \sigma^2}{2}\right) \quad \text{and} \quad \mathbb{V}\text{ar}(\mu_t) = \left[\exp\left(\sum_{j=0}^{\infty} \psi_j^2 \sigma^2\right) - 1\right] \exp\left(\sum_{j=0}^{\infty} \psi_j^2 \sigma^2\right) \end{aligned}$$

where LN represents a log-normal distribution. The conditional expected value and variance of Y_t are $\mathbb{E}(Y_t|Z_t) = \mathbb{V}\text{ar}(Y_t|Z_t) = \mu_t$. The unconditional expected value and variance of Y_t are

$$\begin{aligned} \mathbb{E}(Y_t) &= \mathbb{E}[\mathbb{E}(Y_t|Z_t)] = \exp\left(\frac{\sum_{j=0}^{\infty} \psi_j^2 \sigma^2}{2}\right), \\ \mathbb{V}\text{ar}(Y_t) &= \mathbb{E}[\mathbb{V}\text{ar}(Y_t|Z_t)] + \mathbb{V}\text{ar}[\mathbb{E}(Y_t|Z_t)], \quad \text{using the law of total variance} \\ &= \mathbb{E}(\mu_t) + \mathbb{V}\text{ar}(\mu_t). \end{aligned}$$

In addition, the $\text{Cov}(Y_t, Y_{t+h}) = \mathbb{E}(Y_t Y_{t+h}) - \mathbb{E}(Y_t) \mathbb{E}(Y_{t+h})$.

$$\begin{aligned} \mathbb{E}(Y_t Y_{t+h}) &= \mathbb{E}[\mathbb{E}(Y_t Y_{t+h} | Z_t)] \\ &= \mathbb{E}(\mu_t \mu_{t+h}) \\ &= \mathbb{E} \left[\exp \left(\sum_{j=0}^{\infty} \psi_j Z_{t-j} + \sum_{k=0}^{\infty} \psi_k Z_{t+h-k} \right) \right]. \end{aligned}$$

The mean functions are

$$\mu_t = \exp(\psi_0 Z_t + \psi_1 Z_{t-1} + \psi_2 Z_{t-2} + \dots),$$

$$\mu_{t+h} = \exp(\psi_0 Z_{t+h} + \psi_1 Z_{t+h-1} + \psi_2 Z_{t+h-2} + \dots + \psi_{h-1} Z_{t+1} + \psi_h Z_t + \psi_{h+1} Z_{t-1} + \psi_{h+2} Z_{t-2} + \dots).$$

Then

$$W_t = \exp \left(\sum_{j=0}^{\infty} \psi_j Z_{t-j} + \sum_{k=0}^{\infty} \psi_k Z_{t+h-k} \right) \stackrel{\text{i.i.d}}{\sim} \text{LN} \left(0, \sigma^2 \left\{ \sum_{j=0}^{h-1} \psi_j^2 + \sum_{j=0}^{\infty} [\psi_{h+j} + \psi_j]^2 \right\} \right),$$

since

$$\begin{aligned} \log(W_t) &= \sum_{j=0}^{\infty} \psi_j Z_{t-j} + \sum_{k=0}^{\infty} \psi_k Z_{t+h-k} = \sum_{j=0}^{h-1} \psi_j Z_{t+h-j} + \sum_{j=0}^{\infty} [\psi_{h+j} + \psi_j] Z_{t-j} \\ &\stackrel{\text{i.i.d}}{\sim} \text{N} \left(0, \sigma^2 \left\{ \sum_{j=0}^{h-1} \psi_j^2 + \sum_{j=0}^{\infty} [\psi_{h+j} + \psi_j]^2 \right\} \right), \\ \mathbb{E}(Y_t Y_{t+h}) &= \exp \left(\frac{\sigma^2}{2} \left\{ \sum_{j=0}^{h-1} \psi_j^2 + \sum_{j=0}^{\infty} [\psi_{h+j} + \psi_j]^2 \right\} \right), \end{aligned}$$

and

$$\mathbb{E}(Y_t) \mathbb{E}(Y_{t+h}) = \exp \left(\frac{\sigma^2}{2} \sum_{j=0}^{\infty} \psi_j^2 \right) \exp \left(\frac{\sigma^2}{2} \sum_{j=0}^{\infty} \psi_j^2 \right) = \exp \left(\sigma^2 \sum_{j=0}^{\infty} \psi_j^2 \right).$$

Hence, we can obtain the autocovariance function (ACF) in Equation (3.11). The spectral density function of the GLGARMA-PD model can be derived by applying a Fourier transform on the ACF of Equation (3.11).

Proposition 3.2.2 (Spectral density function of GLGARMA-PD). *Given a GLGARMA-PD process $\mathbf{Y}_{t \in \{1,2,3,\dots,T\}}$, the spectral density function can be expressed as*

$$\begin{aligned} f_s(\lambda) = & \exp \left(\sigma^2 \sum_{j=0}^{\infty} \psi_j^2 \right) \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \exp \left(\sigma^2 \sum_{j=0}^{\infty} \psi_{h+j} \psi_j \right) e^{-i\lambda h} \\ & - \exp \left(\sigma^2 \sum_{j=0}^{\infty} \psi_j^2 \right) \delta_h(\lambda), \end{aligned} \quad (3.12)$$

where the delta function is defined as $\delta_h(\lambda) = \mathfrak{F}_h(1) = \int_{-\infty}^{\infty} e^{-i\lambda h} dh$.

3.3 Spectral density for non-Gaussian model

The spectral density function in Equation (3.12) could be alternatively reduced to

$$\sum_{h=-\infty}^{\infty} \exp \left(\sigma^2 \sum_{j=0}^{\infty} \psi_{h+j} \psi_j \right) e^{-i\lambda h}, \quad (3.13)$$

if the term

$$\exp \left(\sigma^2 \sum_{j=0}^{\infty} \psi_j^2 \right) \quad (3.14)$$

is bounded since the other terms in Equation (3.12) except Equation (3.13) are irrelevant to the oscillatory properties. Subsequently, we can then focus on Equation (3.13) as the modified spectral density function. We begin with proving Equation (3.14) is bounded.

3.3.1 Approximation to sum of Gegenbauer coefficients

In order to prove that $\exp \left(\sigma^2 \sum_{j=0}^{\infty} \psi_j^2 \right)$ is bounded, we need to study the properties for the product of two Gegenbauer coefficients (Askey, 1975).

Proposition 3.3.1 (Product of two Gegenbauer coefficients). *We write $\psi_j \psi_k = C_j^d(u) C_k^d(u)$ to highlight the dependency on u and d , then we have*

$$\begin{aligned}
C_j^d(u) C_k^d(u) &= \sum_{i=0}^{\min(j,k)} [(-2i+j+k+d)\Gamma(-2i+j+k+1)\Gamma(i+d) \\
&\quad \Gamma(-i+j+d)\Gamma(-i+k+d)\Gamma(-i+j+k+2d)C_{-2i+j+k}^d(u)] \\
&\quad / [\Gamma(i+1)\Gamma(-i+j+1)\Gamma(-i+k+1)(\Gamma(d))^2 \\
&\quad \Gamma(-i+j+k+d+1)\Gamma(-2i+j+k+2d)] \\
&= \sum_{i=0}^{\min(j,k)} G(i, j, k, d) C_{-2i+j+k}^d(u). \tag{3.15}
\end{aligned}$$

Remark 3.3.1. According to Equation (3.15), we can conclude that any higher order product can be decomposed into linear combination of first order terms. In addition, this product of $C_j^d(u)$ is consistent in the u , which means that the period controlled by u is constant.

Moreover, the explicit representation of $G(i, j, k, d)$ is

$$\begin{aligned}
G(i, j, k, d) &= \frac{(-2i+j+k+d)}{(\Gamma(d))^2} \cdot \frac{\Gamma(-2i+j+k+1)}{\Gamma(-2i+j+k+2d)} \cdot \frac{\Gamma(i+d)}{\Gamma(i+1)} \cdot \\
&\quad \frac{\Gamma(-i+j+d)}{\Gamma(-i+j+1)} \cdot \frac{\Gamma(-i+k+d)}{\Gamma(-i+k+1)} \cdot \frac{\Gamma(-i+j+k+2d)}{\Gamma(-i+j+k+d+1)}. \tag{3.16}
\end{aligned}$$

We simplify this representation using the Stirling's formula which states that

$$x! \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x.$$

Applying this result to one of the term in Equation (3.16) of the form

$$\begin{aligned}
\frac{\Gamma(x+1+\beta)}{\Gamma(x+1+\alpha)} &\sim \frac{\sqrt{2\pi(x+\beta)} \left(\frac{x+\beta}{e}\right)^{x+\beta}}{\sqrt{2\pi(x+\alpha)} \left(\frac{x+\alpha}{e}\right)^{x+\alpha}} \\
&= \left(1 + \frac{\gamma}{x+\alpha}\right)^{x+\alpha+1/2} \left(1 + \frac{\beta}{x}\right)^\gamma \left(\frac{x}{e}\right)^\gamma \\
&\approx e^\gamma \left(\frac{x}{e}\right)^\gamma = x^\gamma,
\end{aligned}$$

where $\gamma = \beta - \alpha$, $e^u \approx (1 + \frac{u}{n})^n$ and $1 + \frac{u}{n} \approx 1$ for large n , we get in general

$$\frac{\Gamma(x+\beta)}{\Gamma(x+\alpha)} \approx x^{\beta-\alpha}.$$

Applying the result, we obtain the following asymptotic result.

Proposition 3.3.2 (Asymptotic behaviour of $G(i, j, k, d)$ in Equation (3.16)). *For $k = j + h$, $h \in (0, \infty)$, $0 < d < 0.5$, $i = 0, 1, 2, \dots, j$, as $j \rightarrow \infty$, we have*

$$G(i, j, k, d) \leq j^{2d-2}.$$

The Proposition 3.16 can be proved by following steps

$$\begin{aligned} G(i, j, k, d) &\leq \frac{(j+k+d)}{(\Gamma(d))^2} \cdot \frac{\Gamma(j+k+1)}{\Gamma(j+k+2d)} \cdot \frac{\Gamma(j+d)}{\Gamma(j+1)} \\ &\quad \frac{\Gamma(j+d)}{\Gamma(j+1)} \cdot \frac{\Gamma(k+d)}{\Gamma(k+1)} \cdot \frac{\Gamma(j+k+2d)}{\Gamma(j+k+d+1)} \\ &\approx jj^{1-2d}j^{d-1}j^{d-1}j^{d-1}j^{d-1} \\ &\approx j^{2d-2}. \end{aligned}$$

For the computation of Equation (3.14), we investigate the asymptotic behavior by splitting the sum into two parts, one being truncated at a higher order j and the other from $j+1$ to infinity.

Proposition 3.3.3 (Asymptotic behaviour of $\exp\left(\sigma^2 \sum_{j=0}^{\infty} \psi_j^2\right)$). *Consider large n , $k = j$, $0 < d < 0.5$ and $u < |1|$, the following inequality holds*

$$\exp\left(\sigma^2 \sum_{j=0}^{\infty} \psi_j^2\right) \leq \exp\left(\sigma^2 \sum_{j=0}^n \sum_{i=0}^j G(i, j, j, d) C_{2(j-i)}^d(u) + \sigma^2 \sum_{j=n}^{\infty} j^{4d-3}\right), \text{ as } n \uparrow \infty.$$

Since the range of long memory parameter satisfies $0 < d < 0.5$, we have $-3 < 4d-3 < -1$. The series $\sum_{j=n}^{\infty} j^{4d-3}$ converges. Hence, the term $\exp\left(\sigma^2 \sum_{j=0}^{\infty} \psi_j^2\right)$ converges.

The Proposition 3.3.3 can be proved by following steps

$$\begin{aligned} \exp\left(\sigma^2 \sum_{j=0}^{\infty} \psi_j^2\right) &= \exp\left(\sigma^2 \sum_{j=0}^n \sum_{i=0}^j G(i, j, j, d) C_{2(j-i)}^d(u) + \sigma^2 \sum_{j=n}^{\infty} \sum_{i=0}^j G(i, j, j, d) C_{2(j-i)}^d(u)\right) \\ &\leq \exp\left(\sigma^2 \sum_{j=0}^n \sum_{i=0}^j G(i, j, j, d) C_{2(j-i)}^d(u) + \sigma^2 \sum_{j=n}^{\infty} \sum_{i=0}^j G(i, j, j, d) \cdot \frac{\Gamma(2j-2i+2d)}{\Gamma(2j-2i+1)\Gamma(2d)}\right) \\ &\stackrel{n \uparrow \infty}{\approx} \exp\left(\sigma^2 \sum_{j=0}^n \sum_{i=0}^j G(i, j, j, d) C_{2(j-i)}^d(u) + \sigma^2 \sum_{j=n}^{\infty} j^{2d-2} \cdot j^{2d-1}\right) \\ &\stackrel{n \uparrow \infty}{\approx} \exp\left(\sigma^2 \sum_{j=0}^n \sum_{i=0}^j G(i, j, j, d) C_{2(j-i)}^d(u) + \sigma^2 \sum_{j=n}^{\infty} j^{4d-3}\right). \end{aligned} \tag{3.17}$$

The behaviour of $G(i, j, j, d)$ across i at different truncated length $n = 500, 2500, 5000$ can be demonstrated in Figure 3.2 when $d = 0.45$. It clearly shows that $G(i, j, j, d)$ decays to zero very fast. Hence, the effect of $G(i, j, j, d)$ will get less as j tends to ∞ . Next, we study the behaviour of truncated sum $\sum_{i=0}^j G(i, j, j, d) C_{2(j-i)}^d(u)$. The black circle in Figure 3.3 shows the behaviour of $\sum_{i=0}^j G(i, j, j, d) C_{2(j-i)}^d(u)$ whereas the grey circle indicates the Gegenbauer coefficient $C_j^d(u)$ in grey colour as a comparison. The two sequences of circles show that the coefficient $G(i, j, j, d)$ oscillates in greater magnitude than the sum. The plot in the bottom of Figure 3.3 shows that the product $\sum_{i=0}^j G(i, j, j, d) C_{2(j-i)}^d(u)$ decays slowly to zero again with periodic oscillations.

Finally, we study the behaviour of truncation in Equation (3.17) which expresses the sum within the exponential function in two parts. We show that the second sum is bounded in Proposition 3.3.3. Figure 3.4 plots the exponentiation of the first sum, $\exp\left(\sigma^2 \sum_{j=0}^n \psi_j^2\right)$ is also bounded and hence Equation (3.14) is again bounded.

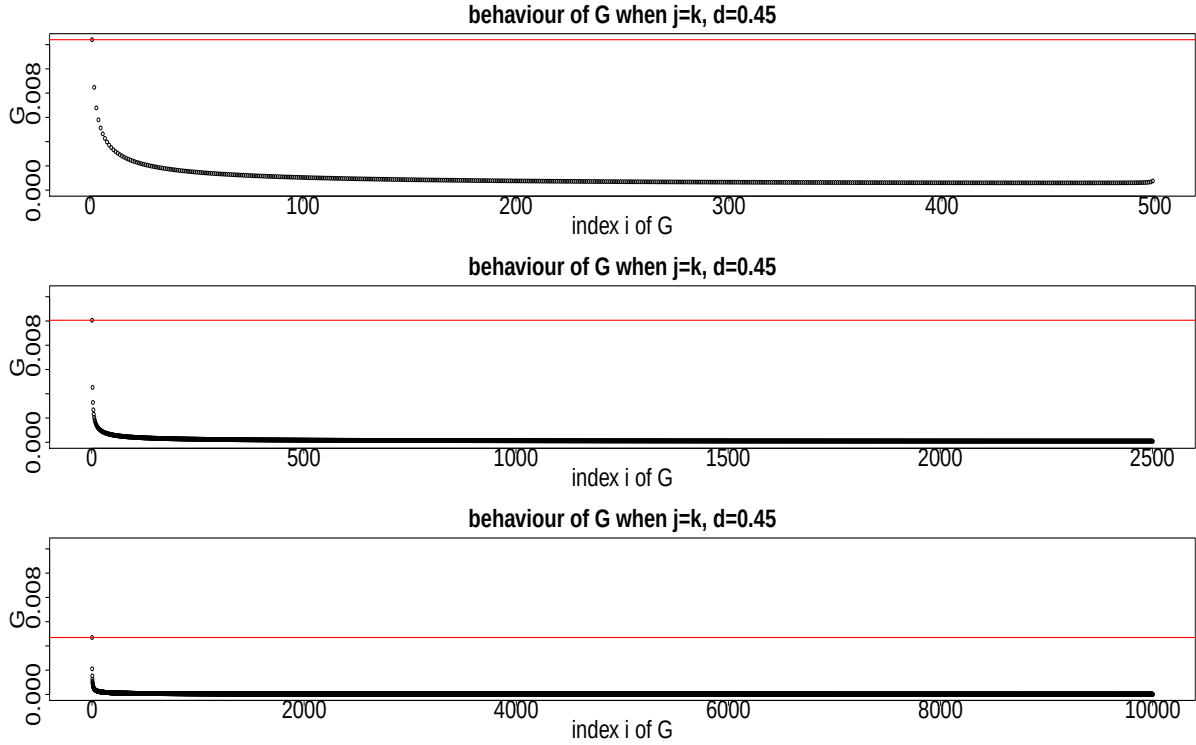


Figure 3.2. Behaviour of $G(i, j, k, d)$ with $j = k$ and different truncation $i = n = 500, 2500, 5000$ and the maximum is marked by a red horizontal line.

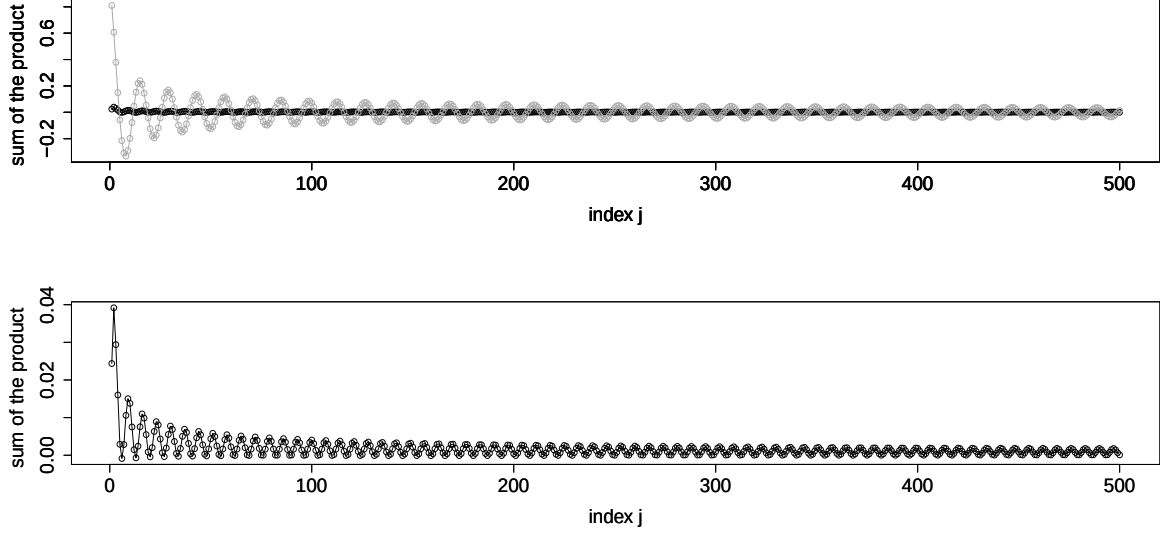


Figure 3.3. The top plot shows $\sum_{i=0}^j G(i, j, j, d)C_{2(j-i)}^d(u)$ in circle ($u = 0.9$) and $C_j^d(u)$ in grey circle; the bottom plot shows $\sum_{i=0}^j G(i, j, j, d)C_{2(j-i)}^d(u)$ with an enlarged scale.

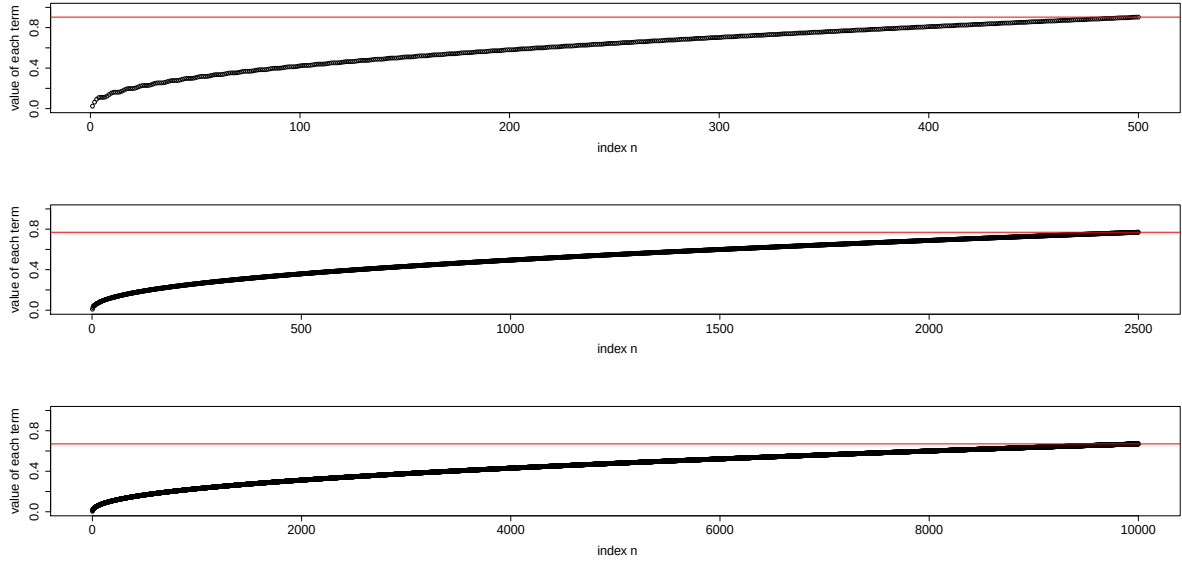


Figure 3.4. Behaviour of $\sum_{j=0}^n \sum_{i=0}^j G(i, j, k, d)C_{2(j-i)}^d(u)$ with different truncation $n = 500, 2500, 5000$ and the maximum is marked by a red horizontal line.

3.3.2 Taylor approximation to the modified spectral density function

Consider the term $\sum_{h=-\infty}^{\infty} \exp\left(\sigma^2 \sum_{j=0}^{\infty} \psi_{h+j} \psi_j\right) e^{-i\lambda h}$, the exponential function can be approximated by Taylor's expansion at centre 0

$$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots.$$

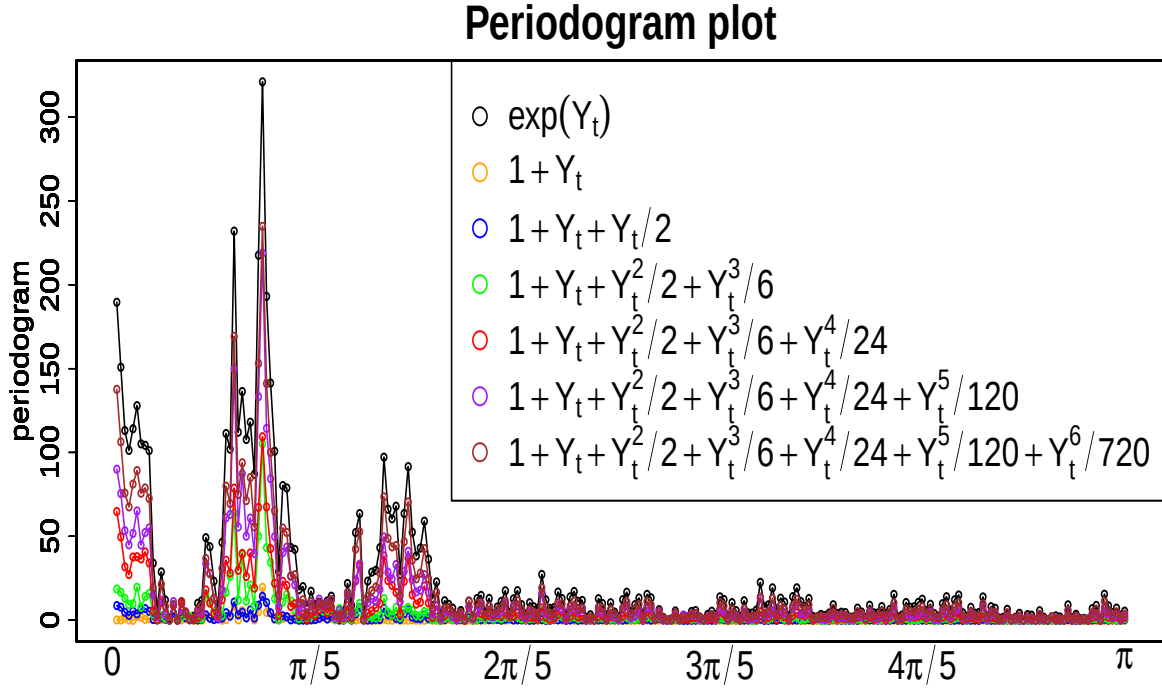


Figure 3.5. Different types of periodogram for data simulated from GARMA model with $d = 0.4$ and $u = 0.9$.

Figure 3.5 shows the periodograms of different transformations, namely, $\exp(Y_t)$, $1+Y_t$, $1+Y_t+\frac{Y_t^2}{2}$, $1+Y_t+\frac{Y_t^2}{2}+\frac{Y_t^3}{6}$, $1+Y_t+\frac{Y_t^2}{2}+\frac{Y_t^3}{6}+\frac{Y_t^4}{24}$, $1+Y_t+\frac{Y_t^2}{2}+\frac{Y_t^3}{6}+\frac{Y_t^4}{24}+\frac{Y_t^5}{120}$, $1+Y_t+\frac{Y_t^2}{2}+\frac{Y_t^3}{6}+\frac{Y_t^4}{24}+\frac{Y_t^5}{120}+\frac{Y_t^6}{720}$ for data simulated from GARMA model with $d = 0.4$ and $u = 0.9$. It is clear that transformations involving higher order terms provide periodogram which approximates better the periodogram of $\exp(Y_t)$. Hence,

we expand $\exp\left(\sigma^2 \sum_{j=0}^{\infty} \psi_{h+j} \psi_j\right)$ into

$$\begin{aligned} \exp\left(\sigma^2 \sum_{j=0}^{\infty} \psi_{h+j} \psi_j\right) &= 1 + \sigma^2 \sum_{j=0}^{\infty} \psi_{h+j} \psi_j + \frac{\left(\sigma^2 \sum_{j=0}^{\infty} \psi_{h+j} \psi_j\right)^2}{2} \\ &\quad + \frac{\left(\sigma^2 \sum_{j=0}^{\infty} \psi_{h+j} \psi_j\right)^3}{6} + \frac{\left(\sigma^2 \sum_{j=0}^{\infty} \psi_{h+j} \psi_j\right)^4}{24} \\ &\quad + \frac{\left(\sigma^2 \sum_{j=0}^{\infty} \psi_{h+j} \psi_j\right)^5}{120} + \frac{\left(\sigma^2 \sum_{j=0}^{\infty} \psi_{h+j} \psi_j\right)^6}{720} + \dots \quad (3.18) \end{aligned}$$

Figure 3.6 describes the different truncation types of Equation (3.18) with different h values. From the figure, we conclude that we should keep at least the first six order terms to obtain a good approximation, that is, $e^x \approx 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$. Hence, the sum of approximated terms to Equation (3.18) can be represented as

$$\begin{aligned} \sum_{h=-\infty}^{\infty} \exp\left(\sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}\right) &\approx \sum_{h=-\infty}^{\infty} \left[1 + \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} + \frac{\sigma^4}{2} \left(\sum_{j=0}^{\infty} \psi_j \psi_{j+h}\right)^2 \right. \\ &\quad + \frac{\sigma^6}{6} \left(\sum_{j=0}^{\infty} \psi_j \psi_{j+h}\right)^3 + \frac{\sigma^8}{24} \left(\sum_{j=0}^{\infty} \psi_j \psi_{j+h}\right)^4 \\ &\quad \left. + \frac{\sigma^{10}}{120} \left(\sum_{j=0}^{\infty} \psi_j \psi_{j+h}\right)^5 + \frac{\sigma^{12}}{720} \left(\sum_{j=0}^{\infty} \psi_j \psi_{j+h}\right)^6 \right] \quad (3.19) \end{aligned}$$

To visualise the accuracy of the approximation, we plot in Figure 3.6 the following cumulative sums

$$\mathfrak{E}^K(h) = \sum_{k=0}^K \mathfrak{S}(h, k), \quad \text{where } \mathfrak{S}(h, k) = \frac{\sigma^{2k} \left(\sum_{j=0}^n \psi_{h+j} \psi_j\right)^k}{k!} \quad \text{and } \mathfrak{S}(h, 0) = 1 \quad (3.20)$$

as well as

$$\mathfrak{E}(h) = \mathfrak{E}^{\infty}(h) = \exp\left(\sigma^2 \sum_{j=0}^n \psi_{h+j} \psi_j\right). \quad (3.21)$$

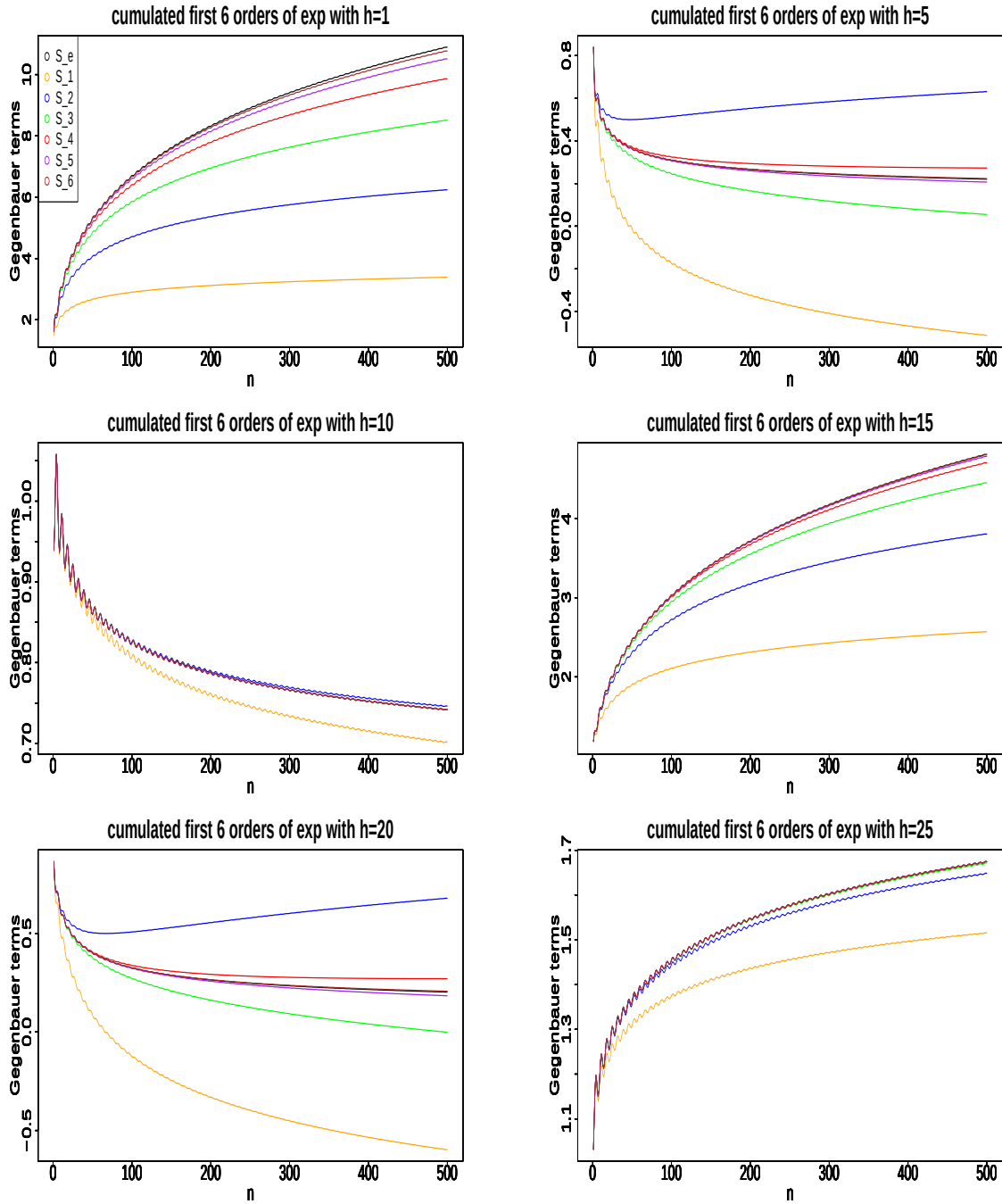


Figure 3.6. We denote exponentiated sum $S_e = \mathfrak{C}(h)$ and cumulated sum $S_K = \mathfrak{C}^K(h)$ of higher order terms against the number of terms n in the sum for six levels of lag h and six order K .

3.3.3 Linear transformation to higher order sum

Furthermore, Equation (3.19) can be rewritten into a linear representation and then the parameters can be estimated using the idea of tensor cubic spline. Furthermore, according to Remark 3.3.1 that products of two Gengenbauer coefficients G can be expressed as a linear function of G . We can assume that there exist a linear relationship between higher order of Ψ^n and first order Ψ , that is, $\Psi^k = a_k(d, u, h) + b_k(d, u, h)\Psi$ where $\Psi = \sum_{j=0}^{\infty} \psi_j \psi_{j+h}$. Hence Equation (3.19) can be written as.

$$\begin{aligned}
& \sum_{h=-\infty}^{\infty} \exp \left(\sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} \right) \\
& \approx \sum_{h=-\infty}^{\infty} \left[1 + \sum_{k=1}^6 \frac{\sigma^{2k}}{k!} \left(\sum_{j=0}^{\infty} \psi_j \psi_{j+h} \right)^k \right] \\
& \approx \sum_{h=-\infty}^{\infty} \left[1 + \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} + \sum_{k=2}^6 \frac{\sigma^{2k}}{k!} \left(\mathcal{A}_k(d, u, h) + \mathcal{B}_k(d, u, h) \sum_{j=0}^{\infty} \psi_j \psi_{j+h} \right) \right]. \quad (3.22)
\end{aligned}$$

Hence, the modified spectral density function in Equation (3.13) can be written as

$$\begin{aligned}
& \sum_{h=-\infty}^{\infty} \exp \left(\sigma^2 \sum_{j=0}^{\infty} \psi_{h+j} \psi_j \right) e^{-i\lambda h} \\
& \approx \sum_{h=-\infty}^{\infty} \left[e^{-i\lambda h} + \sigma^2 \left(\sum_{j=0}^{\infty} \psi_{j+h} e^{-i\lambda(j+h)} \psi_j e^{i\lambda j} \right) + \right. \\
& \quad \left. \sum_{k=2}^6 \frac{\sigma^{2k}}{k!} \left(\mathcal{A}_k(d, u, h) e^{-i\lambda h} + \mathcal{B}_k(d, u, h) \sum_{j=0}^{\infty} \psi_{j+h} e^{-i\lambda(j+h)} \psi_j e^{i\lambda j} \right) \right].
\end{aligned}$$

3.4 The tensor cubic spline approximation

In order to solve the coefficient functions

$$\Psi^k(d, u, h) = \mathcal{A}_k(d, u, h) + \mathcal{B}_k(d, u, h) \times \Psi(d, u, h) \quad \text{where} \quad \Psi(d, u, h) = \sum_{j=0}^{\infty} \psi_j \psi_{j+h},$$

we use the tensor cubic spline in which a cubic spline is defined as

$$B(j) = \sum_{i=0}^I \left(a_i^{(0)} + a_i^{(1)}(j - \tau_i) + a_i^{(2)}(j - \tau_i)^2 + a_i^{(3)}(j - \tau_i)^3 \right) \cdot \mathbf{1}_{j \in [\tau_i, \tau_{i+1}]}$$

Hence the basis function of tensor cubic spline function for $\mathcal{A}_k(d, u, h)$ and $\mathcal{B}_k(d, u, h)$ are defined as

$$\begin{aligned} \mathcal{A}_k(d, u, h) &= \sum_{i_d=0}^{I_d^{(a)}} \sum_{i_u=0}^{I_u^{(a)}} \sum_{i_h=0}^{I_h^{(a)}} \omega_{i_d, i_u, i_h}^{(a)} \times \\ &\quad \left(a_{i_d}^{(d0)} + a_{i_d}^{(d1)}(d - \tau_{i_d}^{(da)}) + a_{i_d}^{(d2)}(d - \tau_{i_d}^{(da)})^2 + a_{i_d}^{(d3)}(d - \tau_{i_d}^{(da)})^3 \right) \cdot \mathbf{1}_{d \in [\tau_{i_d}^{(da)}, \tau_{i_d+1}^{(da)}]} \\ &\quad \left(a_{i_u}^{(u0)} + a_{i_u}^{(u1)}(u - \tau_{i_u}^{(ua)}) + a_{i_u}^{(u2)}(u - \tau_{i_u}^{(ua)})^2 + a_{i_u}^{(u3)}(u - \tau_{i_u}^{(ua)})^3 \right) \cdot \mathbf{1}_{u \in [\tau_{i_u}^{(ua)}, \tau_{i_u+1}^{(ua)}]} \\ &\quad \left(a_{i_h}^{(h0)} + a_{i_h}^{(h1)}(h - \tau_{i_h}^{(ha)}) + a_{i_h}^{(h2)}(h - \tau_{i_h}^{(ha)})^2 + a_{i_h}^{(h3)}(h - \tau_{i_h}^{(ha)})^3 \right) \cdot \mathbf{1}_{h \in [\tau_{i_h}^{(ha)}, \tau_{i_h+1}^{(ha)}]}, \\ \mathcal{B}_k(d, u, h) &= \sum_{i_d=0}^{I_d^{(b)}} \sum_{i_u=0}^{I_u^{(b)}} \sum_{i_h=0}^{I_h^{(b)}} \omega_{i_d, i_u, i_h}^{(b)} \times \\ &\quad \left(b_{i_d}^{(d0)} + b_{i_d}^{(d1)}(d - \tau_{i_d}^{(db)}) + b_{i_d}^{(d2)}(d - \tau_{i_d}^{(db)})^2 + b_{i_d}^{(d3)}(d - \tau_{i_d}^{(db)})^3 \right) \cdot \mathbf{1}_{d \in [\tau_{i_d}^{(db)}, \tau_{i_d+1}^{(db)}]} \\ &\quad \left(b_{i_u}^{(u0)} + b_{i_u}^{(u1)}(u - \tau_{i_u}^{(ub)}) + b_{i_u}^{(u2)}(u - \tau_{i_u}^{(ub)})^2 + b_{i_u}^{(u3)}(u - \tau_{i_u}^{(ub)})^3 \right) \cdot \mathbf{1}_{u \in [\tau_{i_u}^{(ub)}, \tau_{i_u+1}^{(ub)}]} \\ &\quad \left(b_{i_h}^{(h0)} + b_{i_h}^{(h1)}(h - \tau_{i_h}^{(hb)}) + b_{i_h}^{(h2)}(h - \tau_{i_h}^{(hb)})^2 + b_{i_h}^{(h3)}(h - \tau_{i_h}^{(hb)})^3 \right) \cdot \mathbf{1}_{h \in [\tau_{i_h}^{(hb)}, \tau_{i_h+1}^{(hb)}]}, \end{aligned}$$

where $\omega_{i_d, i_u, i_h}^{(m)}$, $m \in \{a, b\}$; $i_j = 1, \dots, I_j$; $j \in \{d, u, h\}$ are the weight parameters, $a_{i_j}^{(jl)}$, $i_j = 1, \dots, I_j$; $j \in \{d, u, h\}$; $k = 1, \dots, 6$; $l = 0, \dots, 3$ are the coefficients of the cubic spline and $\tau_{i_j}^{(j)}$, $i = 1, \dots, I_j$; $j \in \{d, u, h\}$ are the knot points. The tensor cubic spline can be implemented using R package, such as **GAM**.

3.5 Conclusion

This chapter studies the spectral density and Whittle likelihood estimation method for Gaussian and non-Gaussian models. For Gaussian model, it provides estimator of d using local periodogram and local Whittle likelihood. It also reports the asymptotic normality of Whittle likelihood estimator. Then it derives the spectral density for the

GLGARMA-PD model. The derivation is challenging because of the log link function which is a non-linear transformation. We approximate the product of two Gegenbauer coefficients and prove the convergence of a certain sum through which we provide a modified spectral density function. By approximating the exponential function from the log link function with the Taylor expansion up to six order and linearising the power terms using tensor cubic spline, we are able to provide an approximated spectral density function. This theoretical procedures will be testify using simulated and real data in future research.

CHAPTER 4

Analysis of CFTC data - measuring long memory

Chapter 2 and 3 propose the generalised linear Gegenbauer autoregressive moving average (GLGARMA) model and introduce the properties in both time and frequency domains. This chapter and the next chapter apply the model to study open interest series from Commodity Futures Trading Commission (CFTC) database. We first state clearly the importance of analysing this data. Then we describe the structure of this data in details. The analysis begins in this chapter with the identification of long memory using Hurst exponent which is a non-parametric approach to estimate long memory strength. Many estimators of long-range dependence have been proposed in the literature. In particular, we take advantages of these different estimators to show that they all can detect long memory structure in data. We provide overwhelming statistical evidence via three estimators of long memory: rescaled range analysis, detrended fluctuation analysis and periodogram regression methods. To measure the strength of long memory, we estimate H using all three estimators presented and we only report the results of the Rescaled Range R/S estimator since the other two estimators provide very similar results.

4.1 Financial reviews

From the financial perspective, it is essential to understand the dynamics of open interest of futures contracts in commodity markets. The open interest on futures contracts is the number of futures contracts that are not matured, i.e. expired with the physical quantity delivered, and held by traders and investors. These are categorised according to long and short positions, which correspond to the purchase and sales of such futures contracts respectively. Summaries of such open positions are reported weekly by the Commodity Futures Trading Commission (CFTC) by major market participant types. One can think of open interest as a less reactive measure of liquidity than other popular measures such as traded volume. In effect the trading volume, which is the number of futures contracts exchanged between buyers and sellers, is a direct measure of the activity of futures contracts. However, the open interest that we study is generally viewed as a measure of the flow of money in and out of the futures market. Another perspective on these two measures is that whilst volume measures the number of contracts that exchanged hands during the trading session, thereby capturing directly market activity. Instead, open interest is the total number of outstanding contracts and as such it can be used to understand market participation. In addition to providing information on monetary flows, one can also use future forecasts of open interest as an indicator of momentum in futures markets. We have several aims in analysing CFTC data.

Firstly, we are particularly interested in time variation of market participation by participant type on a weekly time scale historically over-time. It is valuable in practice to be able to forecast changes in market participant behaviours for a range of financial modelling applications. We also note that since open interest tends to be less stochastically volatile than traded volume, we believe that forecasting of the open interest time series for market participants can be achieved very accurately, when incorporating appropriate model structures such as long-memory structures, as demonstrated in the manuscript. We believe the study of such open interest by market participant type in the U.S. treasury bond futures is novel. To date there have been very few studies undertaken to explore liquidity dynamics, as captured through speculative open interest on long, short and net positions of futures markets. Therefore, we aim to fill the gap in

the literature by applying forecasting models to study the dynamics of such monetary flows in the US treasury futures market.

Secondly, our analysis allows us to assess if one can characterize market participant behaviours and show that these behaviours form very systematic behavioural patterns from a statistical perspective, demonstrated according to particular long memory structures in their serial correlation and persistence in their trading activities. To understand what we mean by market participant type, we note that at a basic level, it is common to classify traders into two basic categories: informed traders and uninformed traders. Within these categories, there have been numerous studies on how information is absorbed by such trader types and its follow on effects to aspects of liquidity and price volatility. A range of models based on liquidity and traded volumes are available to attempt to explain the relations between volatility and traded volume by traders with divergent beliefs and asymmetric information ((Shalen, 1993) and (Harris and Raviv, 1993)). Daigler and Wiley (1999) studied volatility-volume relationships in futures markets using volume data categorized by type of traders. However, to date there has been little study of the net cash flow dynamics into such markets as captured by open interest dynamics and how this is influence by individual market participant type.

Furthermore, the selection of two types of U.S. security in this study is undertaken purposefully for the following reasons. In Amihud and Mendelson (1991) it was observed that the maturity can influence liquidity, for instance short term instruments such as U.S. treasury bills (T-bills), intermediate maturity instruments such as U.S. T-notes and long term maturity instruments such as U.S. T-bonds have different market dynamics and liquidity profiles since it is common in practice for traders to specialise solely in one type of these government securities, leading to differences in trading behaviours in each market. We therefore, select intermediate and long term instruments to study the differences in behaviour between market participants. We extend these prior conjectures and studies in a detailed manner by studying individual market participants in both markets, which allows us to compare differences and commonalities in market participant behaviours in short and long term debt markets, disaggregated at a very fine level of granularity by market participant type, including their proclivity to act in certain systematic manners in long or short positions over time and their propensity to maintain such behaviours in the long term.

4.2 Data description

The data consist of 136 individual U.S. Commodity Futures Trading Commission (CFTC) time series of counts which are obtained from U.S. CFTC which releases weekly reports for Futures-Only Commitments of Traders and for Futures-and-Options-Combined Commitments of Traders on every Friday at 3:30 p.m. eastern time. The reports contain details of every Tuesday's open interest.

The data covers the period from 13 June 2006 to 20 January 2015 resulting in 450 weekly observations. It contains open interest for *futures-only* and *futures-and-options* crossed classified with four fixed income securities provided by the U.S. federal government: three *T-notes* and one *T-bond* which are issued with full faith and credit of the U.S. government. The T-notes are usually offered with 2, 5, or 10-year terms whereas the T-bonds with longer maturity are offered to investors with either a 20 or 30-year term. Because of the much shorter investment period for T-notes, their interest rates are lower and fluctuate less than T-bonds. U.S. T-notes and T-bills are distinct fixed-income securities. T-bills are short-term discount bonds whereas T-notes are coupon-bearing bonds with far longer maturities. Within 6 months to maturity, however, T-notes become short-term single-payment securities like T-bills.

Apart from the length of terms, the data are further divided into *reportables* and *non-reportables*. Under the reportables, there are four different categories of traders, namely, *asset managers*, *dealers*, *leveraged funds* and *other reportables*. These trader types can be understood as follows: Asset managers are institutional investors including pension funds, endowments, insurance companies, mutual funds and those portfolio/investment managers whose clients are predominantly institutional; Dealers are traders in the sell side of the market who use futures markets to manage the risk of commodity swap transactions; Leveraged funds usually refer to registered commodity trading advisors (CTA), registered commodity pool operators (CPO) or unregistered fund identified by CFTC. They manage organised futures trading on behalf of clients. Any traders who are not classified in these three categories belong to other reportables. On the other hand, non-reportables refer to trades such that the commercial/non-commercial classification of traders are unknown.

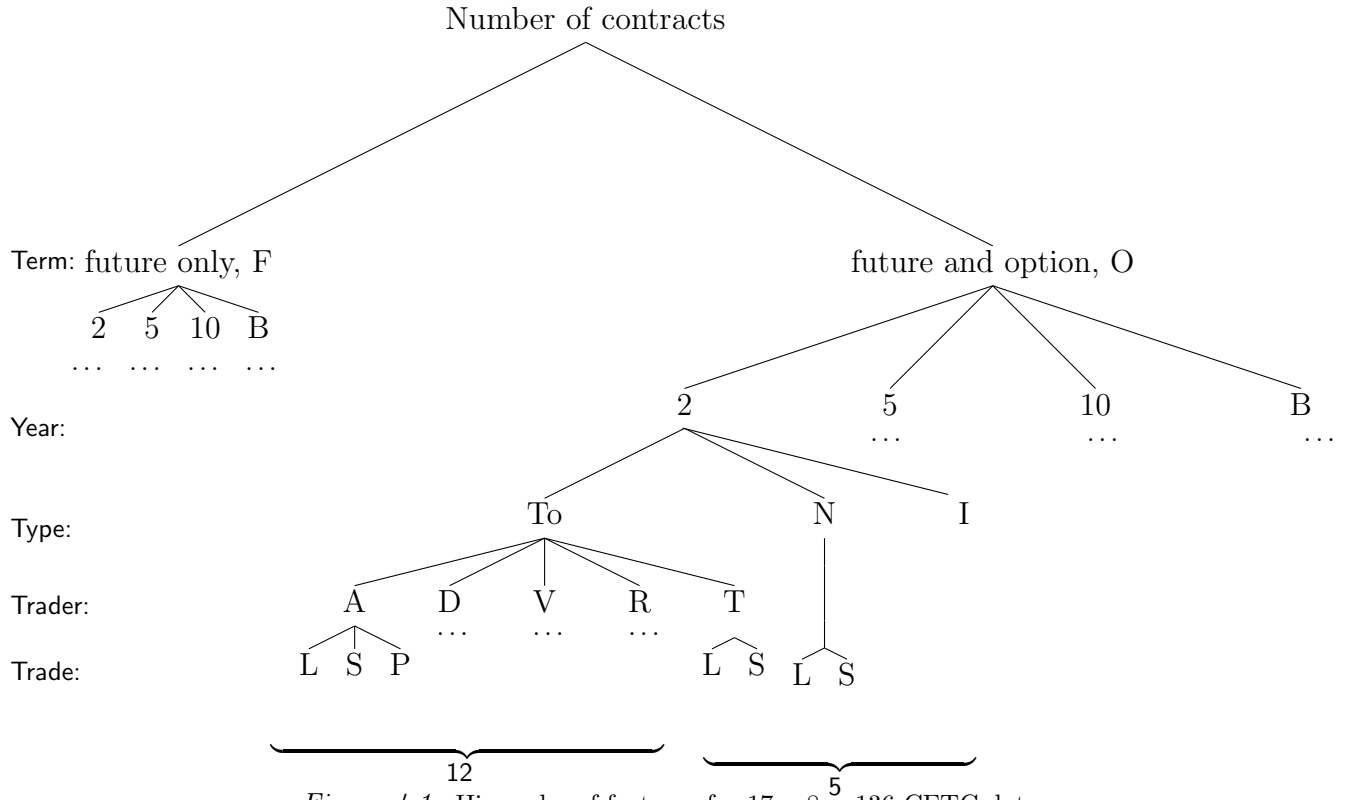
Under both types of reportables and non-reportables, there are trades of *long*-call, *short*-put and *spread* being the difference between long-call and short-put. The long and short open interest for non-reportables are derived by subtracting total long and short reportables from *open interest all* being the sum of reportables and non-reportables. Accordingly, open interest all is the total of all futures and/or option contracts entered into and not yet offset by a transaction, by delivery, by exercise, etc. Long-call and short-put open interest are converted to long futures-equivalent open interest. For the futures-only report, spread measures the extent to which each non-commercial trader holds equal long and short futures positions.

Table 4.1 summaries the abbreviation for different names that describe the characteristics of each data set. For example, the 5-year U.S. T-notes dealer spread future only will be abbreviated according to the short-hand given by F-5-D-P. Figure 4.1 summarizes the hierarchy of different characteristics for the totally 136 CFTC data sets analysed in this study and reports the number of data sets under each sub-category. In addition, the units of A-L, A-P, D-S, D-P, V-P, R-S and R-P data are ten thousand whereas the units of A-S, D-L, V-L, V-S, N-L, N-S, I, R-L, T-L and T-S data are thousand.

Table 4.1. Abbreviation for data names. (Column No. refers to the number of data sets in each sub-category classification of the data.)

Group: Full name			Group: Full name		
Abbrev. No.			Abbrev. No.		
Term:	future only	F 68	Trader:	Asset manager	A‡ 24
	future and option	O 68		Dealer	D‡ 24
Year:	2-year Treasury notes	2 34		Leveraged funds	V‡ 24
	5-year Treasury notes	5 34		Other reportables	R‡ 24
	10-year Treasury notes	10 34		Sum (A to R)	T‡ 16
	Treasury bonds	B 34	Trade:	long	L† 48
Type:	Total reportables	*To 112		short	S† 48
	Non-reportables	N‡ 16		spread	P† 32
	Open interest all (sum T & N)	I†‡ 8			

Groups with † and ‡ each contain 136 data sets totally. *To contains A to T.



4.3 Nonparameter measures of long memory

To motivate the application of the proposed long memory count time series models to the CFTC data, we first demonstrate empirical evidence for long memory features in the data. We achieve this with a simple analysis of long term persistence for each open interest series cross-classified according to the trading activities. To capture this persistence, we apply a Hurst exponent H estimation procedure introduced by [Hurst \(1951\)](#) to summarise the long memory features in CFTC data. [Davidson and De Jong \(2000\)](#) further demonstrated that the relationship between d and H is given by a simple equation $d = H - 0.5$. Consequently, the Hurst exponent H can also be adopted to measure the strength of long memory in a given time series.

4.3.1 Hurst exponent

The Hurst exponent H also known as the index of long-range dependence was proposed by [Hurst \(1951\)](#). Following this, [Mandelbrot and Van Ness \(1968\)](#) defined fractional Brownian motion

$$W_H = \{W_H(t) : 0 \leq t < \infty, W_H(t) \in \mathbb{R}\},$$

by its stochastic representation given below.

Definition 4.3.1 (fractional Brownian motion). *Let $W_H(t)$ be a random process defined as the solution to following diffusion*

$$W_H(t) := \frac{1}{\Gamma(H + \frac{1}{2})} \left[\int_{-\infty}^0 \left((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right) dW(s) + \int_0^t (t-s)^{H-\frac{1}{2}} dW(s) \right],$$

where the Gamma function $\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} \exp(-x) dx$, $W(t)$ denotes standard Brownian motion and $H \in (0, 1)$.

In addition, $W_H(t)$ reduces to a standard Brownian motion $W(t)$ when $H = 0.5$. A fractional Brownian motion $W_H(t)$ is Gaussian distributed with stationary increments. Given the initial value $W_H(0) = 0$ at $t = 0$, the expected value and variance are given by

$$\mathbb{E}(W_H(t)) = 0 \quad \text{and} \quad \mathbb{V}\text{ar}(W_H^2(t)) = t^{2H}, \quad \text{for } t > 0,$$

respectively. The covariance function derived by [Nualart \(2006\)](#) is given by

$$\text{Cov}_H(t, s) = \mathbb{E}[W_H(s)W_H(t)] = \frac{1}{2} (t^{2H} + s^{2H} - (t-s)^{2H}), \quad \text{for } 0 < s \leq t.$$

An increment of the process in an interval $[s, t]$ has a normal distribution with zero mean and variance

$$\mathbb{E}[(W_H(t) - W_H(s))^2] = |t - s|^{2H}.$$

For $H = 1/2$, the covariance $\text{Cov}_{1/2}(t, s) = \min(s, t)$ and increments of the process in disjoint intervals are independent. For $H \neq 1/2$, increments are not independent. Consider a unit increment discrete time process $\{\Xi_n := W_H(n) - W_H(n-1), n \geq 1\}$, which gives a Gaussian stationary sequence with unit variance and autocorrelation function (ACF)

$$\begin{aligned} \rho_H(n) &= \frac{1}{2} [(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}] \\ &\approx H(2H-1)n^{2H-2} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

For $\frac{1}{2} < H < 1$ and large enough n , we have $\rho_H(n) > 0$ and $\sum_{n=1}^{\infty} \rho_H(n) \rightarrow \infty$. For $0 < H < \frac{1}{2}$ and large enough n , we have $\rho_H(n) < 0$ and $\sum_{n=1}^{\infty} |\rho_H(n)| < \infty$. Hence, a value of H in the range $(\frac{1}{2}, 1)$ indicates long memory in a time series, meaning that a high value in the series will more likely be followed by another high value and such an effect is likely to continue a long time into the future. A value in the range $(0, \frac{1}{2})$ indicates a time series which is more likely to switch between high and low values in adjacent pairs, and such anti persistence will last a long time into the future. This process can be used to model sequences with intermittency. A value of $H = \frac{1}{2}$ can indicate a standard Brownian motion which is a short memory process.

4.3.2 Relationship between long memory d and Hurst exponent H

This section explains the relationship between H and d . To understand this relationship, one can show a weak convergence result for the discrete time fractional time series and its convergence to a fractional Brownian motion. To achieve this, we define a fractional difference characterizing process

$$Q_t = (1 - B)^{-d} \varepsilon_t, \quad \varepsilon_t \stackrel{\text{i.i.d}}{\sim} N(0, 1), \quad |d| < 1/2,$$

then the Q_t is a stationary and weakly dependent process. Now consider the following scaled partial sum constructed from this process,

$$Z_T(\xi) := \frac{\sum_{t=1}^{[T\xi]} Q_t}{\sigma_T}, \quad 0 \leq \xi \leq 1,$$

where $\sigma_T^2 = \mathbb{E} \left[\left(\sum_{t=1}^T Q_t \right)^2 \right]$ and $[\cdot]$ denotes integer part. Using this partial sum, [Davidson and De Jong \(2000\)](#) showed the following theorem

Theorem 4.3.1. *The following weak convergence holds for all $|d| < 1/2$ and all $0 \leq \xi \leq 1$,*

$$Z_T(\xi) \xrightarrow{d} V_d^{-\frac{1}{2}} W_d(\xi), \quad \text{as } T \rightarrow \infty,$$

with the scale constant

$$V_d = \frac{1}{\Gamma(d+1)^2} \left[\frac{1}{2d+1} + \int_0^\infty ((1+\tau)^d - \tau^d)^2 d\tau \right],$$

chosen to ensure that $\mathbb{E}[W_d(1)^2] = 1$. For $|d| < 1/2$, $W_d(\xi)$ is a fractional Brownian motion with the following representation

$$W_d(\xi) = \frac{1}{\Gamma(d+1)} \left(\int_{-\infty}^0 ((\xi-s)^d - (-s)^d) dW(s) + \int_0^\xi (\xi-s)^d dW(s) \right),$$

where W is the standard Brownian motion.

The relationship between d and H is then given by $d = H - 0.5$. As a consequence, one can use existing estimators of the Hurst exponent H to also estimate the long memory parameter d . In the next section we will present three examples of such estimators.

4.3.3 Estimator of Hurst exponent

4.3.3.1 Rescaled Range Analysis

To measure the intensity of long-range dependence in a time series, the first Hurst exponent estimator was developed by [Hurst \(1951\)](#) in the rescaled range R/S analysis. Next, we give a clear definition of H . H is an effective measure for the level of persistence in a time series and is defined according to the equation

$$cn^H = \frac{R_n}{S_n},$$

where R_n is the range of a time series of length n , S_n is the standard deviation and c is a constant. Under the co-called *simplified R over S* approach, we can obtain an estimator for H via log transformation and least squares estimation ([Rasheed and Qian, 2004](#)). Given a time series $\mathbf{Y}_{t \in \{1,2,3,\dots,T\}}$, the sample mean and the standard deviation process are given by

$$\bar{Y}_T = \frac{1}{T} \sum_{j=1}^T Y_j \quad \text{and} \quad S_t = \sqrt{\frac{1}{t-1} \sum_{j=1}^t (X_j)^2}, \quad (4.1)$$

where the mean adjusted series $X_t = Y_t - \bar{Y}_T$. Then a cumulative sum series is given by $Z_t = \sum_{j=1}^t X_j$ and the cumulative range based on these sums is

$$R_t = \text{Max}(0, Z_1, \dots, Z_t) - \text{Min}(0, Z_1, \dots, Z_t). \quad (4.2)$$

The following proposition proposes an estimator of H as derived in [Mandelbrot \(1975\)](#).

Proposition 4.3.1. *Consider a time series $Y_t \in \mathbb{R}$ and define S_t and R_t in Equations (4.1) and (4.2) respectively, then $\exists C \in \mathbb{R}$ such that following asymptotic property of the rescaled range R/S holds*

$$[R/S](T) = \frac{1}{T} \sum_{t=1}^T R_t/S_t \sim CT^H, \text{ as } T \rightarrow \infty.$$

In addition, for small sample size T , the rescaled range R/S can also be approximated by following equation (Annis and Lloyd, 1976)

$$[R/S](T) = \begin{cases} \frac{T^{-1/2} \Gamma((T-1)/2)}{\sqrt{\pi(T/2)}} \sum_{j=1}^{T-1} \sqrt{\frac{T-j}{j}}, & \text{for } T \leq 340, \\ \frac{T^{-1/2}}{\sqrt{T\pi/2}} \sum_{j=1}^{T-1} \sqrt{\frac{T-j}{j}}, & \text{for } T > 340, \end{cases}$$

where the $\frac{T-1/2}{T}$ term was added by Peters (1994). The H estimate can be obtained by a simple linear regression

$$\log[R/S](T) = \log C + H \log T.$$

Hence, we have the following definition for the estimator of H .

Definition 4.3.2 (Estimator $\hat{H}$ by R/S). *The estimator $\hat{H}$ based on the rescaled range R/S analysis is given by*

$$\hat{H}_{R/S} = \frac{T(\sum_{t=1}^T \log[R/S](t) \log t) - (\sum_{t=1}^T \log[R/S](t))(\sum_{t=1}^T \log t)}{T(\sum_{t=1}^T (\log t)^2) - (\sum_{t=1}^T \log t)^2}. \quad (4.3)$$

The empirical confidence interval of $\hat{H}$ given in Equation (4.3) with sample length $T = 2^N$ (Weron, 2002) is

$$(0.5 - \exp[-7.33 \log(\log N) + 4.21], \exp[-7.20 \log(\log N) + 4.04] + 0.5),$$

In this thesis, we apply the R package called `pracma` to estimate the value of H adopting the R/S analysis.

Remark 4.3.1. *A drawback of the R/S analysis is the fact that no asymptotic distribution theory has been derived for the estimated Hurst parameter H but we can apply bootstrap methods to find related properties to test for statistical significance of the estimates in order to detect the long memory properties.*

4.3.3.2 Detrended Fluctuation Analysis

Detrended Fluctuation Analysis (DFA) proposed by [Peters \(1994\)](#) is a powerful tool to test the presence of long memory in a time series and to quantify such long-range dependence. As implied by its name, it was conceived as a method for detrending local variability in a time series which will then provide insight into long-term variations in the data sets. A time series $\mathbf{Y}_{t \in \{1,2,3,\dots,T\}}$ is divided into K non-overlapped subseries $Y_{l,k}$ of length L for $l = 1, 2, \dots, L$ and $k = 1, 2, \dots, K$. Given a new cumulative time series $Z_{j,k} = \sum_{l=1}^j Y_{l,k}$ with $j = 1, \dots, L$, the sample average of the root mean square fluctuation for all K subseries of length L is obtained by

$$\bar{F}(L) := \frac{1}{K} \sum_{k=1}^K \sqrt{\frac{1}{L} \sum_{j=1}^L (Z_{j,k} - a_k j - b_k)^2}, \quad (4.4)$$

where $a_k j + b_k$ is the least-squares line $\tilde{Z}_k(j) = a_k j + b_k$ fitted to points $(j, Z_{j,k}), j = 1, \dots, L$. The following proposition proved by [Taqqu et al. \(1995\)](#) has been used to estimate H .

Proposition 4.3.2. *Consider a time series Y_t and $\bar{F}(L)$ defined in Equation (4.4), $\exists C \in \mathbb{R}$ such that the following asymptotic property for $\bar{F}(L)$ holds:*

$$\bar{F}(L) \sim CL^H, \quad \text{as } L \rightarrow \infty.$$

As a consequence of this asymptotic result, the estimated value of H can be obtained by running a linear regression

$$\log(\bar{F}(L)) = \log(C) + H \log(L),$$

and this leads to an alternative statistical estimator for H .

Definition 4.3.3 (Estimator $\hat{H}$ by DFA). *The estimator $\hat{H}$ developed by adopting DFA approach is given by*

$$\hat{H}_{DFA} = \frac{L(\sum_{l=1}^L \log \bar{F}(l) \log l) - (\sum_{l=1}^L \log \bar{F}(l))(\sum_{l=1}^L \log l)}{L(\sum_{l=1}^L (\log l)^2) - (\sum_{l=1}^L \log l)^2}. \quad (4.5)$$

The empirical confidence interval of $\hat{H}$ given in Equation (4.5) with sample length $T = 2^N$ for $N \in [10, 50]$ ([Weron, 2002](#)) is

$$(0.5 - \exp(-2.33 \log N + 3.25), \exp(-2.46 \log N + 3.38) + 0.5),$$

and the empirical confidence interval for $N > 50$ is

$$(0.5 - \exp(-2.93 \log N + 4.45), \exp(-3.10 \log N + 4.77) + 0.5).$$

Remark 4.3.2. *There is no asymptotic distribution theory derived for the DFA statistics. Hence, explicit hypothesis testing can not be performed however again one can apply bootstrap procedures.*

4.3.3.3 Periodogram regression

The spectral density function of a general fractionally integrated model with parameter d is identical to that of a fractional Gaussian noise with Hurst exponent $H = d + 0.5$. Hence, periodogram regression was proposed by [Geweke and Porter-Hudak \(1983\)](#) to estimate H . A periodogram of time series $\mathbf{Y}_{t \in (1,2,3,\dots,T)}$ with frequencies $\omega_j = \frac{j}{T}$ and $j = 1, \dots, [T/2]$ can be represented as

$$I_T(\omega_j) = \frac{1}{T} \left| \sum_{t=1}^T Y_t e^{-2\pi i(t-1)\omega_j} \right|^2. \quad (4.6)$$

A simple linear regression at low frequencies ω_j , $j = 1, \dots, J \leq [T/2]$ can be applied to estimate H according to the linear model,

$$\log(I_T(\omega_j)) = C - d \log\left(4 \sin^2\left(\frac{\omega_j}{2}\right)\right) + \varepsilon_j,$$

where C is a constant and define $X_t = \log\left(4 \sin^2\left(\frac{\omega_j}{2}\right)\right)$. Consequently, the least squares estimate of the slope yields another estimator for d .

Definition 4.3.4 (Estimator $\hat{d}$ by periodogram regression). *Consider a time series Y_t with the periodogram $I_T(\omega_j)$ defined in Equation (4.6), the estimator $\hat{d}$ developed by adopting periodogram regression is given by*

$$\hat{d} = \frac{J \left[\sum_{j=1}^J \log(I_T(\omega_j)) \log\left(4 \sin^2\left(\frac{\omega_j}{2}\right)\right) \right] - \left[\sum_{j=1}^J \log(I_T(\omega_j)) \right] \left[\sum_{j=1}^J \log\left(4 \sin^2\left(\frac{\omega_j}{2}\right)\right) \right]}{J \sum_{j=1}^J [\log\left(4 \sin^2\left(\frac{\omega_j}{2}\right)\right)]^2 - \left[\sum_{j=1}^J \log\left(4 \sin^2\left(\frac{\omega_j}{2}\right)\right) \right]^2}, \quad (4.7)$$

with asymptotic distribution

$$\hat{d} \sim N \left(d, \frac{\pi^2}{6 \sum_{t=1}^T (Y_t - \bar{Y})^2} \right).$$

Hence, an estimate of H using periodogram regression can be calculated by $\hat{H} = \hat{d} + 0.5$ and the 95% empirical confidence interval for $\hat{d}$ calculated in Equation (4.7) with the sample length $L = 2^N$ and $J = \lfloor L^{0.5} \rfloor$ (Weron, 2002) is

$$(0.5 - \exp(-0.71 \cdot N^{2/3} + 2.04), \exp(-0.68 \cdot N^{2/3} + 1.78) + 0.5).$$

Remark 4.3.3. *The periodogram regression approach is the only approach with known asymptotic properties.*

4.4 Empirical evidence for long memory in trading behaviours

Among the three estimators, the estimator we adopt is the rescaled range R/S estimator which is available from the R package called `pracma`. We also test other estimators which provide similar H estimates. We can say that a time series has long memory if H lies within (0.5,1) and $H = 0.5$ indicates absence of long memory.

Figure 4.2 depicts boxplots of H cross-classified by terms and years of open interest for all CFTC data. The prevalence of long memory is obvious in all classes of open interest series, despite the fact that the open interest for 2-year term has slightly lower H indicating the slightly lower level of long memory as measured by H . Hence, there is clearly convicting evidence to study this feature further with the long memory time series count models we develop and to explore how our proposed models can improve forecast performance.

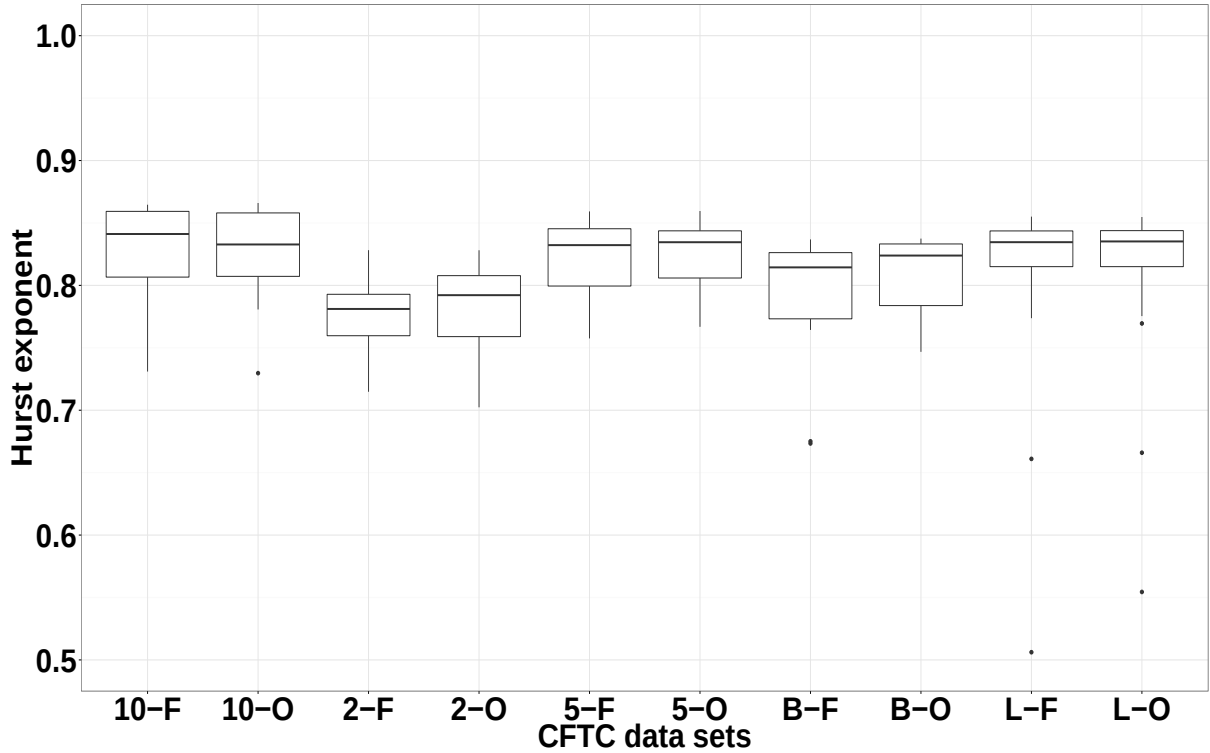


Figure 4.2. Hurst exponent for all CFTC data cross-classified by term and year sub-categories.

4.5 Conclusion

This chapter starts the analysis of CFTC data by describing the financial relevance of analysing open interest and then introducing the CFTC data. We claim that there are forecastable patterns in trading behaviour and liquidity of open interest via the consistently observed long-memory feature which can be measured by H . We define H , describe some properties of H , state its relationship with the long memory parameter d and lastly provide some methods to estimate H . As the application of H to CFTC data confirms clearly the presence of long memory in the open interest data, we model the open interest using our proposed GLGARMA model in the next chapter.

CHAPTER 5

Analysis of CFTC data - modelling long memory

5.1 Background

Chapter 4 clearly documents the presence of long memory in CFTC data consistently across term and year sub-categories. In this chapter, we first identify and summarise the long memory features through ACFs and periodograms into four types, namely, basic, oscillatory, combined and irregular. Then forty GLGARMA models with different choice of count distribution, model type, long memory type and order are fitted to each of 136 open interest series. To estimate the models, we adopt a Bayesian approach implemented using R package Rstan. In addition, we provide the posterior predictive distribution for forecasting.

We perform simulation to test the performance of the estimator and apply deviance information criterion and other model selection criteria to select some best fitting models among the forty models. Then four representative data for the four types of long memory are selected to undertake forecasting experiment along with many other competing models. The superiority of GLGARMA is demonstrated as it provides consistently the best forecast performance for all four types of long memory open interest series. We also remark that the trade type spread with oscillatory long memory possibly due the seasonal (contract) rollover effect is still best modelled by Gegenbauer long memory instead of seasonal long memory and this result applies to short and long position of basic type which do not possess obvious oscillatory Gegenbauer long memory.

5.2 Empirical evidence for long memory features in trading behaviours

Section 4.4 confirms the presence of long memory in CFTC data. We further demonstrate in this section that not only is long memory prevalent in the CFTC data, but in addition, there are a range of different classes of long memory structures presented in the data. This is also relevant to our proposed models, as we demonstrate that our models can capture different structures of long memory and these structures can be characterised and parameterised with particular orthogonal polynomial basis functions.

Figure 5.1 below further shows four different types of long memory patterns, namely, basic, oscillatory, combined and irregular called type I to IV, respectively. The plots in the left panel show the autocorrelation function (ACF) for four different data sets and the periodograms in the right are estimates of spectral densities. The first ACF plot illustrates a clear ARFIMA type (Type I; basic) long memory pattern. The second ACF shows a Gegenbauer type (Type II; oscillatory) with apparent oscillatory structure. The third ACF displays a combination of ARFIMA and GARMA (Type III; combined) features. There is an obvious long memory decay trend together with oscillatory fluctuations. The last ACF does not show clear ARFIMA nor GARMA (Type IV; irregular) features. The periodograms illustrate these long memory features in the frequency domain. Because of the symmetric property of the spectral density, a range of $[0, \frac{\pi}{2}]$ for the frequency in the periodograms is enough to present the features of different special densities. The first periodogram shows only one clear peak around 0 which is a typical spectral density pattern for ARFIMA models. The second plot shows a predominant peak at a certain non-zero frequency which aligns with the features for GARMA models. The third plot shows a predominant peak around 0 and another weaker peak demonstrating a combination of ARFIMA and GARMA features. The last one shows irregularities at the beginning, and hence is classified as a irregular type. These four data are used to represent data with these four types of long memory in subsequent analyses.

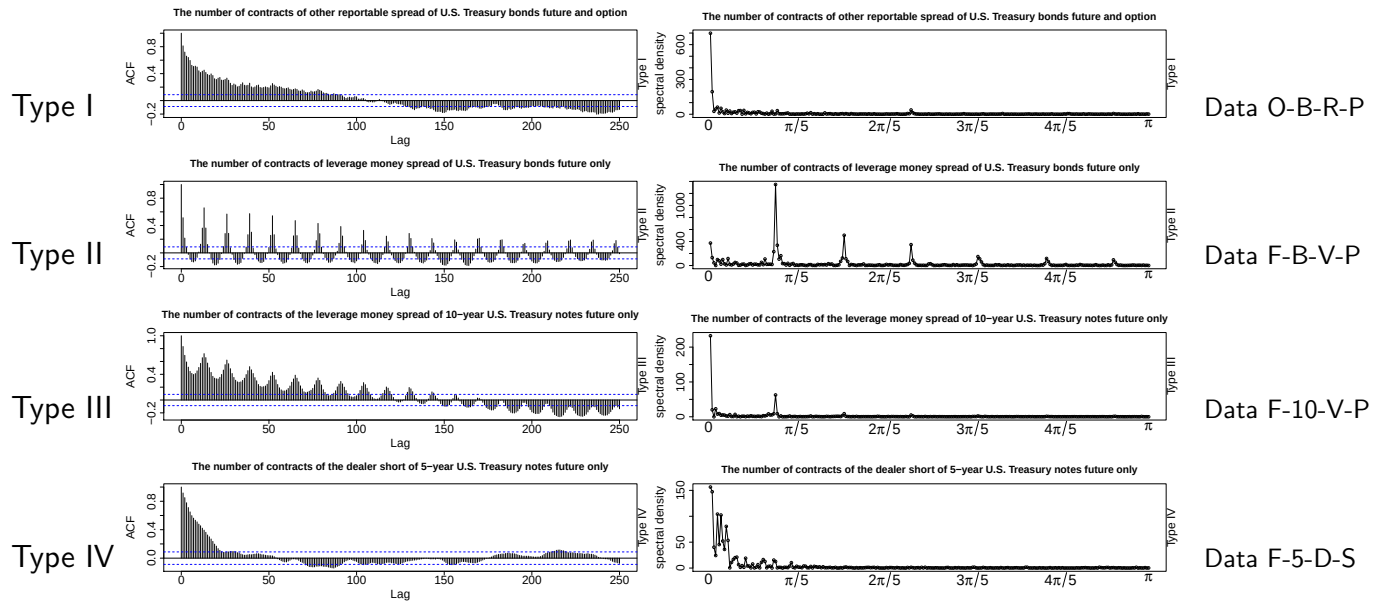


Figure 5.1. Four types of ACF (left) and periodogram (right) for four selected CFTC data.

Table 5.1 reports the distribution of data across different data groups for the four types of ACF. It shows that basic (type I) and irregular (type IV) are the most common types of long memory. Oscillatory (type II & III) long memory occurs only in trade type P and trader type D and V. In addition, aggregated groups such as R, N, T and I do not display oscillatory long memory.

Table 5.1. Classification of types of ACF across sub-category classification of CFTC data.

	Term		Year				Type/Trader								Trade		
Data type	F	O	2	5	10	B	I	N	T	A	D	V	R	L	S	P	
Type I	37	38	19	24	20	12	5	11	8	17	6	12	16	27	29	14	
Type II	4	0	1	1	0	2	0	0	0	0	2	2	0	0	0	4	
Type III	4	6	3	2	3	2	0	0	0	0	6	4	0	0	0	10	
Type IV	23	24	11	7	11	18	3	5	8	7	10	6	8	21	19	4	

5.3 Bayesian inference

To model the different long memory features, we apply the GLGARMA model using a Bayesian approach. Bayesian approach has a clear advantage that it allows one to incorporate apriori beliefs on model structures such as the presence or absence of particular forms of long memory into the state space structures proposed. In addition to allowing one to incorporate apriori beliefs on model structures such as the presence or absence of particular forms of long memory into the state space structures proposed, there are other computational advantages in using a Bayesian model for the general models proposed in this thesis. For instance, one may avoid the computation complexities in evaluating the marginal likelihood function for PD model which involve latent values and high-dimensional integration. Therefore, we develop a Bayesian modelling framework for the analysis. Moreover, as we will show, this is naturally very beneficial for obtaining posterior predictive distributions that provide distributional forecast summaries for such state space models.

Given a set of observed values $\mathbf{y}_{1:T} = (y_1, y_2, \dots, y_T)$, with each $y_t \in \mathbb{Z}^+ \cup \{0\}$ and the vector of unknown parameters $\boldsymbol{\vartheta}^*$, the posterior distribution for $\boldsymbol{\vartheta}^*$ conditional on $\mathbf{y}_{1:T}$ is given by

$$\pi(\boldsymbol{\vartheta}^* | \mathbf{y}_{1:T}) = \frac{f(\mathbf{y}_{1:T} | \boldsymbol{\vartheta}^*) \pi(\boldsymbol{\vartheta}^*)}{\int f(\mathbf{y}_{1:T} | \boldsymbol{\vartheta}^*) \pi(\boldsymbol{\vartheta}^*) d\boldsymbol{\vartheta}^*} \propto f(\mathbf{y}_{1:T} | \boldsymbol{\vartheta}^*) \pi(\boldsymbol{\vartheta}^*),$$

which is proportional to the likelihood function $f(\mathbf{y}_{1:T} | \boldsymbol{\vartheta}^*)$ and the prior densities $\pi(\boldsymbol{\vartheta}^*)$ estimated using available information or past data. Even if there is no prior information, we can still apply non-informative or reference priors. Furthermore, the Bayesian approach generates posterior distributions for all parameters of interest and hence allows construction of creditable intervals for these parameters.

5.3.1 Bayesian model

Let $\boldsymbol{\vartheta}^* = (\boldsymbol{\vartheta}, \mathbf{z})$ denote the vector of all model parameter $\boldsymbol{\vartheta} = (c, u, d, \theta_j, \phi_j, \sigma^2, \nu)$ and state parameter $\mathbf{z} = \mathbf{z}_{1:T} = (z_1, z_2, \dots, z_T)$, with each $z_t \in \mathbb{R}$. The likelihood function $f(\mathbf{y}_{1:T} | \boldsymbol{\vartheta}^*)$ is based on the model assumptions (2.10) to (2.12) and the following priors

$\pi(\boldsymbol{\vartheta})$:

$$\begin{aligned} \nu &\sim D_\nu(a_\nu, b_\nu), \quad u \sim U(-1, 1), \quad d \sim U(0, 1/2), \quad c \sim N(0, \sigma_c^2), \\ \phi_j &\sim N(0, \sigma_\phi^2), \quad \theta_j \sim N(0, \sigma_\theta^2) \quad \text{and} \quad \sigma^2 \sim \Gamma(a, b), \end{aligned} \quad (5.1)$$

are adopted in which $U(a_u, b_u)$ denotes the uniform priors on the range (a_u, b_u) for the long memory parameters u and d , $\Gamma(a, b)$ denotes the gamma prior with shape and scale parameters a and b respectively for the scale parameter σ^2 in the PD models and $D_\nu(a_\nu, b_\nu)$ represents gamma priors for the shape parameters of NB and DP data distributions and uniform prior for the shape parameter of GP data distribution respectively. As an example, the joint posterior distribution for the PD GLGARMA(p, d, q) model with GP data distribution (denoted by GLGARMA-PD-GP(p, d, q)) is

$$\begin{aligned} f(\boldsymbol{\vartheta}^* | \mathbf{y}_{1:T}) &= f(\mathbf{y}_{1:T} | \mathbf{z}_{1:T}, \boldsymbol{\vartheta}) f(\mathbf{z}_{1:T} | \boldsymbol{\vartheta}) \pi(\boldsymbol{\vartheta}) \\ &\propto \prod_{t=1}^T \left[\frac{\exp(c + \sum_{j=0}^{\infty} \psi_j z_{t-j})(1 - \nu) [\exp(c + \sum_{j=0}^{\infty} \psi_j z_{t-j})(1 - \nu) + y_t \nu]^{y_t - 1}}{\Gamma(y_t + 1) \exp(-y_t \nu) - \exp(c + \sum_{j=0}^{\infty} \psi_j z_{t-j})(1 - \nu)} \cdot \frac{1}{\sigma} \exp\left(-\frac{z_t^2}{2\sigma^2}\right) \right] \\ &\quad \times \exp\left(-\frac{c^2}{2\sigma_c^2}\right) \cdot \prod_{j=1}^p \exp\left(-\frac{\phi_j^2}{2\sigma_\phi^2}\right) \cdot \prod_{j=1}^q \exp\left(-\frac{\theta_j^2}{2\sigma_\theta^2}\right) \cdot (\sigma^2)^{a-1} e^{-(\sigma^2)b} I_u(-1, 1) I_d(0, 0.5) I_\nu(-1, 1), \end{aligned} \quad (5.2)$$

where the hyperparameters are set to be $\sigma_c^2 = \sigma_\phi^2 = \sigma_\theta^2 = 10$, $a = 3$ and $b = 1$. For the OD model, the term $\frac{1}{\sigma} \exp\left(-\frac{z_t^2}{2\sigma^2}\right)$ for normal errors should be dropped, the error term z_{t-j} should change to e_{t-j} in Equation (2.12) and the vector of parameters becomes $\boldsymbol{\vartheta}^* = \boldsymbol{\vartheta} = (c, u, d, \theta_j, \phi_j, \nu)$.

5.3.2 Implementation with Rstan

To implement the proposed models efficiently, we choose the Bayesian R package **Rstan** (Team, 2016) which utilises the **stan** program within R developed in the C++ language. The sampling method applied in **Rstan** is the Hamiltonian Monte Carlo (HMC) sampler ((Duane et al., 1987) and (Neal, 1994)) that belongs to the class of Markov chain Monte Carlo (MCMC) sampling methods (Metropolis et al., 1953). For some complicated models with many parameters, the HMC sampler converges faster than the conventional samplers such as random-walk Metropolis (Metropolis et al., 1953) and Gibbs sampler (Geman and Geman, 1984) because the HMC sampler transforms the simulation of high-dimensional target distribution into the simulation of Hamiltonian dynamics (Neal,

2011). By using the gradient information, Hamiltonian dynamics can produce distant proposals for the Metropolis algorithm, thereby avoiding the slow exploration of the state space that results from the diffusive behaviour of random-walk proposals. The Hamiltonian dynamics is defined as

$$H(\mathbf{q}, \mathbf{p}) = U(\mathbf{q}) + K(\mathbf{p}),$$

where the interpretation of $U(\mathbf{q})$ and $K(\mathbf{p})$ in physics are the potential and kinetic energy respectively and the Hamiltonian dynamic across time t satisfies the following conditions:

$$\frac{dq_i}{dt} = \frac{dH}{dp_i} \quad \text{and} \quad \frac{dp_i}{dt} = -\frac{dH}{dq_i}, \quad (5.3)$$

where $\mathbf{q} = (q_1, q_2, \dots, q_d)$ is a d -dimensional position vector and $\mathbf{p}$ is a d -dimensional auxiliary momentum vector. In the context of Bayesian approach, $U(\mathbf{q})$ and $K(\mathbf{p})$ refer to minus log posterior density and minus log density (plus a constant) of the Gaussian distribution with mean zero and variance τ_i^2 given by respectively

$$U(\boldsymbol{\vartheta}^*) = -\log [f(\boldsymbol{\vartheta}^*|\mathbf{y})] \quad \text{and} \quad K(\mathbf{p}) = \sum_{i=1}^d \frac{p_i^2}{2\tau_i^2}, \quad (5.4)$$

where $\boldsymbol{\vartheta}^*$ corresponding to $\mathbf{q}$ is a vector of parameters and p_i are auxiliary variables drawn from the Gaussian distribution given $\boldsymbol{\vartheta}^*$ (Neal, 2011). The HMC sampler alternates simple updates of p_i from the Gaussian distribution with Metropolis updates of ϑ_i^* to a new state by computing a trajectory according to Hamiltonian dynamics under the **leapfrog** method. A proposed state ϑ_i^* can be distant from the current state but nevertheless has a high probability of acceptance.

For computational convenience, the differential equations of Hamiltonian dynamics in Equation (5.3) are implemented using discretized time (an analogy for iterations) $0, \xi, 2\xi, 3\xi$, etc with leapfrog stepsize ξ and steps L . The leapfrog method starts with a half step for the momentum variables $p_i(t + \xi/2)$, then performs a full step for ϑ_i^* , using the new $p_i(t + \xi/2)$, and finally does another half step to $p_i(t + \xi)$, using the new $\vartheta_i^*(t + \xi)$.

The whole scheme is summarized as

$$\begin{aligned} p_i(t + \xi/2) &= p_i(t) - (\xi/2) \frac{\partial U}{\partial \vartheta_i^*}(\vartheta_i^*(t)), \\ \vartheta_i^*(t + \xi) &= \vartheta_i^*(t) + \xi \frac{p_i(t + \xi/2)}{\tau_i^2}, \\ p_i(t + \xi) &= p_i(t + \xi/2) - (\xi/2) \frac{\partial U}{\partial \vartheta_i^*}(\vartheta_i^*(t + \xi)). \end{aligned}$$

An analogous scheme can be used with any kinetic energy function $\partial K / \partial p_i$ replacing p_i / τ_i in Equation (5.4). Lastly, at the end of this L -step trajectory, the momentum variables are negated with a proposed state (q^*, p^*) and the acceptance rate is calculated as

$$\min[1, \exp(-H(q^*, p^*) + H(q, p))] = \min[1, \exp(-U(q^*) + U(q) - K(p^*) + K(p))].$$

The two user-specified parameters ξ and L need to be tuned adequately for optimal efficiency. In particular, [Homan and Gelman \(2014\)](#) pointed out that the efficiency may be lower and becomes similar to random walk if L is set to be too small. On the other hand, the computation will be wasted if L is set to be too large. In this case, the No-U-Turn sampler as an extension of HMC is taken into consideration because the number-of-steps parameter L and the step size parameter ξ will be automatically tuned.

The HMC sampler needs to be closely monitored. To assess the dependence, precision and convergence of posterior sample, three measures are reported in **Rstan**. The first measure is the *number of effective samples* which indicates dependence within a Monte Carlo sample. The second measure is the *Monte Carlo standard error* (MCSE)

$$\text{MCSE} = \frac{\text{posterior standard deviation}}{\sqrt{\text{number of effective samples}}},$$

which reports the error of estimation for the posterior mean. To monitor the convergence for $k > 2$ chains of length $2n$ each, [Gelman and Rubin \(1992\)](#) proposed $\hat{R}$ which is defined as

$$\hat{R} = \frac{\hat{V}}{W} \cdot \frac{df}{df - 2},$$

where

$$\hat{V} = \frac{n-1}{n}W + \frac{k+1}{kn}B, \quad W = \sum_{i=1}^k \frac{s_i^2}{k}, \quad B = n \sum_{i=1}^k \frac{(\bar{\vartheta}_i - \bar{\vartheta}_{..})^2}{k-1}, \quad df = \frac{2\hat{V}^2}{\widehat{\text{Var}}(\hat{V})}, \quad (5.5)$$

$$\begin{aligned}\widehat{\text{Var}}(\widehat{V}) &= \left(\frac{n-1}{n}\right)^2 \frac{1}{k} \widehat{\text{Var}}(s_i^2) + \left(\frac{k+1}{kn}\right)^2 \frac{2}{k-1} B^2 + \\ &\quad 2 \frac{(k+1)(n-1)}{kn^2} \cdot \frac{n}{k} [\widehat{\text{Cov}}(s_i^2, \bar{\vartheta}_{i.}^2) - 2\bar{\vartheta}_{..} \widehat{\text{Cov}}(s_i^2, \bar{\vartheta}_{i.})],\end{aligned}\quad (5.6)$$

s_i is the within-chain variance and ϑ_{ij} is the j -th parameter in chain i . If $\widehat{R}$ is close to 1, the parameter ϑ has converged.

For the analysis of CFTC data in Chapter 5, HMD data in Chapter 7 and Chapter 8, we run $k = 1$ chain and hence $B = 0$ in Equations (5.5) and (5.6). For each chain, we run totally 50,000 iterations, discard the first 5,000 iterations as burn-in and sample the subsequent $L = 45,000$ iterations with thin set to 1. We check $\widehat{R}$ and the history plot carefully to ensure that all parameters meet the convergence condition. Take the forecast models in Figure 5.11 as examples, the number of effective sample ranges from 15,000 to 36,000 across parameters and the range of $\widehat{R}$ is between 1.0000 and 1.0003 indicating moderate dependency and clear convergence.

5.3.3 Model selection

The performance of each model is evaluated through two popular Bayesian model selection criteria, namely, deviance information criterion (DIC) (Spiegelhalter et al., 2014) and Watanabe-Akaike information criterion (WAIC) (Watanabe, 2010). As a generalisation of Akaike's Information Criterion (AIC), DIC can deal with models containing informative priors, such as hierarchical models. As the priors can effectively restrict the freedom of model parameters, the number of parameters as required in the calculation of AIC is generally unclear. DIC overcomes such problem by providing an estimate for the effective number of parameters. However, DIC may provide a negative effective number of parameters and is inappropriate for singular models. On the other hand, WAIC is invariant to parametrisation (Vehtari and Gelman, 2014). Since both DIC and WAIC select the best-fitted model allowing for model complexity, they both avoid models being excessively adjusted to match the historical observations and hence resulting in poor forecasts. For the analysis of CFTC data, as both DIC and WAIC provide consistent results in model selection, we only discuss DIC for convenience.

The DIC can be calculated using the equation

$$\text{DIC} = \bar{D} + p_D = 2\bar{D} - D(\bar{\boldsymbol{\vartheta}}^*),$$

where the deviance is defined as $D(\boldsymbol{\vartheta}^*) = -2 \log(f(\mathbf{y}|\boldsymbol{\vartheta}^*))$, $\bar{D} = E_{\theta|\mathbf{y}}[-2 \log(f(\mathbf{y}|\boldsymbol{\vartheta}^*))]$ measures the model fit and the estimated number of parameters $p_D = \bar{D} - D(\bar{\boldsymbol{\vartheta}}^*)$ measures model complexity (Spiegelhalter et al., 2002).

To evaluate the forecast performance, three types of measures, namely residuals $r_t = y_t - \hat{y}_t$, percentage errors $p_t = \frac{r_t}{y_t} \times 100$ and scaled errors q_t defined in Equation (5.7), are adopted. The forecasts $\hat{y}_t$ are given by the posterior mean or median. Based on r_t and p_t , three popular criteria, namely the mean absolute error (MAE), root mean squared error (RMSE) and mean absolute percentage error (MAPE), are defined respectively below

$$\text{MAE} = \frac{1}{m} \sum_{t=1}^m |r_{T+t}|, \quad \text{RMSE} = \sqrt{\frac{1}{m} \sum_{t=1}^m r_{T+t}^2} \quad \text{and} \quad \text{MAPE} = \frac{1}{m} \sum_{t=1}^m |p_{T+t}|.$$

However r_t are scale-dependent making comparison difficult. Although p_t are scale-free, they are sensitive to observations close to zero. Hence the fourth criterion we adopt is the mean absolute scaled error (MASE) defined as

$$\text{MASE} = \frac{1}{m} \sum_{t=1}^m |q_{T+t}|,$$

making use of the scaled errors

$$q_{T+t} = \frac{r_{T+t}}{\frac{1}{m-1} \sum_{t=2}^m |y_{T+t} - y_{T+t-1}|}, \quad (5.7)$$

proposed by Hyndman and Koehler (2006). Furthermore, similar approach can also be applied to evaluate estimated results $\hat{\mu}_{x,t}$ calculated by the posterior mean or median. Hence, the residuals $r_{x,t}^s = \mu_{x,t} - \hat{\mu}_{x,t}$, percentage errors $p_{x,t}^s = \frac{r_{x,t}^s}{\mu_{x,t}} \times 100$ and scaled errors

$$g_{x,t}^s = \frac{r_{x,t}^s}{\frac{1}{m-1} \sum_{t=2}^m |\mu_{x,t} - \mu_{x,t-1}|},$$

can construct similar criteria by the same formulas.

5.3.4 Bayesian forecasting

Bayesian forecasting is another application of Bayesian inference. One significant advantage of Bayesian inference is its provision of posterior predictive distributions for all forecasts. Suppose we want m -step forecasts $\mathbf{y}_{T+1:T+m}$ via a sequence of 1-step ahead forecasts of y_{T+s} , $s = 1, \dots, m$ using sliding window where the observed data filtration for each window is $\mathcal{F}_{s:T+s-1}$ (see Equation (2.5) for the definition). The posterior predictive distribution for y_{T+s} , $s = 1, \dots, m$ is defined as

$$\begin{aligned} f(y_{T+s}|\mathcal{F}_{s:T+s-1}) &= \int \dots \int f(y_{T+s}|\mu_{T+s}, \boldsymbol{\vartheta}, \mathcal{F}_{s:T+s-1}) f(\mu_{T+s}|\boldsymbol{\mu}_{s:T+s-1}, \boldsymbol{\vartheta}, \mathcal{F}_{s:T+s-1}) \\ &\quad \times f(\boldsymbol{\vartheta}|\mathcal{F}_{s:T+s-1}) d\boldsymbol{\mu}_{s:T+s-1} d\boldsymbol{\vartheta}, \end{aligned} \quad (5.8)$$

and this integral can be approximated by the Monte Carlo estimator, constructed from posterior samples according to:

$$\hat{f}(y_{T+s}|\mathcal{F}_{s:T+s-1}) = \frac{1}{L} \sum_{l=1}^L f(y_{T+s}|\boldsymbol{\mu}_{s:T+s-1}^{(l)}, \boldsymbol{\vartheta}_s^{(l)}, \mathcal{F}_{s:T+s-1}), \quad (5.9)$$

where in the forecast of CFTC data in Section 5.5, we set $L = 45,000$ as the number of iterations after burn-in in each MCMC sampler run given the current window of information $\mathcal{F}_{s:T+s-1}$ and $\boldsymbol{\mu}_{s:T+s-1}^{(l)}$ and $\boldsymbol{\vartheta}_s^{(l)}$ are the l -th draw in the posterior sample of $\boldsymbol{\mu}_{s:T+s-1}$ and $\boldsymbol{\vartheta}$, respectively. In particular, the density $f(\mu_{T+s}|\boldsymbol{\mu}_{s:T+s-1}, \boldsymbol{\vartheta}, \mathcal{F}_{s:T+s-1})$ in Equation (5.8) for the OD model degenerates to the delta function $\delta(\mu_{T+s}|\boldsymbol{\mu}_{s:T+s-1}, \boldsymbol{\vartheta}, \mathcal{F}_{s:T+s-1})$ because μ_t is a deterministic function. Hence, the posterior predictive distribution for y_{T+s} can be approximated by

$$\hat{f}(y_{T+s}|\mathcal{F}_{s:T+s-1}) = \frac{1}{L} \sum_{l=1}^L f(y_{T+s}|\boldsymbol{\vartheta}_s^{(l)}, \mathcal{F}_{s:T+s-1}), \quad (5.10)$$

based on Equation (5.9). Apart from these fully Bayesian forecasts that produce the posterior predictive distribution, the resultant posterior predictive point estimators and predictive credible intervals, one may also adopt a different class of forecast estimators in the Bayesian setting. These are typically known as empirical Bayes forecasts which can be obtained in a computationally more efficient manner. [Robins \(1983\)](#) studied empirical Bayes approaches and illustrated features by providing some examples. Furthermore, [Knox et al. \(2001\)](#) compared the features of frequentist and empirical Bayes forecasts. Rather than integrating out posterior (in sample) parameter uncertainty from the predictive distribution and resultant forecast estimators, one may instead condition

upon selected posterior (in sample) point estimators, generically denoted by $\tilde{\boldsymbol{\vartheta}}$ and $\tilde{\boldsymbol{\mu}}_{s:T+s}$. This produces the empirical Bayes predictive distributions given by

$$\hat{f}^{EB}(y_{T+s}|\mathcal{F}_{s:T+s-1}) = f(y_{T+s}|\tilde{\boldsymbol{\mu}}_{s:T+s}, \tilde{\boldsymbol{\vartheta}}_s, \mathcal{F}_{s:T+s-1}). \quad (5.11)$$

Typically the point estimators used in $\tilde{\boldsymbol{\vartheta}}_s$ (similarly for $\tilde{\boldsymbol{\mu}}_{s:T+s}$) are either formed from the maximum-a-posteriori estimate (MAP) or the estimate which minimizes the posterior expected loss. The concept of MAP is similar to the maximum likelihood (ML) estimate when the priors are uninformative since in this case $\tilde{\boldsymbol{\vartheta}}_s$ is the mode of the posterior distribution

$$\tilde{\boldsymbol{\vartheta}}_{s,MAP} = \arg \max_{\boldsymbol{\vartheta}_s} f_{\boldsymbol{\vartheta}_s}(\boldsymbol{\vartheta}_s|\mathcal{F}_{s:T+s-1}).$$

Another Bayes estimator which minimises the posterior expected loss (PEL) is defined as

$$\tilde{\boldsymbol{\vartheta}}_{s,PEL} = \arg \min_{\boldsymbol{\vartheta}_s} \mathbb{E}(L(\boldsymbol{\vartheta}_s, \tilde{\boldsymbol{\vartheta}}_s|\mathcal{F}_{s:T+s-1})),$$

where $L(\boldsymbol{\vartheta}_s, \tilde{\boldsymbol{\vartheta}}_s|\mathcal{F}_{s:T+s-1})$ is the loss function. One example is the commonly used minimum mean square error (MSE) estimator defined as

$$\tilde{\boldsymbol{\vartheta}}_{s,MSE} = \arg \min_{\boldsymbol{\vartheta}_s} \mathbb{E}([\boldsymbol{\vartheta}_s - \tilde{\boldsymbol{\vartheta}}_s]^2|\mathcal{F}_{s:T+s-1}),$$

where $\tilde{\boldsymbol{\vartheta}}_{s,MSE}$ corresponds to the posterior mean $\mathbb{E}(\boldsymbol{\vartheta}_s|\mathcal{F}_{s:T+s-1}) = \bar{\boldsymbol{\vartheta}}_s$. If the minimum absolute error (AE) estimator $L(\boldsymbol{\vartheta}_s, \tilde{\boldsymbol{\vartheta}}_s) = |\boldsymbol{\vartheta}_s - \tilde{\boldsymbol{\vartheta}}_s|$ is used, it gives $\tilde{\boldsymbol{\vartheta}}_{s,AE} = \boldsymbol{\vartheta}_{s,0.5}$ which is the posterior median.

5.4 Model fitting

In this section, we perform two analyses, fitting all 136 data each to 40 models to evaluate best model structures and fitting oscillatory long memory (type II) data to seasonal models to illustrate the fact that seasonal long memory model cannot replace Gegenbauer long memory model even for spread data which display seasonal (contract) roll over effect.

5.4.1 Proposed models fitted to all data

As long memory patterns are clearly identified in the CFTC data, we conduct a large scale analysis to study how the patterns are associated with different trading activities. In particular, we fit each of the 136 CFTC data to 40 different models, starting with fitting 16 (0,0) order models with 4 distribution choices (Poisson, NB, GP, DP), 2 error types (PD or OD) and 2 long memory types (ARFIMA and GARMA). As GP is found to be consistently the best distribution choice, we further fit 6 more higher order models, namely (0,1), (1,0), (1,1), (1,2), (2,1) and (2,2), for GLARFIMA and GLGARMA models with PD and OD errors keeping the GP distribution choice (totally $6 \times 4 = 24$ models). Totally, there are 5,440 in-sample model fits which is a non-trivial experimental design requiring extensive use of super computing resources. Each model takes roughly 150-200 hours to run which corresponds to 1,088,000 compute hours. DIC is used to choose the best fitting model among the 16 (0,0) order models and the 24 higher order models.

Fitting the (0,0) order models to capture the general long memory feature, Figure 5.2 displays the counts of best fitting models for all CFTC data sets. It shows clearly that the GLARFIMA-PD-GP models provide the best in-sample fit to most data. The shape parameter of GP distribution indicates under-dispersion for most open interest after allowing for the mean function. We also note that the popularity of GLARFIMA model agrees with the prevalence of the basic or ARFIMA-type long memory as shown in Table 5.2. However, despite such prevalence, Figure 5.3 further shows that the differences in DIC relative to GLGARMA-PD-GP model (with positive values indicate preferences of GLGARMA-PD-GP over GLARFIMA-PD-GP model) are actually very small for most of the CFTC data since GLARFIMA model is the limiting case of GLGARMA model. On the other hand, GLGARMA model clearly outperforms GLARFIMA model for data with oscillatory long memory. To quote an example, the U.S. T-bond future only leverage money spread (F-B-V-P) data with oscillatory long memory has DIC of 971.72 when fitted to GLGARMA model ($\hat{d} = 0.481$ and $\hat{u} = 0.937$) as compared to the DIC of 982.01 for GLARFIMA model.

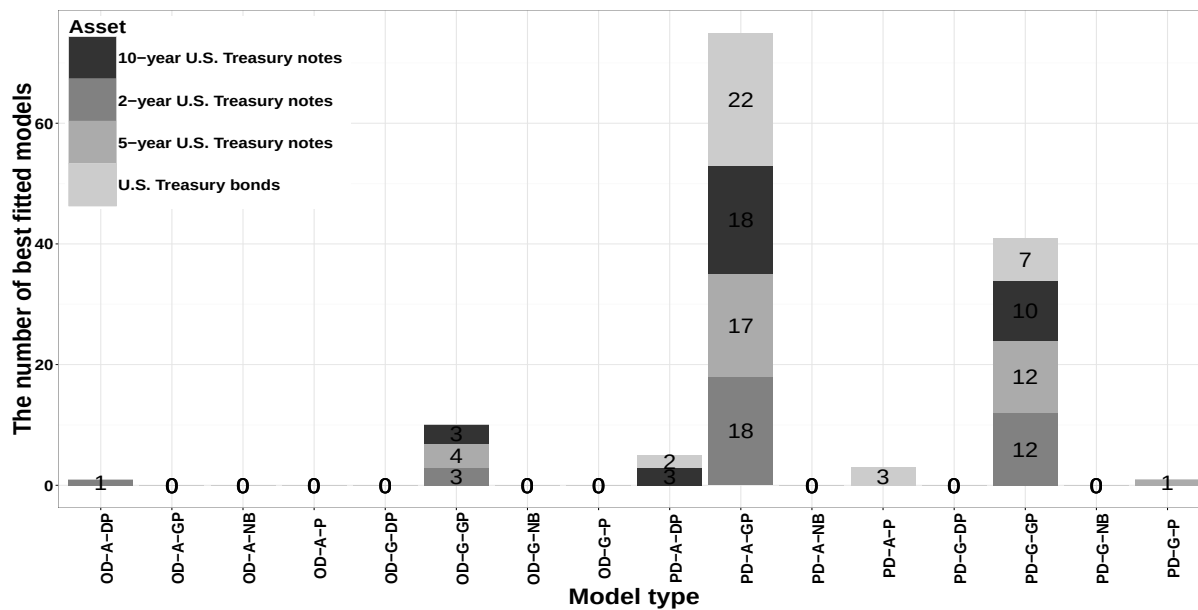


Figure 5.2. Counts of best fitted (0,0) order models for all CFTC data (PD-A-P stands for GLARFIMA-PD-P model with Poisson distribution)

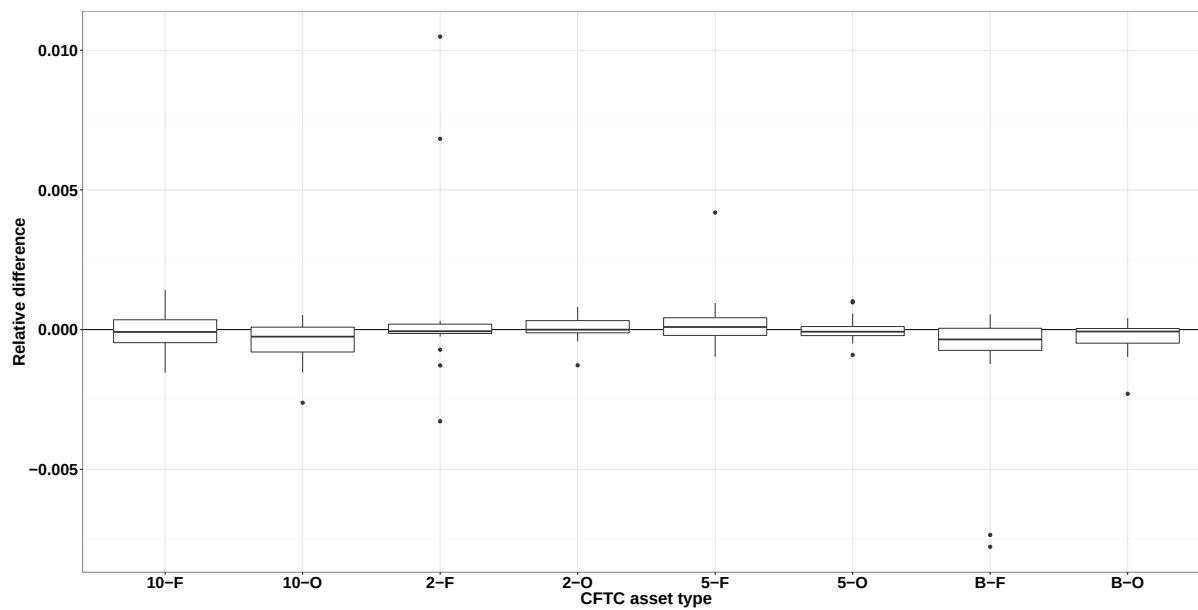


Figure 5.3. Relative difference in DIC between GLARFIMA-PD-GP and GLGARMA-PD-GP models with respect to GLARFIMA-PD-GP model

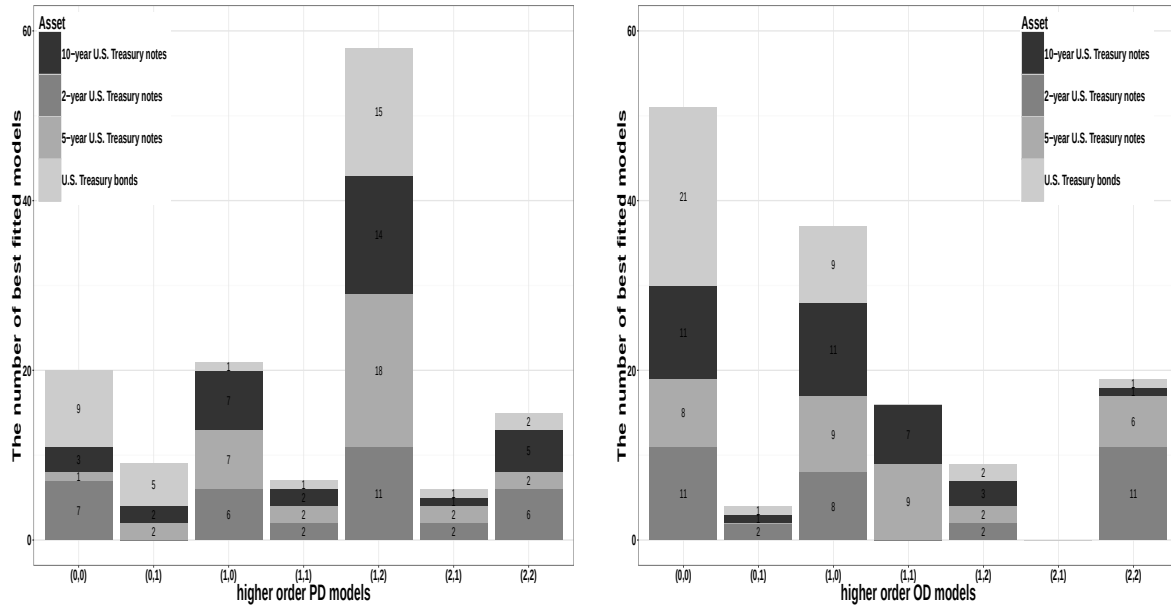


Figure 5.4. Number of best fitting higher order PD (left) and OD (right) GLGARMA-GP models.

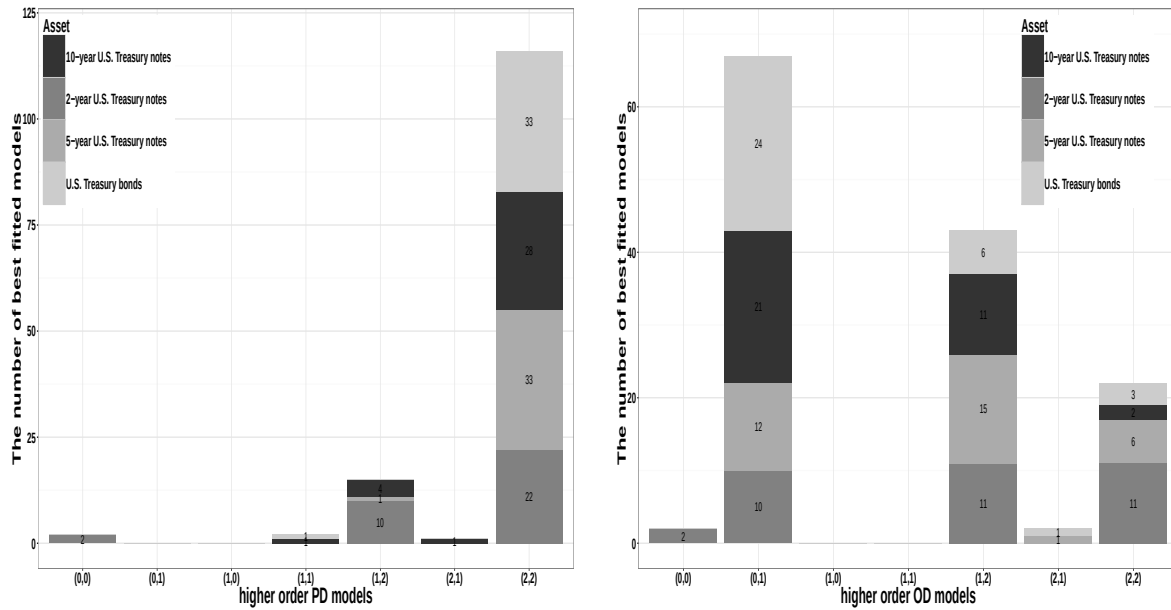


Figure 5.5. Number of best fitting higher order PD (left) and OD (right) GLARFIMA-GP models.

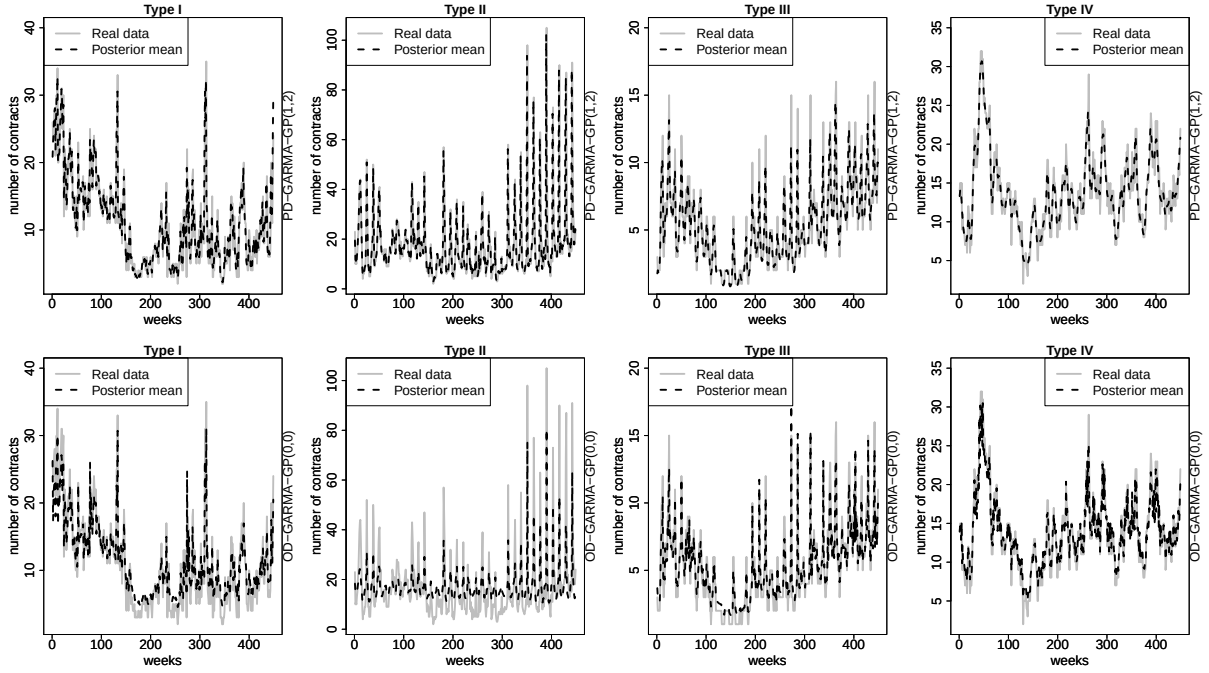


Figure 5.6. In-sample model fits for the best GLGARMA-PD-GP (first row) and GLGARMA-OD-GP (second row) models using the four chosen data sets for each type of long memory

Apart from considering the (0,0) order model, it is natural to extend the model to higher orders as illustrated in Equation (2.11), since many data display a clear upward trend. Figure 5.4 reports the count of best fitting GLGARMA-GP models across the year categories and it is clear that the most common best order for PD model is (1,2) whereas it is (0,0) for OD model. For GLARFIMA-GP models, Figure 5.5 shows that the most common best order is (2,2) for PD model whereas it is (0,1) for OD model.

Figure 5.6 compares the in-sample fit between the best GLGARMA-PD-GP (first row) and GLGARMA-OD-GP (second row) models for the four selected data with their ACFs displayed in Figure 5.1. It is clear that GLGARMA-PD-GP models generally provide better in-sample fit than GLGARMA-OD-GP models.

Figures 5.7 and 5.8 plot the posterior mean of $\hat{d}$ and $\hat{u}$ across terms (3 for T-note and 1 for T-Bond) and traders, respectively, using GLGARMA-PD-GP model. Despite the very close agreement of $\hat{u}$ around 1 (no oscillatory feature) and $\hat{d}$ around 0.5 (long memory), there are a few outliers of $\hat{u}$ with values below 1 indicating the presence of oscillatory long memory for trader type dealers and leveraged funds advisors/operators and trade type spread. This result also agrees with the result using ACF as reported in

Table 5.1. Hence we conclude that long memory feature is prevalent in dealer spread and leverage money spread.

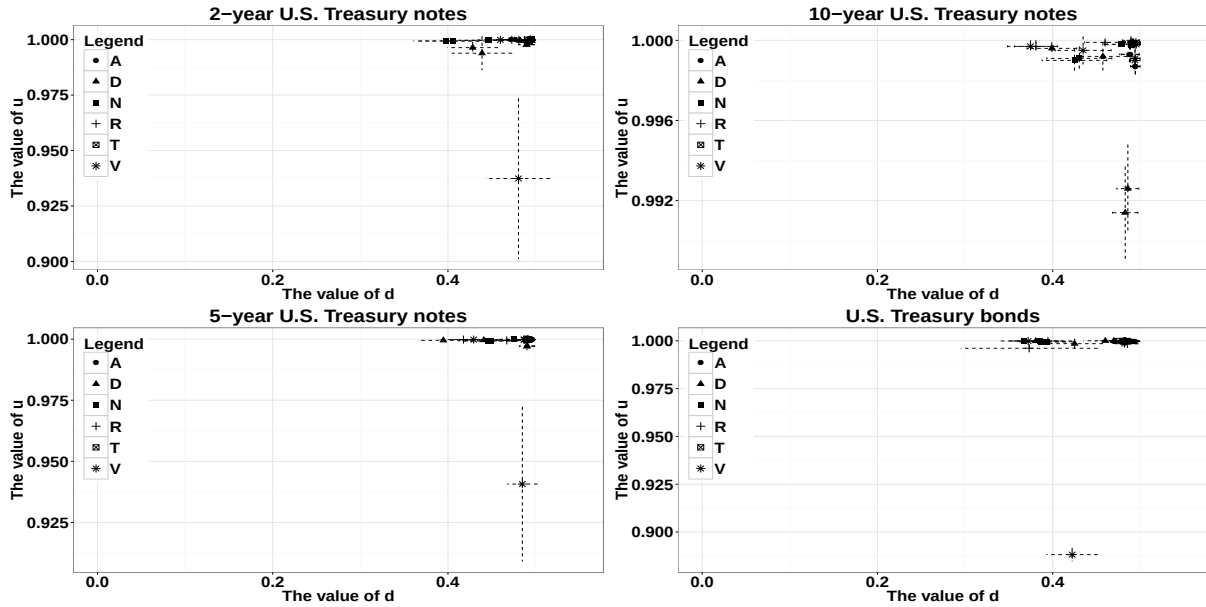


Figure 5.7. Estimates of d and u for different trader types

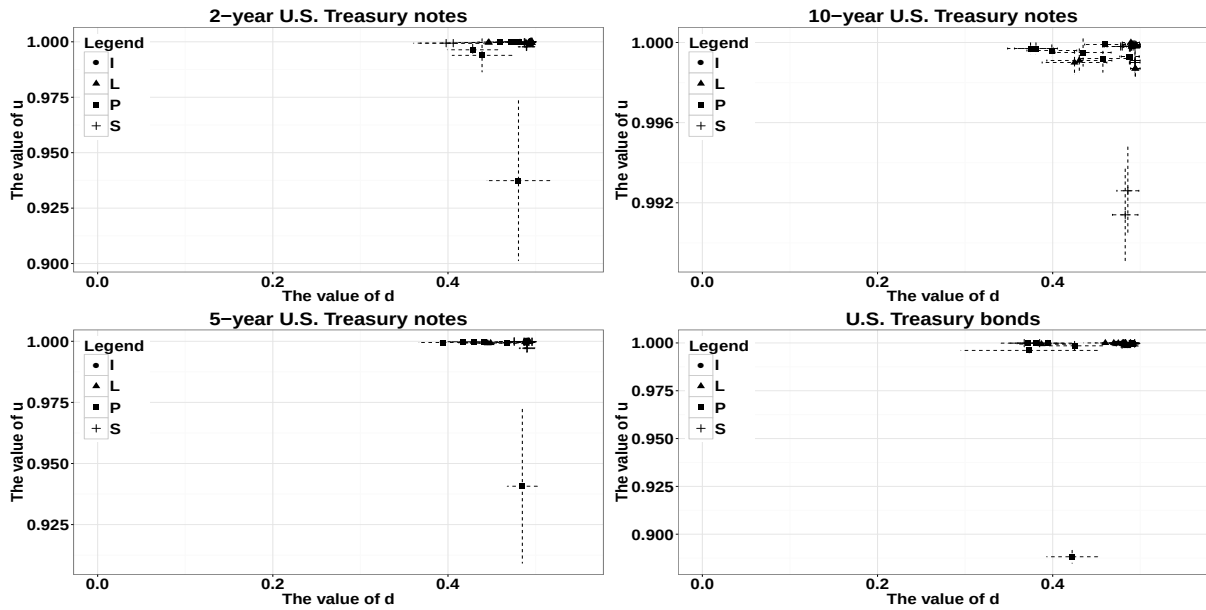


Figure 5.8. Estimates of d and u for different trading types

5.4.2 Seasonal models fitted to a Gegenbauer (type II) oscillatory long memory data

In this section we consider data sets classified according to the ACF as Type II, in which the long memory patterns observed for these data are oscillatory in nature. We seek to determine whether this feature arises from a specific attribute of how the underlying contracts, associated with these open interest data are traded. When studying open interest, we observe this feature for the front contract. This is the contract with highest liquidity that is actively traded in the market and typically corresponds to the contract closest to maturity. As this contract gets close to maturity, it is often the case that the traders in the market will shift their trading activity from this front contract to the next back contract, corresponding to the next closest to maturity contract. The timing at which they perform this contract roll is a random variable, often depending on several attributes, however it is typically very close to seasonal in structure. This process is called contract roll and affects the price series, the traded liquidity and the open interest.

Therefore, it could be possible that seasonality is present in the CFTC data due to potential futures contract rollover effects versus a more fundamental underlying stochastic structure based on long-memory oscillation that is non-periodic as obtained via a Gegenbauer type long memory structure. What we aim to demonstrate in this section is that even if we account for this seasonal type structure in the open interest series, arising from contract roll, when we compare a pure seasonal feature explanation to long memory and long memory seasonal explanation, we see that the best fitting model is still the model simply containing Gegenbauer long memory. This tells us that the seasonal roll effect is not the main driver for the long memory oscillations we see in our open interest analysis and is a very interesting finding for practitioners previously unknown. Most practitioners would attribute this feature to contract roll, we prove in this section definitely that this feature is driven by a different cause.

To capture this rollover effect, we need to study the seasonal component of the data. Typically one would perform such a study via first undertaking a deseasonalisation of the data through a seasonal period difference operator and test if such deseasonalisation can remove the oscillatory behavior in the ACF of a Gegenbauer long memory time series. To do this, we perform a small case study in which we generate data from two

different time series structures, one with a known periodic seasonal structure and the other with oscillatory long memory ACF structure. We then apply a seasonal difference operator to each of the time series, even though the Gegenbauer long memory time series do not have seasonal pattern in the ACF.

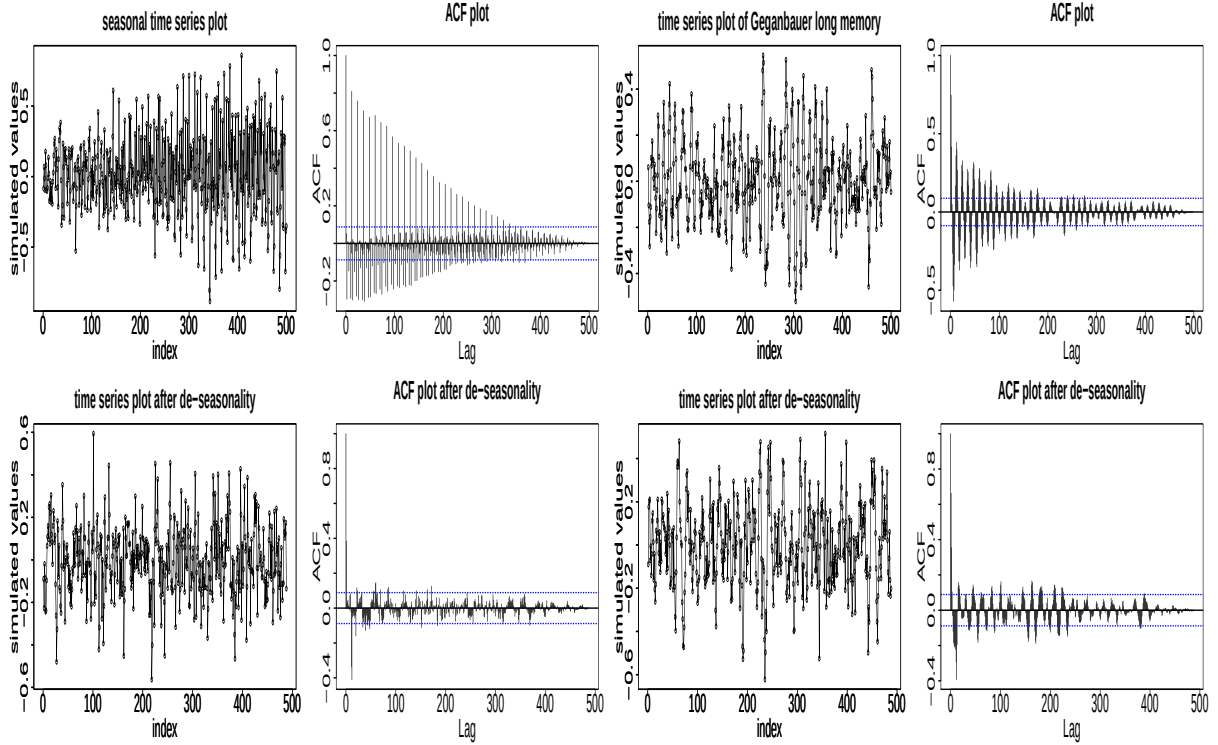


Figure 5.9. Plot of simulated seasonal ARMA(1,1) and GARMA(0,0) time series with their ACFs before (first row) and after (second row) deseasonalisation. The simulation is based on Equation (2.24) with $S = 13$ and Equation (2.16) replacing $\ln \mu_t$ by Y_t and with $u = 0.885$, $d = 0.45$. According to equation $u = \cos(\lambda_0) = \cos\left(\frac{2\pi}{\varpi}\right)$, the oscillation $\varpi = 13$ when $u = 0.885$.

Figure 5.9 shows the findings of the analysis which demonstrate, as we expect, that a seasonal differencing cannot remove the oscillatory behavior in the ACF of a Gegenbauer long memory time series (the last ACF plot). This verifies our argument that periodic oscillation in the ACF, which represents seasonality, is not the same feature as Gegenbauer long memory oscillation as it is not strictly periodic. Furthermore, we note that since we are working with count data a simple seasonal differencing operator will destroy the structure of the data rendering it not applicable to count time series

modelling, which requires positive integers for observations. To overcome this complication and verify the actual data under study does not contain a seasonal periodicity but instead a Gegenabuer oscillatory time series structure, we add additional models, as described previously in Section 2.5 (with Y_t replaced by $\ln \mu_t$ for count model), to our analysis which also incorporate periodic seasonal component. We fit these additional models to verify that these seasonal models do not provide the best fit. We report such results in Table 5.2 in which we consider a range of model structures and they include

- Seasonal GLARMA(1,1) - model (2.24) with periodic seasonality and short memory;
- Seasonal GLARFIMA(0,0) - model (2.27) with periodic seasonality and non-oscillatory long memory;
- Seasonal GLGARMA(0,0) - model (2.28) with periodic seasonality and oscillatory long memory; and
- GLGARMA(0,0) - model (2.16) with no periodic seasonality but with oscillatory long memory.

In section 5.4, we consider detailed analysis of all 136 CFTC data and plot the in-sample fit of four specific representative data sets (O-B-R-P, F-B-V-P, F-10-V-P and F-5-D-S), which corresponded to type I through to type IV of our classifications of long memory ACF structures. In this additional analysis, we fit the type II F-B-V-P data by the seasonal models with GP distribution and the period of the seasonal model is set to be 13 because of the potential quarterly periodic feature that may correspond to roll effects. This further study shows that for a data set with an oscillatory ACF feature such as F-B-V-P, the performance of GLGARMA model is better than seasonal models including seasonal GLARMA (SGLARMA) model, seasonal GLARFIMA (SGLARFIMA) model and seasonal GLGARMA (SGLGARMA) model.

Table 5.2. DIC of F-B-V-P data fitted by four types of models

Model	SGLARMA(1,1)	SGLARFIMA(0,0)	SGLGARMA(0,0)	GLGARMA(0,0)
DIC	2100.453	2133.784	2053.766	1965.988

5.5 Forecast

We calculate one-step ahead forecast for $m = 45$ time points based on the posterior predictive distributions in Equation (5.9) for PD model and Equation (5.10) for OD model and the posterior sample size of $L = 45,000$. Two sets of forecast are obtained from data with four types of long memory and with short and long positions. Overall, there are 2,520 out-of-sample forecast model fits in this section and they take 504,000 computer hours.

5.5.1 Forecast performance of the four long memory type data

The four data each represents the four types of long memory are applied to the proposed GLGARMA-PD-GP(0,0), GLGARMA-PD-GP(1,2), GLGARMA-OD-GP(0,0) and GLGARMA-OD-GP(1,0) models to evaluate the one-step ahead forecasts. To demonstrate the performances, they are compared to the one-step ahead forecasts using four competitive models, namely, the ACP, GLARMA, SGLARMA(1,1) and GAS models with Poisson distribution (GAS-P) and short memory (see Section 2.2 for a review) estimated by the Bayesian and maximum likelihood (ML) approaches. Tables 5.3 and 5.4 report the four forecast performance criteria for the four data displayed in Figures 5.1 and 5.6 and chosen to represent the four types of long memory calculated based on posterior predictive mean and median estimators. We note that NA represents a lack of relevance of a parameter for a given model, NA* indicates that the R package used for these comparative model analysis does not provide this point estimator as an output. Results show that PD models give better forecasts in general.

Table 5.3. Forecast performance criteria and estimates of u and d averaged over forecast period for four selected CFTC data with type I to IV ACF (first part).

Estimator		mean						median					
Data	model	MAE	RMSE	MAPE	MASE	$\hat{d}$	$\hat{u}$	MAE	RMSE	MAPE	MASE	$\hat{d}$	$\hat{u}$
Ö-B-R-P (Type I)	GLGARMA-PD-GP(0,0)	1.633	2.034	0.129	0.339	0.382	1.000	1.933	2.314	0.132	0.401	0.383	1.000
	GLGARMA-PD-GP(1,2)	2.643	3.835	0.138	0.549	0.320	0.245	2.689	3.952	0.136	0.558	0.320	0.123
	GLGARMA-OD-GP(0,0)	2.229	2.667	0.152	0.463	0.093	1.000	2.311	2.805	0.148	0.480	0.094	1.000
	GLGARMA-OD-GP(1,0)	5.111	7.065	0.282	1.061	0.084	0.134	5.089	7.041	0.279	1.056	0.088	0.113
	ACP(0,1)	1.833	2.203	0.107	0.381	NA	NA	1.911	2.329	0.107	0.397	NA	NA
	ACP(1,0)	8.265	10.008	0.478	1.715	NA	NA	8.533	10.330	0.489	1.771	NA	NA
	ACP(1,1)	1.950	2.342	0.118	0.405	NA	NA	2.089	2.530	0.121	0.434	NA	NA
	GLARMA-PD-P(0,1)	6.099	7.410	0.344	1.266	NA	NA	6.622	8.184	0.350	1.374	NA	NA
	GLARMA-PD-P(1,0)	2.777	3.241	0.183	0.576	NA	NA	3.022	3.682	0.177	0.627	NA	NA
	GLARMA-PD-P(1,1)	2.632	3.073	0.169	0.546	NA	NA	2.956	3.599	0.168	0.613	NA	NA
	SGLARMA(1,1)	8.072	9.424	0.507	1.675	NA	NA	8.556	10.221	0.497	1.776	NA	NA
	GLARMA-OD-P(0,1)	6.623	8.236	0.411	1.375	NA	NA	NA*	NA*	NA*	NA*	NA	NA
	GLARMA-OD-P(1,0)	6.319	7.988	0.400	1.312	NA	NA	NA*	NA*	NA*	NA*	NA	NA
	GAS-P	8.260	9.999	0.478	1.714	NA	NA	NA*	NA*	NA*	NA*	NA	NA
F-B-V-P (Type II)	GLGARMA-PD-GP(0,0)	2.742	3.705	0.173	0.619	0.418	0.888	2.711	4.284	0.145	0.612	0.417	0.887
	GLGARMA-PD-GP(1,2)	2.793	4.153	0.157	0.630	0.324	0.075	2.756	4.174	0.153	0.622	0.363	-0.350
	GLGARMA-OD-GP(0,0)	4.220	5.854	0.279	0.952	0.039	0.886	3.644	5.959	0.205	0.822	0.039	0.886
	GLGARMA-OD-GP(1,0)	3.705	5.386	0.246	0.836	0.038	0.537	3.178	5.475	0.177	0.717	0.038	0.537
	ACP(0,1)	2.504	3.631	0.171	0.565	NA	NA	2.511	3.697	0.168	0.567	NA	NA
	ACP(1,0)	7.057	10.328	0.474	1.592	NA	NA	7.111	10.354	0.478	1.605	NA	NA
	ACP(1,1)	2.499	3.306	0.174	0.564	NA	NA	2.511	3.350	0.567	0.567	NA	NA
	GLARMA-PD-P(0,1)	4.347	6.404	0.277	0.981	NA	NA	4.067	6.904	0.217	0.918	NA	NA
	GLARMA-PD-P(1,0)	2.717	4.449	0.179	0.613	NA	NA	2.556	5.031	0.129	0.577	NA	NA
	GLARMA-PD-P(1,1)	2.466	3.211	0.174	0.556	NA	NA	2.267	3.789	0.121	0.511	NA	NA
	SGLARMA(1,1)	6.310	9.592	0.432	1.424	NA	NA	5.644	9.691	0.332	1.274	NA	NA
	GLARMA-OD-P(0,1)	5.694	9.428	0.362	1.285	NA	NA	NA*	NA*	NA*	NA*	NA	NA
	GLARMA-OD-P(1,0)	5.627	9.391	0.356	1.270	NA	NA	NA*	NA*	NA*	NA*	NA	NA
	GAS-P	7.045	10.322	0.473	1.590	NA	NA	NA*	NA*	NA*	NA*	NA	NA

Note: For each model assessment, the best fitting model is indicated by italic and bold face. Values in white background are for our proposed models. Values shaded in light grey are for the competitive models estimated by Bayesian approach whereas values shaded in darker grey are for the competitive models estimated by ML approach.

Table 5.4. Forecast performance criteria and estimates of u and d averaged over forecast period for four selected CFTC data with type I to IV ACF (second part).

Estimator		mean						median					
Data	model	MAE	RMSE	MAPE	MASE	$\hat{d}$	$\hat{u}$	MAE	RMSE	MAPE	MASE	$\hat{d}$	$\hat{u}$
F-10-V-P (Type III)	GLGARMA-PD-GP(0,0)	0.684	0.888	0.059	0.335	0.432	1.000	0.867	1.164	0.069	0.424	0.432	1.000
	GLGARMA-PD-GP(1,2)	1.185	1.657	0.096	0.579	0.273	0.754	1.156	1.660	0.093	0.565	0.295	0.886
	GLGARMA-OD-GP(0,0)	1.066	1.755	0.081	0.521	0.148	1.000	1.022	1.713	0.076	0.500	0.147	1.000
	GLGARMA-OD-GP(1,0)	5.152	7.143	0.422	2.519	0.096	-0.133	5.044	7.025	0.412	2.466	0.078	-0.159
	ACP(0,1)	0.846	0.965	0.065	0.413	NA	NA	0.978	1.135	0.076	0.478	NA	NA
	ACP(1,0)	6.504	7.545	0.495	3.180	NA	NA	6.778	7.830	0.518	3.314	NA	NA
	ACP(1,1)	0.886	1.004	0.069	0.433	NA	NA	1.044	1.202	0.082	0.511	NA	NA
	GLARMA-PD-P(0,1)	5.271	6.097	0.402	2.577	NA	NA	5.867	6.663	0.454	2.868	NA	NA
	GLARMA-PD-P(1,0)	1.604	2.123	0.127	0.784	NA	NA	1.911	2.512	0.149	0.934	NA	NA
	GLARMA-PD-P(1,1)	1.580	2.021	0.124	0.772	NA	NA	2.044	2.494	0.158	1.000	NA	NA
	SGLARMA(1,1)	3.745	4.876	0.291	1.831	NA	NA	4.267	5.481	0.323	2.086	NA	NA
	GLARMA-OD-P(0,1)	4.328	5.292	0.337	2.116	NA	NA	NA*	NA*	NA*	NA*	NA	NA
	GLARMA-OD-P(1,0)	3.764	4.701	0.288	1.840	NA	NA	NA*	NA*	NA*	NA*	NA	NA
	GAS-P	6.506	7.548	0.496	3.181	NA	NA	NA*	NA*	NA*	NA*	NA	NA
F-5-D-S (Type IV)	GLGARMA-PD-GP(0,0)	1.078	1.307	0.053	0.461	0.490	0.999	1.111	1.382	0.054	0.475	0.490	0.999
	GLGARMA-PD-GP(1,2)	0.747	0.955	0.037	0.319	0.433	0.999	0.667	0.943	0.032	0.285	0.441	0.999
	GLGARMA-OD-GP(0,0)	1.418	1.783	0.064	0.606	0.119	1.000	1.422	1.814	0.064	0.608	0.120	1.000
	GLGARMA-OD-GP(1,0)	10.875	23.528	0.470	4.646	0.118	-0.730	10.800	23.372	0.467	4.614	0.119	-0.736
	ACP(0,1)	1.085	1.309	0.047	0.464	NA	NA	1.200	1.445	0.052	0.513	NA	NA
	ACP(1,0)	6.698	7.844	0.292	2.861	NA	NA	7.022	8.173	0.307	3.000	NA	NA
	ACP(1,1)	1.128	1.365	0.050	0.482	NA	NA	1.289	1.535	0.057	0.551	NA	NA
	GLARMA-PD-P(0,1)	5.423	6.313	0.237	2.317	NA	NA	5.889	6.781	0.259	2.516	NA	NA
	GLARMA-PD-P(1,0)	2.396	2.913	0.106	1.023	NA	NA	2.778	3.323	0.123	1.187	NA	NA
	GLARMA-PD-P(1,1)	2.507	3.002	0.111	1.071	NA	NA	2.822	3.416	0.123	1.206	NA	NA
	SGLARMA(1,1)	6.281	7.543	0.279	2.683	NA	NA	6.689	7.976	0.296	2.857	NA	NA
	GLARMA-OD-P(0,1)	4.390	4.919	0.181	1.709	NA	NA	NA*	NA*	NA*	NA*	NA	NA
	GLARMA-OD-P(1,0)	4.000	5.416	0.195	1.876	NA	NA	NA*	NA*	NA*	NA*	NA	NA
	GAS-P	6.716	7.852	0.293	2.869	NA	NA	NA*	NA*	NA*	NA*	NA	NA

Note: For each model assessment, the best fitting model is indicated by italic and bold face. Values in white background are for our proposed models. Values shaded in light grey are for the competitive models estimated by Bayesian approach whereas values shaded in darker grey are for the competitive models estimated by ML approach.

Comparing the performance across the proposed long memory models, we found that the (0,0) order GLGARMA-PD-GP and GLGARMA-OD-GP models for type I and III data are better than their corresponding higher order models possibly because there is no obvious trend in the data. For type II data, the GLGARMA-PD-GP(0,0) model shows similar forecast performance as the GLGARMA-PD-GP(1,2) model but the GLGARMA-OD-GP(1,0) model is better than the GLGARMA-OD-GP(0,0) model. The average of $\hat{u}$ confirms the presence of oscillatory long memory. Lastly, for type IV

data, the forecasts using GLGARMA-PD-GP(1,2) model are better than those using GLGARMA-PD-GP(0,0) model but results are reversed for GLGARMA-OD-GP models. In summary, type IV data display some trend dynamics which should be captured by higher order models.

For the competitive models, the forecasts using ACP(0,1), ACP(1,0), ACP(1,1), GLARMA-PD-P(0,1), GLARMA-PD-P(1,0), GLARMA-PD-P(1,1) and SGLARMA(1,1) models are evaluated adopting the Bayesian approach implemented via the **Rstan** package. Their performance criteria (shaded in light grey) are reported in Tables 5.3 and 5.4. Tables 5.3 and 5.4 also displays the forecast performance using GLARMA-OD-P and GAS-P models (shaded in darker grey) estimated using the ML approach via the R packages **glarma** and **GAS** respectively. Results show that our proposed GLGARMA-PD-GP models outperform the four competitive ACP, GLARMA-P, SGLARMA(1,1) and GAS-P models because these models can hardly capture the different long memory features in the data. The only exception is for the type II data where the performances of ACP and GLARMA-PD-P models are slightly better than the GLGARMA-PD-GP model for some criteria. Comparing the forecast values, Figure 5.10 shows that the forecasts using ACP(1,1) and GLARMA-PD-P(1,1) models are very similar and they are also close to the forecasts using GLGARMA-PD-GP(0,0) model. Hence we conclude that the three models have similar forecast performance.

In forecasting open interest, Figures 5.11 to 5.14 plot the observed and forecast values and the 95% predictive intervals for the four data using the best long memory models in Table Tables 5.3 and 5.4. The open interest for type II and III data displays clear oscillatory pattern throughout the forecast period and this feature is adequately captured in the forecasts. The predictive intervals in Figures 5.11 to 5.14 are all heavily right-skewed, as expected.

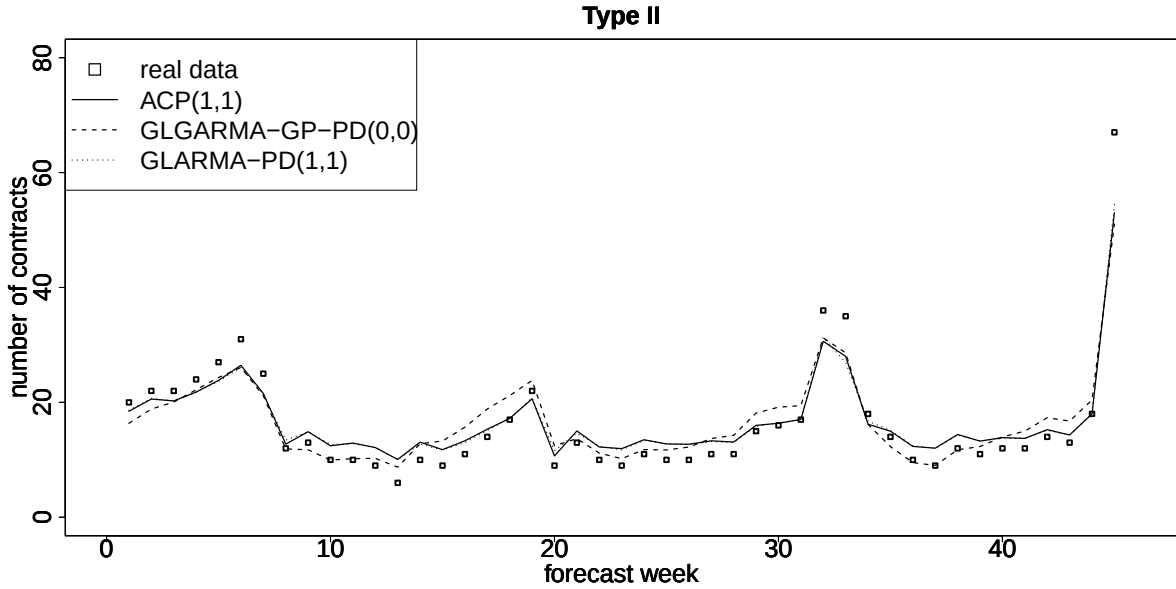


Figure 5.10. Observed and forecast using ACP(1,1), GLARMA-PD(1,1) and GLGARMA-GP-PD(0,0) models for the type II data

To study the long memory features of open interest forecasts, estimates $\hat{u}$ and $\hat{d}$ averaged across forecast period are reported in Tables 5.3 and 5.4 and graphed across forecast period with their credible intervals in Figures 5.15 and Figures 5.16, respectively for the best model. Looking at type I and III data, $\hat{u}$ is consistently one for GLGARMA-PD-GP(0,0) and GLGARMA-OD-GP(0,0) models indicating the absence of oscillatory long memory. This is further confirmed by Figure 5.15 as nearly all credible intervals contain one. We remark that type III long memory is a composite type consisting of a long and a much shorter oscillations as revealed by the ACF in Figure 5.1. It seems that the long oscillation dominates in the estimation of a common u .

Looking at the trend of $\hat{u}$ across forecast period, only type IV data in Figure 5.15 displays an upward trend. In particular, $\hat{u}$ converges to one indicating an increase in the oscillation length to infinity towards the end of the forecast period. On the other hand, the averaged $\hat{\mu}$ of 0.888 for type II data in Tables 5.3 and 5.4 indicates an estimated oscillation $\hat{\omega} = 2\pi / \cos^{-1}(0.888) = 13$ which again agrees closely with the oscillation in the observed data in Figure 5.11 and the ACF in Figure 5.1. Lastly, we note that $\hat{u}$ from higher order models in Tables 5.3 and 5.4 are lower in general but higher order models are not the best fitted models for these data. To further explore

the level of persistence, $\hat{d}$ in Tables 5.3 and 5.4 reflects moderate level for all types of open interest as it ranges from 0.27 to 0.49 for GLGARMA-PD-GP models but $\hat{d}$ from GLGARMA-OD-GP models is much lower. Only type I data show a slightly decreasing trend for $\hat{d}$ indicating decreasing level of persistence across forecast period.

Lastly, comparing the forecast performance between GLARFIMA and GLGARMA models with PD and OD errors, the more general GLGARMA model particularly developed to capture Gegenbauer long memory is preferred and PD model performs slightly better than OD model. Moreover, Figure 5.17 confirms that the GLGARMA-PD model outperforms the GLARFIMA-PD model for type II data with oscillatory long memory. Even for type IV data with $\hat{u}$ around 1, the DIC of GLGARMA model is still better possibly because the effect from the shorter forecast period is outweighed by the elongated in-sample period in the moving window.

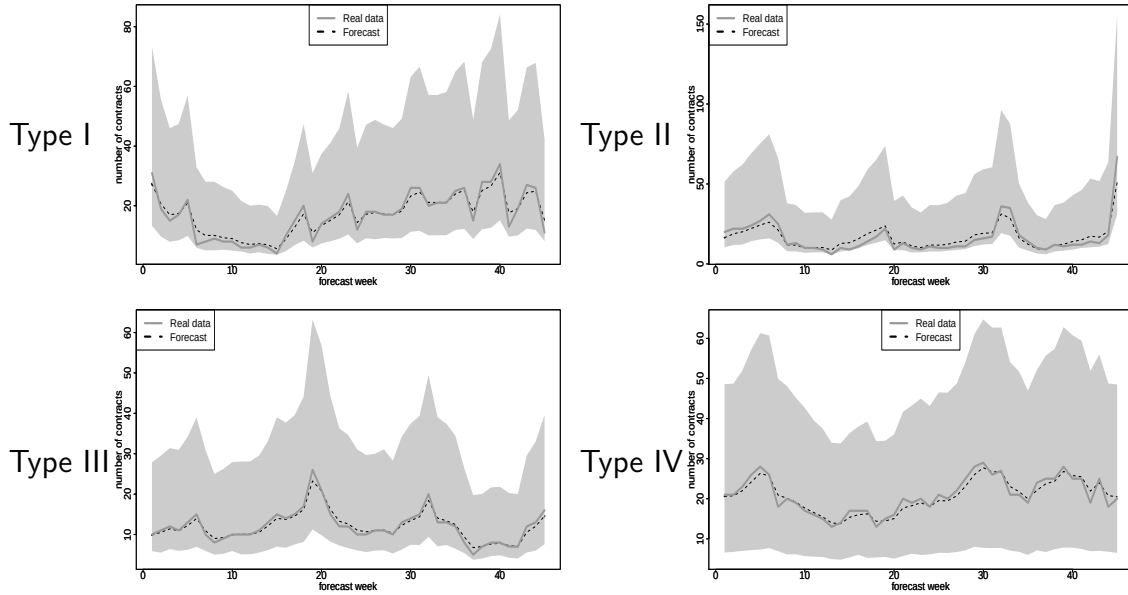


Figure 5.11. One-step ahead forecast for four types of long memory data using GLGARMA-GP-PD00 model

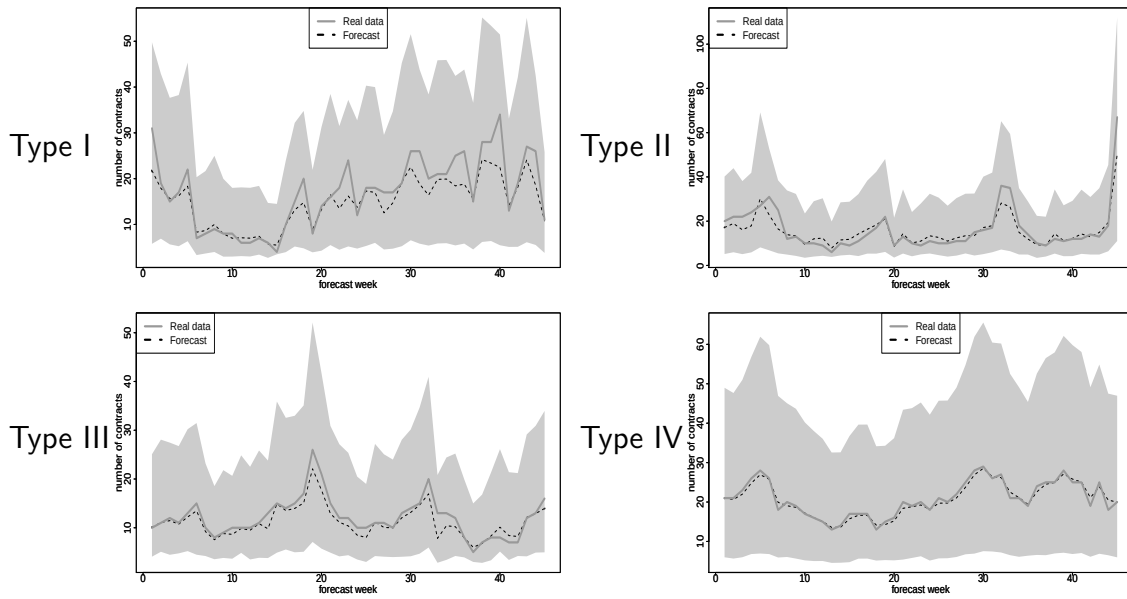


Figure 5.12. One-step ahead forecast for four types of long memory data using GLGARMA-GP-PD12 model

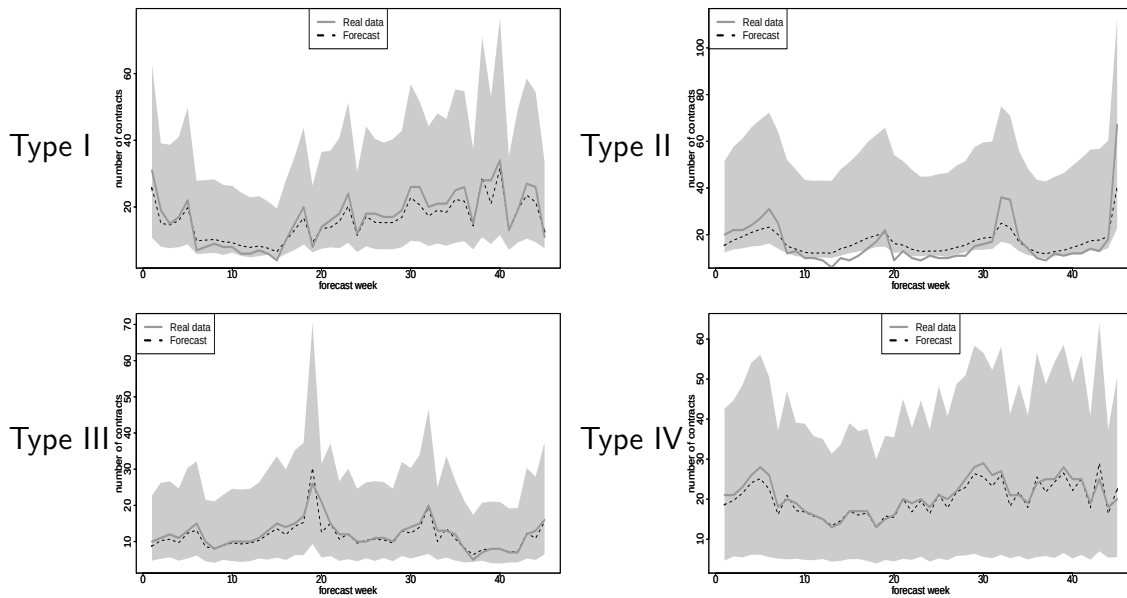


Figure 5.13. One-step ahead forecast for four types of long memory data using GLGARMA-GP-OD00 model

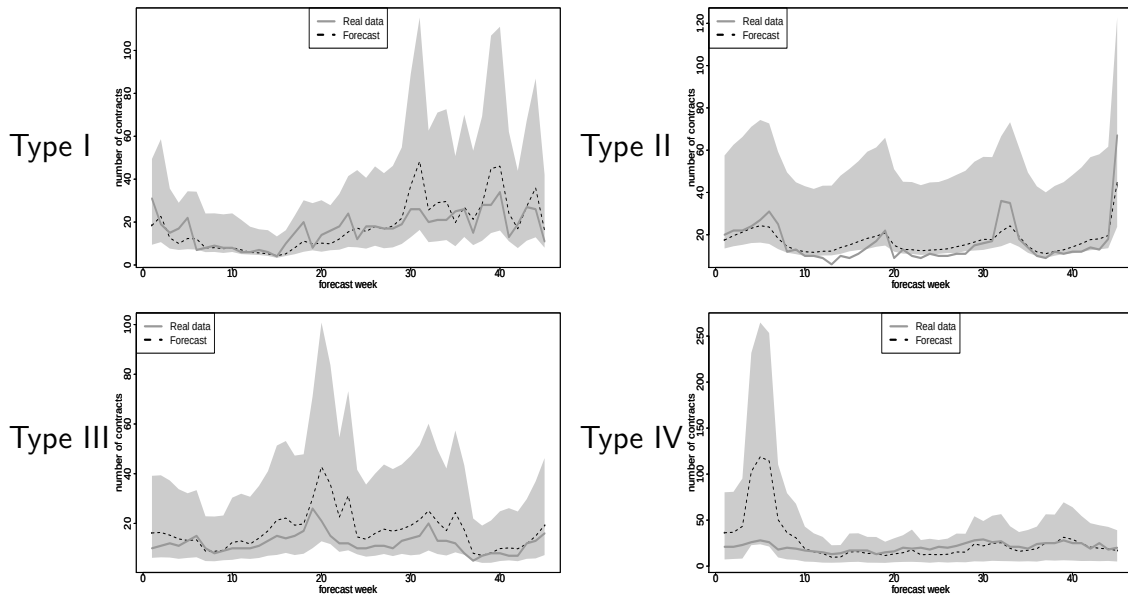


Figure 5.14. One-step ahead forecast for four types of long memory data using GLGARMA-GP-OD10 model

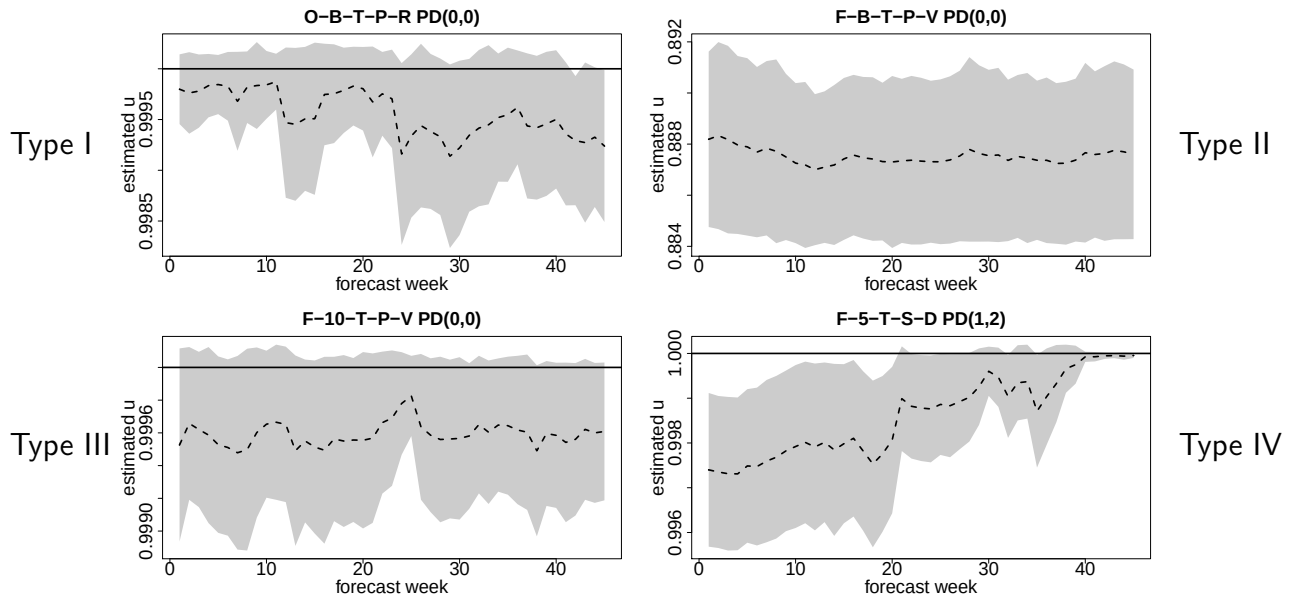


Figure 5.15. Estimates of u in one-step ahead forecast for four types of long memory data

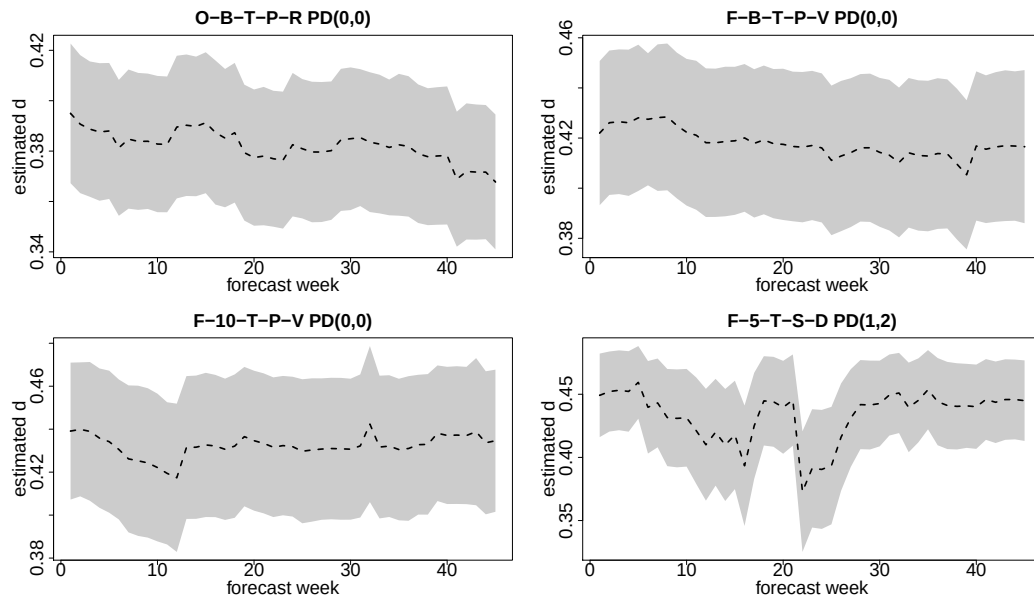


Figure 5.16. Estimates of d in one-step ahead forecast for four types of long memory data

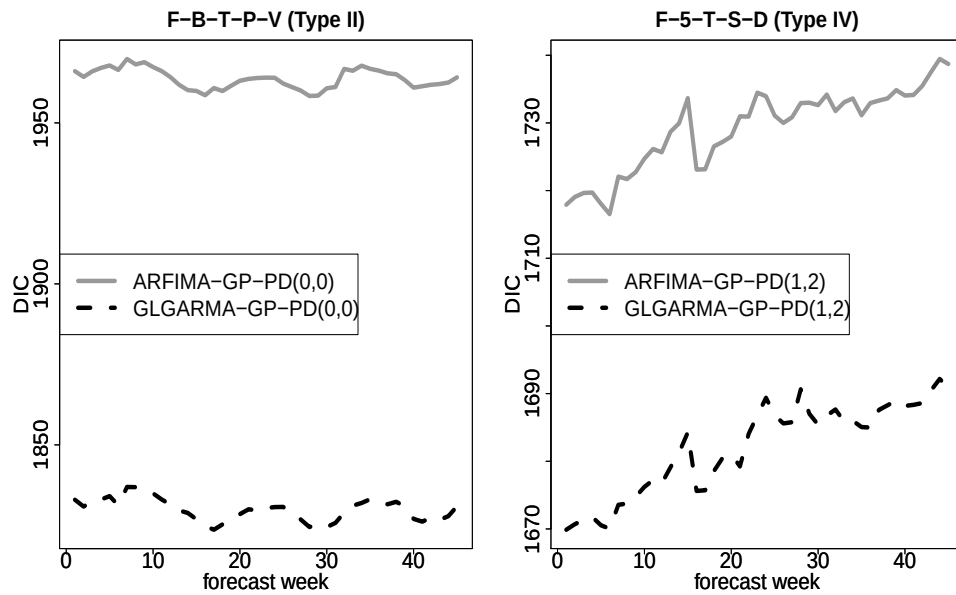


Figure 5.17. DIC of ARFIMA and GLGARMA type II and type IV long memory data

5.5.2 Forecast performance of long and short position

In the previous section, we analyse four open interest time series which represent the four types of long memory as displayed in their ACF plots. In selecting the open interest data sets to explore these long memory features, we concentrate mostly on spread data, which have best fitting models corresponding GLGARMA with an oscillatory Gegenbauer long memory. In this section we explore long and short position data, which may not have the oscillatory phenomena, and we seek to determine if our given GLGARMA model is still optimal. In this study, there are totally 1,260 number of model fits which correspond to 252,000 compute hours on the super computer.

Table 5.5 reports four forecast performance criteria for short (S) and long (L) position data. Similar to Tables 5.3 and 5.4, all 14 models are chosen to compare the forecast performance. The forecasting results from these 28 fitted models in Table 5.5 again show that the proposed Gegenbauer long memory model provides the best fit to both short and long data without obvious seasonal periodicity. The seasonal model again does not provide optimal stochastic structures for these two types of data. The estimates $\hat{d}$ reported in Table 5.5 show strong long memory features because the values of $\hat{d}$ estimated by GLGARMA-PD-GP models are close to 0.5. Furthermore, estimated values of $\hat{u}$ reflect that these two data sets are type I data as they are close to 1.

5.6 Financial outcomes of significance

Fitting the open interest via our proposed GLGARMA model reveals some important features of the CFTC data. Firstly, results of Hurst exponent and in-sample fit confirm that all open interest display long memory properties per market participant type. We further confirm this with detailed and exhaustive model fitting. In effect, we undertake a detailed and comprehensive study of the liquidity provision in futures markets for U.S. treasuries. In all studies, we firstly identify an important, previously unreported, stylised feature of liquidity profiles, by market participant type which is associated with types of long memory. We unequivocally demonstrate that there is a stylised feature of long memory in speculative liquidity provision in U.S. treasury markets. Furthermore, we demonstrate that the type of long memory present in different long, short and net positions of each market participant differs in a systematic and principled manner over

Table 5.5. Forecast performance criteria and estimates of u and d averaged over forecast period for selected long and short CFTC data with type I ACF.

Estimator		mean						median					
Data	model	MAE	RMSE	MAPE	MASE	$\hat{d}$	$\hat{u}$	MAE	RMSE	MAPE	MASE	$\hat{d}$	$\hat{u}$
F-2-A-L	GLGARMA-PD-GP(0,0)	0.563	0.715	0.022	0.451	0.494	0.999	0.489	0.730	0.019	0.391	0.496	0.999
	GLGARMA-PD-GP(1,2)	0.440	0.567	0.018	0.352	0.475	0.999	0.378	0.615	0.015	0.479	0.339	0.999
	GLGARMA-OD-GP(0,0)	5.265	6.300	0.210	4.212	0.101	0.999	5.200	6.240	0.208	4.160	0.101	0.999
	GLGARMA-OD-GP(1,0)	5.327	6.390	0.213	4.261	0.104	0.999	5.222	6.302	0.209	4.178	0.104	0.999
	ACP(0,1)	1.255	1.317	0.040	1.204	NA	NA	1.133	1.365	0.035	1.107	NA	NA
	ACP(1,0)	1.829	2.186	0.074	1.463	NA	NA	1.933	2.324	0.077	1.547	NA	NA
	ACP(1,1)	1.369	1.459	0.045	1.295	NA	NA	1.356	1.596	0.044	1.284	NA	NA
	GLARMA-PD-P(0,1)	1.365	1.629	0.055	1.092	NA	NA	1.733	2.011	0.068	1.387	NA	NA
	GLARMA-PD-P(1,0)	0.772	0.947	0.032	0.618	NA	NA	0.889	1.135	0.035	0.711	NA	NA
	GLARMA-PD-P(1,1)	0.730	0.888	0.030	0.584	NA	NA	0.956	1.145	0.037	0.764	NA	NA
	SGLARMA(1,1)	1.918	2.356	0.078	1.535	NA	NA	2.133	2.616	0.083	1.707	NA	NA
	GLARMA-OD-P(0,1)	1.427	1.810	0.057	1.141	NA	NA	NA*	NA*	NA*	NA*	NA	NA
	GLARMA-OD-P(1,0)	1.490	1.857	0.059	1.192	NA	NA	NA*	NA*	NA*	NA*	NA	NA
	GAS-P	1.828	2.181	0.074	1.462	NA	NA	NA*	NA*	NA*	NA*	NA	NA
F-10-N-S	GLGARMA-PD-GP(0,0)	1.428	1.981	0.044	0.757	0.432	0.999	1.489	2.082	0.045	0.789	0.459	1.000
	GLGARMA-PD-GP(1,2)	1.366	1.854	0.042	0.724	0.342	1.000	1.400	1.856	0.043	0.742	0.339	1.000
	GLGARMA-OD-GP(0,0)	0.960	1.202	0.030	0.509	0.049	1.000	0.844	1.193	0.026	0.448	0.048	1.000
	GLGARMA-OD-GP(1,0)	0.950	1.168	0.030	0.504	0.054	1.000	0.867	1.145	0.027	0.459	0.054	1.000
	ACP(0,1)	6.975	7.438	0.215	3.698	NA	NA	7.067	7.540	0.218	3.746	NA	NA
	ACP(1,0)	7.404	7.979	0.226	3.925	NA	NA	7.911	8.449	0.242	4.194	NA	NA
	ACP(1,1)	6.983	7.512	0.215	3.702	NA	NA	7.111	7.651	0.219	3.770	NA	NA
	GLARMA-PD-P(0,1)	3.833	4.743	0.114	2.032	NA	NA	4.000	4.908	0.119	2.120	NA	NA
	GLARMA-PD-P(1,0)	3.850	4.767	0.114	2.041	NA	NA	4.000	4.900	0.125	2.100	NA	NA
	GLARMA-PD-P(1,1)	3.831	4.739	0.114	2.031	NA	NA	4.000	4.808	0.110	2.000	NA	NA
	SGLARMA(1,1)	3.502	4.424	0.104	1.856	NA	NA	3.600	4.536	0.107	1.908	NA	NA
	GLARMA-OD-P(0,1)	2.751	3.662	0.081	1.458	NA	NA	NA*	NA*	NA*	NA*	NA	NA
	GLARMA-OD-P(1,0)	2.846	3.776	0.084	1.509	NA	NA	NA*	NA*	NA*	NA*	NA	NA
	GAS-P	3.846	4.762	0.114	2.039	NA	NA	NA*	NA*	NA*	NA*	NA	NA

Note: For each model assessment, the best fitting model is indicated by italic and bold face. Values in white background are for our proposed models. Values shaded in light grey are for the competitive models estimated by Bayesian approach whereas values shaded in darker grey are for the competitive models estimated by ML approach.

the last decade of liquidity data from the CFTC. The implications of this finding are marked as important for forecasting and developing greater insight into price formation and market participant dynamics. Furthermore, we demonstrate that such features are persistent across the term-structure of the U.S. treasury securities, both in the short term maturity contracts and the longer dated maturities, where we again identify a range of different long memory structures.

Importantly for this chapter, we demonstrate that by incorporation of this important stylised feature, one can significantly improve the accuracy of the ability to perform forecasts of the liquidity provision of weekly long, short and net positions of each individual market participant type. In our opinion this is an important finding that should aid future studies that aim to study dynamic relationships between liquidity and price formation.

Looking at market participants, open interest from dealers or leveraged funds advisors/operators classified as uninformed traders are more likely to display oscillatory long memory. This oscillatory long memory structure can surely assist and hence improve the forecast performance. Moreover the estimated oscillation for type II data agrees closely with the observed data and its ACF. Table 5.1 also shows that the presence of another group of uninformed traders who display more complicated long memory features for the open interest with multiple and/or time varying oscillations. An example is the chosen type IV long memory data coming from dealers with the forecasted oscillation parameter gradually approaching one indicating the absence of oscillation. This indicates the difficulties to forecast market liquidity which displays multi-oscillation with interaction for this group of uninformed traders. On the other hand, open interest from informed traders like asset managers displays mainly basic long memory without noticeable oscillatory pattern. The inclusion of one in most forecast credible intervals for the oscillation parameter shows that the GLGARMA model can still capture this basic type of long memory feature well. Lastly, Figure 5.4 shows that higher order models are more popular indicating the presence of trend movements although our four chosen data do not display obvious trend pattern.

5.7 Conclusion

This chapter demonstrates that there are four different long memory features present in U.S. Treasury futures relating to different market participant behaviours. This finding motivates an extensive study for 136 CFTC data sets which represent some of the most active futures contracts traded in markets. We demonstrate for the first time that for this important class of U.S. treasury futures, there are clear distinctions between different market participant trading characteristics, that are captured when one observes the processes appropriately through long memory time series structures of different types.

Through analysing their ACFs, we summarise the long memory structures into four types, namely basic, oscillatory, combined and irregular. We find that basic and irregular types are more prevalent. We further clarify that the seasonality which may be present in the spread data due to future roll periodicity is distinct from oscillatory long memory through ACF/periodogram plots, simulation study, in-sample case study and forecast experiments. Further in-sample model fit and forecast experiments confirm that seasonal time series model cannot describe the feature of oscillatory long memory for spread data which may be seasonal and for the long and short data which are not seasonal. Hence we conclude with confidence that the Gegenbauer long memory feature in the CFTC data is distinct from the periodic seasonal feature.

To foster a deeper understanding of how the liquidity dynamics depend on trading activity described by types of trade and traders who possess different access to information, we fit 40 models with different distribution choices, model types, long memory types and orders to each data. Results show that GP is consistently the best distribution choice and GARMA model is preferred to the ARFIMA model due to its greater flexibility. We then select four data which represent the four types of long memory and forecast one-step ahead for 45 time points using the GLGARMA-GP model with submodel and order types PD(0,0), PD(1,2), OD(0,0) and OD(1,0). We also compare the forecasts with three short memory models, namely the ACP, GLARMA and GAS models. Moreover, to demonstrate the difference between seasonal model and Gegenbauer long memory model, we add one more seasonal model and two more long and short position data sets. Results show that the PD model generally provides better forecasts and the order depends on the presence/absence of trends. Based on four performance criteria, our proposed GLGARMA-PD-GP model is shown to provide better open interest forecasts than the GLARFIMA-PD-GP model and the four types of short memory models including the seasonal model for a group of uninformed traders, the dealers and leverage funds advisors/operators with open interest exhibits oscillatory long memory. Lastly, we demonstrate that the GLGARMA-PD-GP model outperforms in forecasts except for the type II data but it still gives comparable forecasts. In summary, this non-trivial experimental design requires 5,444 in-sample model fits and 3,870 out-of-sample forecast model fits which bring the total number of model fits to 9,224 and compute hours to 1,844,000 on super computers. With such comprehensive model fittings and forecasting experiments, we can conclude with confidence that the GLGARMA model provides the best model structure for all types of open interest data.

We have seen how our proposed GLGARMA model is applicable to analyse financial time series. In the next chapter, we further demonstrate the applicability of GLGARMA model in mortality modelling in insurance.

CHAPTER 6

Analysis of mortality data

6.1 Insurance background reviews

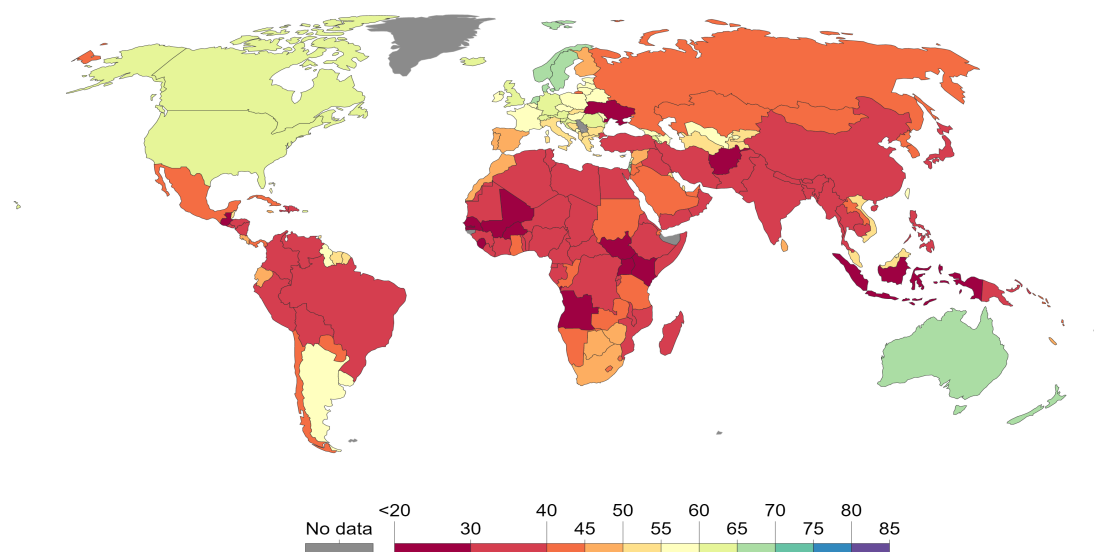
Many developed countries have experienced a significant increase in the life expectancy over the last century due to the improvements in medical technologies and scientific advances in disease control. This is demonstrated in Figure 6.1 which reports the distribution of life expectancies across countries in 1941 and 2015. This increase drives increases in longevity risk since it means that life insurance policyholders and pensioners will live longer than expected. The resulting declining trends in mortality rates also produce great challenges for pricing and reserving in life insurance, especially when expected payout levels need reconsideration to account for the longer than expected life expectancy for policyholders. When this also couples with decreasing fertility rates and aging populations, one further sees a decreasing trend in the number of active taxpayers in developed country populations in the future. This in turn can also cause adverse effects in policy making and pension planning from a government perspective. Consequently, accurate mortality rates forecasting and life expectancy projection have drawn considerable attention not only by life insurance companies and pension funds but also by governments.

As Forecast of mortality rates is important for demographers to project population and actuaries to provide cash flow projections and manage reserves. Subsequently, the forecasting will provide insightful information for policy making (Koissi and Shapiro, 2008). There will be adverse effects on pension funds and annuity insurers if the mortality forecasts are not accurate enough because under or over estimated mortality rates will cause either a high-risk profile for pension funds and annuity insurers or excess

premium costs for policyholders ([Giacometti et al., 2012](#)). As decreasing trend in mortality is observed in many of these studies across countries, genders and age groups, we conjecture that a long memory feature in mortality data time series may be an option to explain this phenomenon adequately and this additional stylised fact of mortality data should be accommodated to the popular Lee-Carter (LC) type mortality models. We argue that the existence of long memory in mortality data improves the understanding of features of mortality and provides a new approach to establish mortality models that will resolve the problem of underestimation of mortality rates. Using these new models with long memory feature, forecasts of mortality rates and hence estimate of life expectancies in life table construction can be more accurate. This Chapter starts the application of long memory models to mortality data by first detecting the presence of this long memory feature in the death counts data of 16 countries downloaded from Human Mortality Database (HMD)([Vladimir et al., 2017](#)).

Life expectancy, 1941

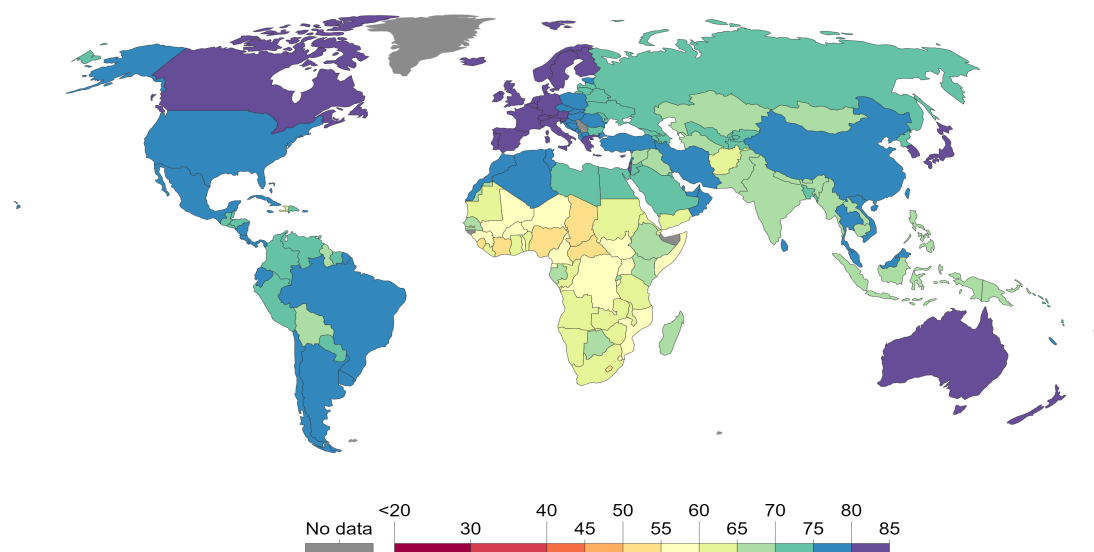
Shown is period life expectancy at birth. This corresponds to an estimate of the average number of years a newborn infant would live if prevailing patterns of mortality at the time of its birth were to stay the same throughout its life



Source: Clio-Infra estimates until 1949; UN Population Division from 1950 to 2015
OurWorldInData.org/life-expectancy-how-is-it-calculated-and-how-should-it-be-interpreted/ • CC BY-SA

Life expectancy, 2015

Shown is period life expectancy at birth. This corresponds to an estimate of the average number of years a newborn infant would live if prevailing patterns of mortality at the time of its birth were to stay the same throughout its life



Source: Clio-Infra estimates until 1949; UN Population Division from 1950 to 2015
OurWorldInData.org/life-expectancy-how-is-it-calculated-and-how-should-it-be-interpreted/ • CC BY-SA

Figure 6.1. The top figure shows global life expectancy by country in 1941, and the bottom figure shows global life expectancy in 2015, source [Roser \(2018\)](#)

6.2 Human Mortality Database

The human mortality data of 16 countries was downloaded from HMD on 8th April 2017. To statistically measure directly the persistence and long memory properties in human mortality data, we adopt the Hurst exponent measure H (see Section 4.3.1). This is theoretically a useful approach as it is directly related to our proposed Gegenbauer long memory processes with no modelling. We can apply existing estimators of H as described in Section 4.3.1 to statistically validate the existence of this stochastic feature in the death count series. We only report the results of the Rescaled Range R/S estimator since the other two estimators provide very similar results.

We perform a comprehensive study of H using on the motivating HMD from 16 countries which provides detailed mortality and population data to the public. Table 6.1 reports the data length for each country together with their abbreviated name.

Table 6.1. Data length and abbreviation for country names

Full name	Australia	Belgium	Canada	Denmark	Finland	France	Iceland	Italy
Abbrev.	AU	BE	CA	DK	FI	FR	IS	IT
Data length	94	170	91	180	138	199	176	141
Full name	Netherlands	Norway	Spain	Sweden	Switzerland	U.K.	U.S.A.	Japan
Abbrev.	NL	NO	ES	SE	CH	GB	US	JP
Data length	163	169	107	264	139	92	83	68

Each data set contains the number of death $y_{x,t}^*$ and the exposure $E_{x,t}^*$ cross-classified by genders and years and aggregated into 21 age groups, namely, age 0 group, age 1-4 group, age 5-10 group, age 11-15 group etc. The values of $y_{x,t}^*$ and $E_{x,t}^*$ are scaled according to the following equation:

$$y_{x,t} = [y_{x,t}^*/c_x] \quad \text{and} \quad E_{x,t} = E_{x,t}^*/c_x, \quad (6.1)$$

where $[\cdot]$ represents rounding to the nearest integer, and the scaling factor c_x is determined by

$$c_x = 10^k \quad \text{if} \quad \min y_{x,t}^* \in (10^{k+1}, 10^{k+2}),$$

such that the range of $y_{x,t}$ is approximately within $(0, 100)$ and k is an integer selected to achieve this. This re-scaling of data only changes the unit and preserves the features.

In other words, if the number of death $y_{x,t}^*$ and the exposure $E_{x,t}^*$ are scaled by same amount c_x , the features of data will maintain. This can be explained by following equation:

$$\mathbb{E}(y_{x,t}^*/c_x|E_{x,t}^*) \approx \mathbb{E}([y_{x,t}^*/c_x]|E_{x,t}^*) = E_{x,t}^*\mu_{x,t}/c_x = E_{x,t}\mu_{x,t}. \quad (6.2)$$

Furthermore, we find statistically significant evidence for the presence of long memory features which is present irrespective of how the data sets are aggregated or disaggregated. To illustrate this, we take Australia female death count as an example. There are different ways to aggregate age groups. For example, taking five age groups for each type, one can form single age $\{0, 1, 2, 3, 4\}$ for Type 1 or alternatively $\{0, 1-4, 5-9, 10-14, 15-19\}$ for Type 2 or $\{0-10, 11-20, 21-30, 31-40, 41-50\}$ for Type 3. Table 6.2 shows that the estimated H for the three types of aggregation are similar. This feature is important to demonstrate since, unlike weak serial correlation in a time series which may be diminished or removed via aggregation, stronger persistence in the form of long memory prevails as a structural feature in aggregation typically.

Table 6.2. Estimated Hurst exponent for three types of aggregation of age group, for example, Age group 2 for Type 1 is 1, for Type 2 is 1-4 and for Type 3 is 11-20.

Aggregated types	Age group 1	Age group 2	Age group 3	Age group 4	Age group 5
Type 1	0.7905	0.7954	0.7905	0.7901	0.7882
Type 2	0.7905	0.7942	0.7825	0.7888	0.7713
Type 3	0.7913	0.7855	0.8089	0.8017	0.7731

Since all H are clearly above 0.5, there is clear evidence for the presence of long memory irrespective of how the data sets are aggregated or disaggregated. Then H is used to study the long memory features across different factors. We also remark that we adopt the Type 2 age group which is the age group provided in HMD data for all subsequent analyses.

6.3 Identifying long memory pattern

In the previous section, groups of mortality data are introduced with the explanation of data scaling and age group aggregation. We demonstrate in this section the prevalence of a long memory feature in the death count data from four perspectives by studying

the pattern of long memory across different countries, age groups, genders and time periods.

All countries demonstrate some degrees of long memory across all age groups. Heatmap in Figure 6.2 displays the estimated H for female and male in 21 age groups across 16 different countries. For Iceland female, compared with other countries, the estimated H is relatively lower between age 25 to 74. It may be the result of a smaller population size such that the mortality process becomes incapable to sustain a stable mortality feature consistently over time. For male population between age 15 to 64, the mortality persistence (estimated H) is lower than female in most of the countries except Belgium, Sweden, Switzerland and U.K.. For U.S. and Japan, both female and male mortality features show generally weaker mortality persistence than other countries. For U.S., this feature can be explained by the multi-cultural and diverse population immigration. For Japan, one possible explanation is that the length of mortality data is too short to provide enough information to estimate H in nonparametric setting.

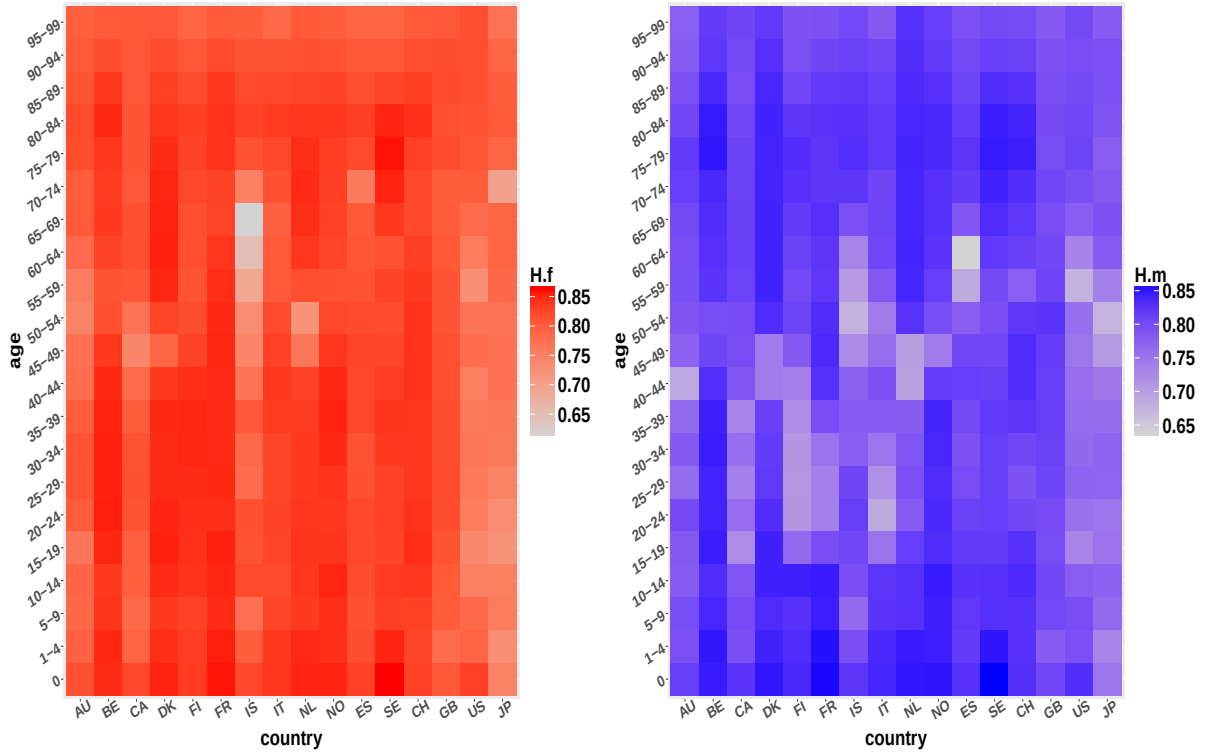


Figure 6.2. Heat map of estimated H across countries and age groups for female (left) and male (right).

6.3.1 Long memory pattern across age group by gender

After an overview of how the three factors of age, gender and country interact in the long memory patterns as present in the mortality data, we then investigate the factors of age and country separately. In this section, we look at age effect by gender on the long memory pattern after aggregating over countries. Figures 6.3 and 6.4 below display the estimated H from both female and male groups respectively in different ages across 15 countries. The trend of long memory as measured by H is quite different between female and male populations. For female populations, the estimated H is in general higher and displays stable values before age 40 and a U-shape trend afterwards. For male populations, the estimated H has a wider spread, and the U-shape trend starts from an earlier age of 15. Results indicate that the mortality change (improvement) over the years is more persistent for female than male and for younger than middle age groups. Moreover, such improvements vary more across countries for male than female. For the senior age groups ranging between 80-99, there is a consistent downward trend of estimated H for both genders. This feature is due to the fact that the mortality persistence is more difficult to detect in a smaller population of seniors. There is also less variation in estimated H across countries because the health deterioration for seniors is a common feature for all countries. To facilitate the comparison of long memory pattern between genders, the boxplots in Figures 6.3 and 6.4 are plotted side by side for each age group in Figure 6.5. The mortality persistence is consistent between genders only in young age groups. For age 15 to 49, the difference is more obvious with male showing a clear downward trend. This observation agrees, of course, with those from Figures 6.3 and 6.4.

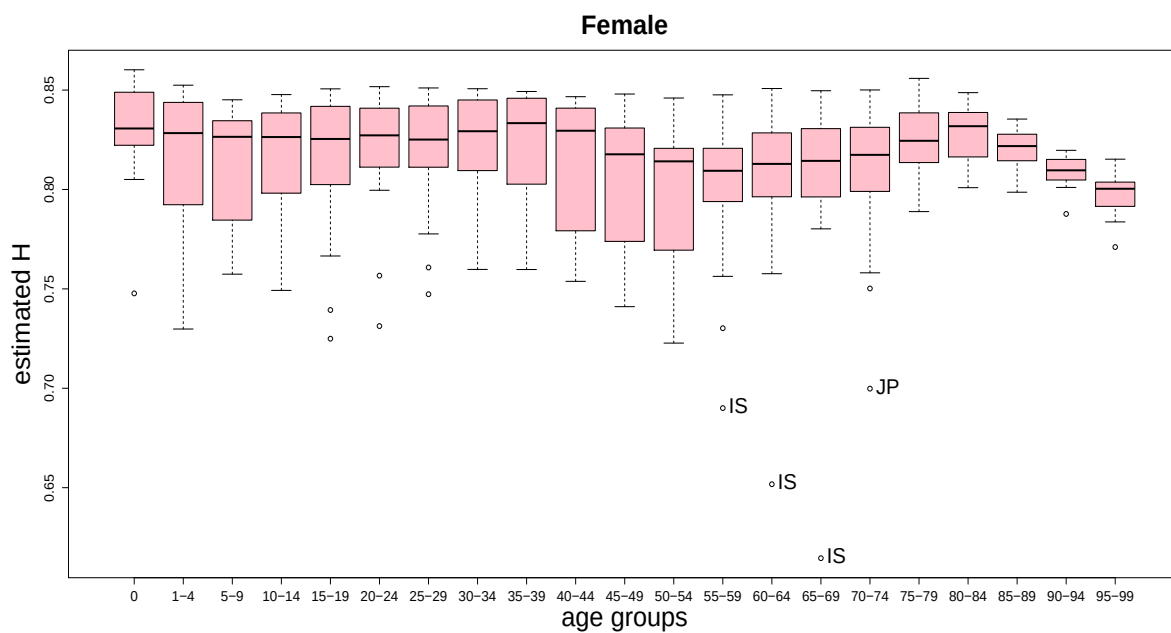


Figure 6.3. Boxplot of estimated H across age groups for female aggregated over countries.

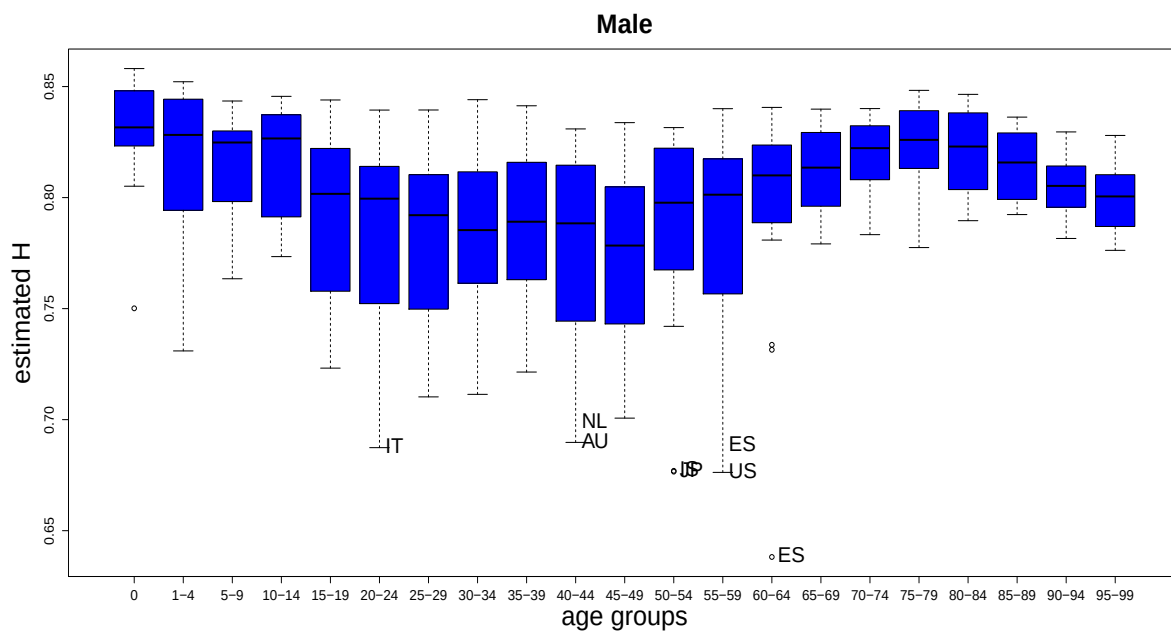


Figure 6.4. Boxplot of estimated H across age groups for male aggregated over countries.

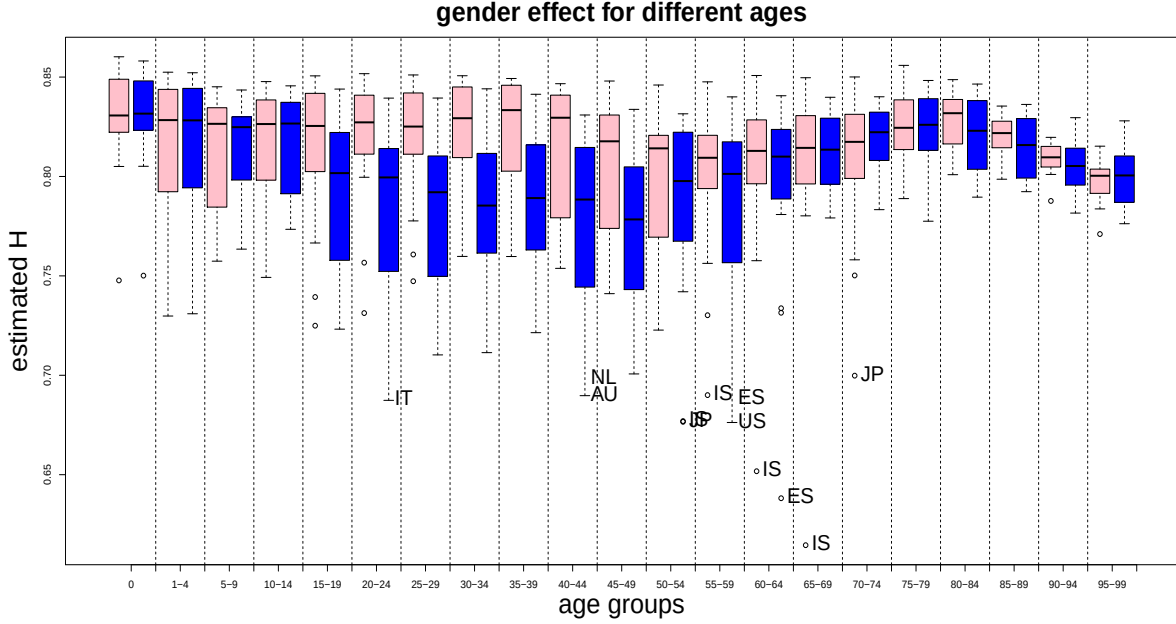


Figure 6.5. Boxplot of estimated H across age groups aggregated over countries to show gender effect. For each age group, the first boxplot is female (pink) and the second is male (blue).

6.3.2 Long memory pattern across countries by gender

Figures 6.6 and 6.7 below compare the estimated H across 16 countries by gender aggregated over 21 age groups, which indicate mortality persistence over the years. Again, the estimated H has lower values and wider spreads for male populations. The exceptions of Iceland, Australia and Japan are marginal. This difference in spread is most apparent for Finland, France, Italy and Netherlands. Some countries like Belgium and Denmark display high level of mortality persistence over the years whereas Iceland, US and Japan typically exhibit lower levels of mortality persistence. Belgium and Denmark are economically and politically stable European countries, and so their high level of mortality persistence is not unexpected. As Iceland is a small country, US is a multi-cultural country, and Japan was seriously affected by world war 2 (WW2), their lower level of mortality persistence is again not surprising.

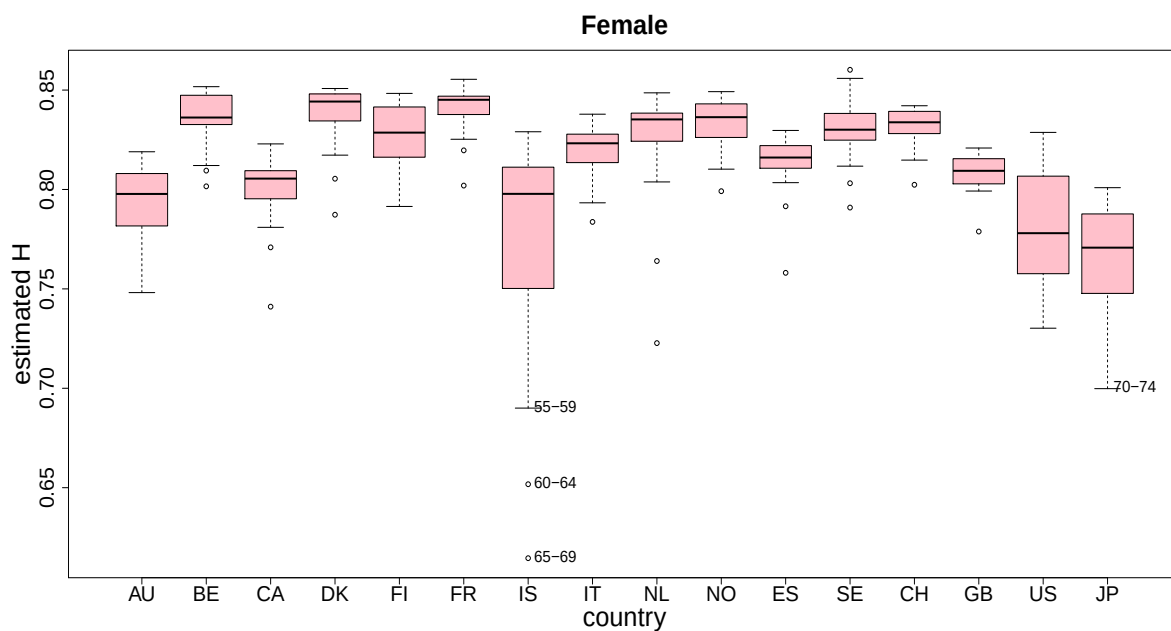


Figure 6.6. Boxplot of estimated H across countries for female aggregated over ages.

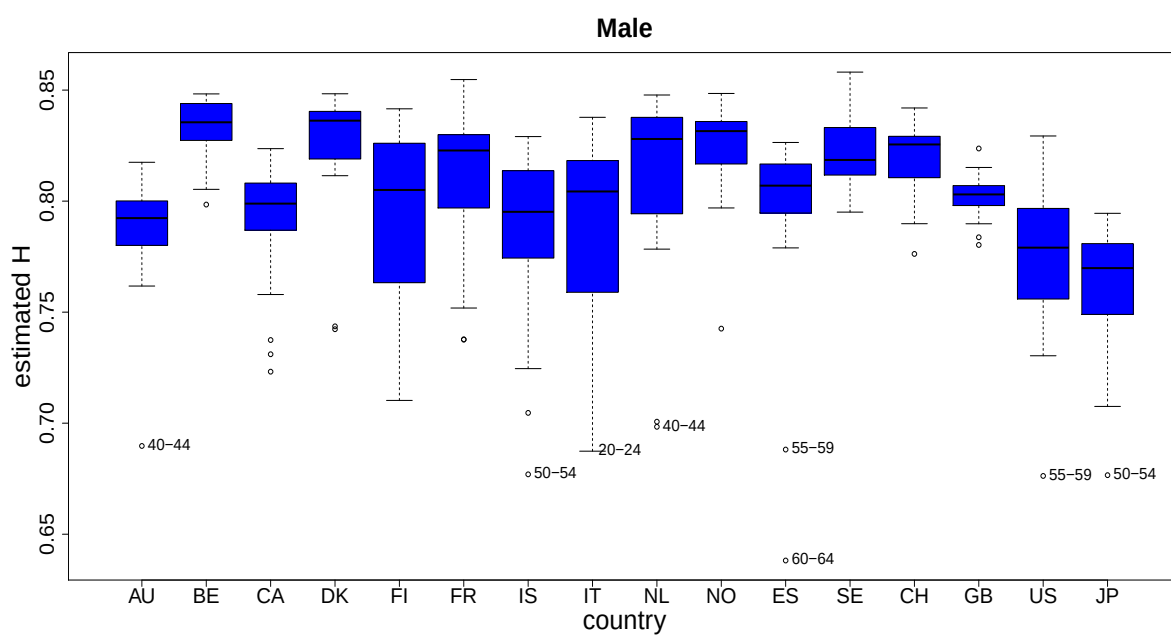


Figure 6.7. Boxplot of estimated H across countries for male aggregated over ages.

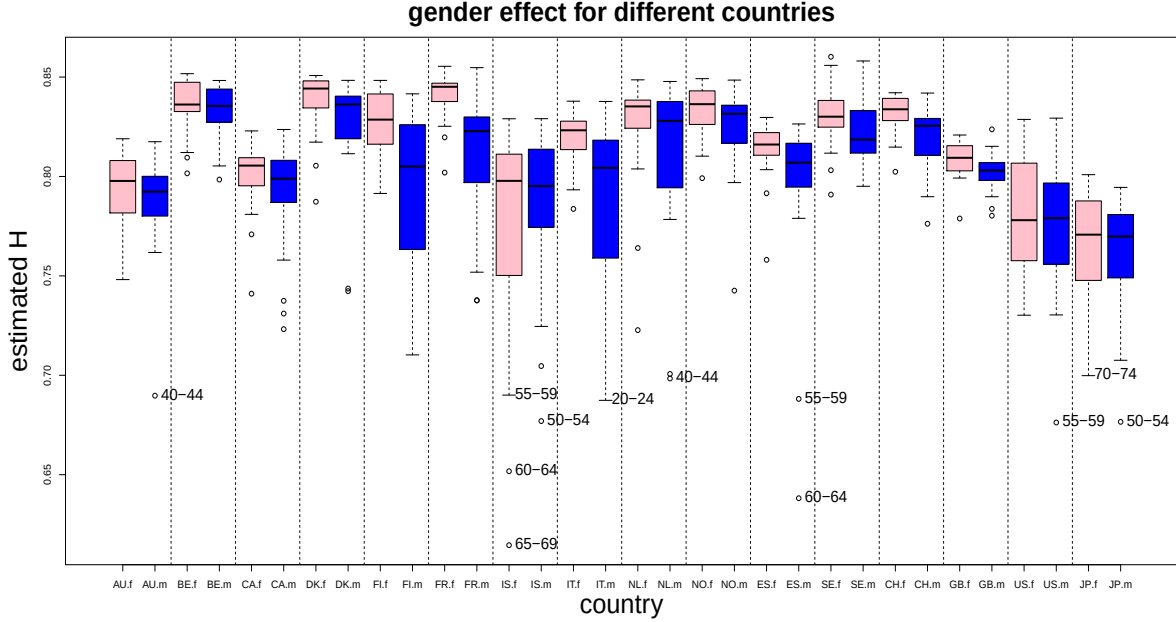


Figure 6.8. Boxplot of estimated H across countries aggregated over ages to show gender effect. For each age group, the first boxplot is female (pink) and the second is male (blue).

Figure 6.8 compares the estimated H for males and females across 15 different countries aggregated over ages. It shows that the spread of H is wider for male than female in Finland, France, Italy, Netherlands and Spain. A possible reason is the effect of the two world wars which affect the mortality of European male more. The next section studies the effect of wars in more details. Again, the spread of H for Iceland is more and the spread is wider for female than male.

6.3.3 Long memory pattern before and after world war two

In this study, we seek to determine the strength of a population mortality shock that alters the persistence features of the mortality processes in a range of countries. In order to facilitate such study, we focus on two population regimes where the post-shock structural properties of the time series may be altered. We show that such a population shock due to the effect of world war one (WW1) and WW2 influences the mortality processes in different countries in various ways. In particular, countries heavily involved in the wars seem to have different structural response in the mortality processes than those

less involved. For instance, we observe evidence similar in the nature to the results obtained in [Fung et al. \(2017a\)](#), which indicates that as a consequence of mortality shock, there is a creation of a cohort effect in European populations. However, despite this, the long memory structure prevails as a statistically significant feature. The features of long memory structures of 21 age groups and 15 countries before and after WW2 are investigated, excluding Japan because there is no fully available record of mortality data before 1947 in Japan. Heatmaps in [Figure 6.9](#) and [Figure 6.10](#) show that the H estimates for all age groups after WW2 increases. Such an increase of mortality persistence indicates a more stable living environment and better living standards including food quality and medical services after WW2. However, the persistence for youth and senior age groups is higher than the age groups 25-40, consistent with our previous observations. To study the age group effect and country effect separately, we aggregate the H estimates in the Heatmap over countries and age groups respectively in [Figure 6.11](#) and [Figure 6.12](#). [Figure 6.11](#) shows that the distribution of H estimates before WW2 spreads more widely than the H after WW2 for all age groups. This observation indicates that the persistence of mortality differs more across countries before WW2. Before WW2, the median of H for the male is mostly lower than the median for female aged from 15-45. However, after WW2, such difference disappears or even reverses. This implies that the mortality persistence for male increases relatively to female after WW2. [Figure 6.12](#) illustrates that the pre-war and post-war periods for these 16 countries are totally different in their mortality persistence. For Australia, Canada, Italy, Spain, U.K. and U.S.A, the median of H before WW2 is smaller than the median after WW2. This agrees with the previous observation. For Belgium, Denmark, Finland, France, Netherlands and Norway, the range of estimated H for male before WW2 is much wider than male after WW2. The mortality persistence for male before WW2 varies more across age groups among these European countries than after WW2.

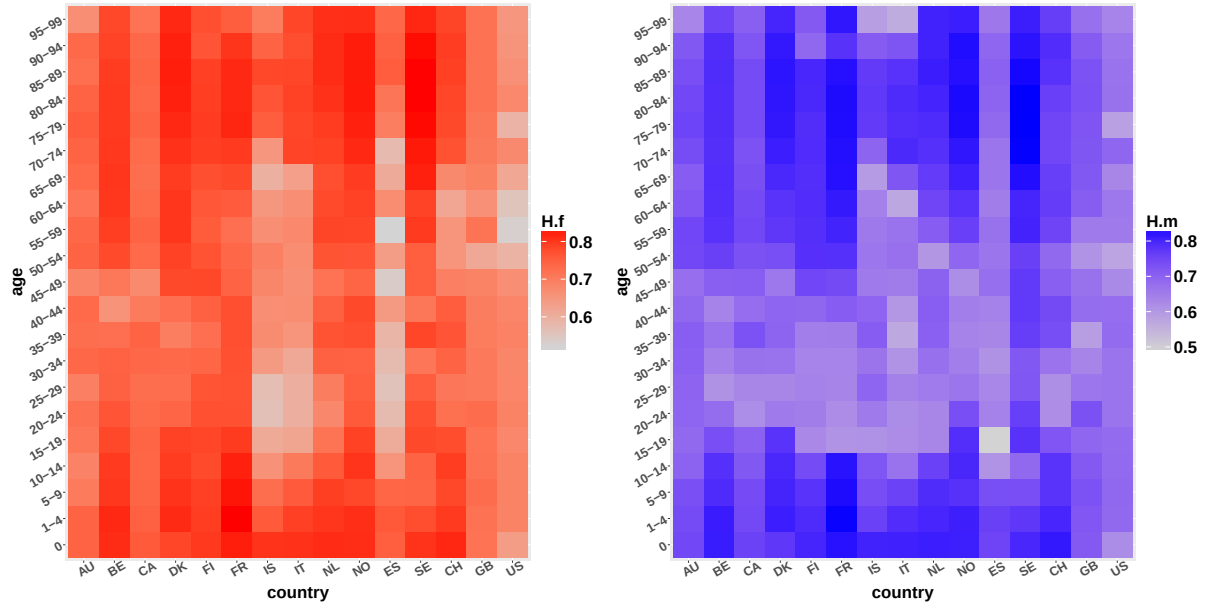


Figure 6.9. Heat map of estimated Hurst exponent across gender, age groups and countries before WW2 for female (left) and male (right).

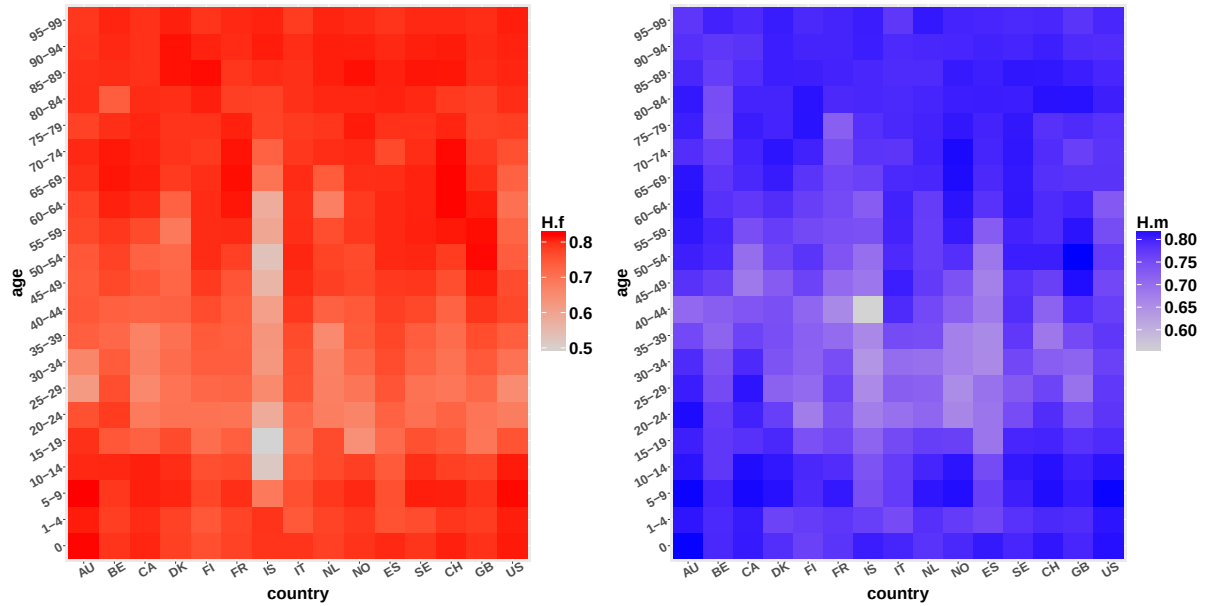


Figure 6.10. Heat map of estimated Hurst exponent across gender, age groups and countries after WW2 for female (left) and male (right).

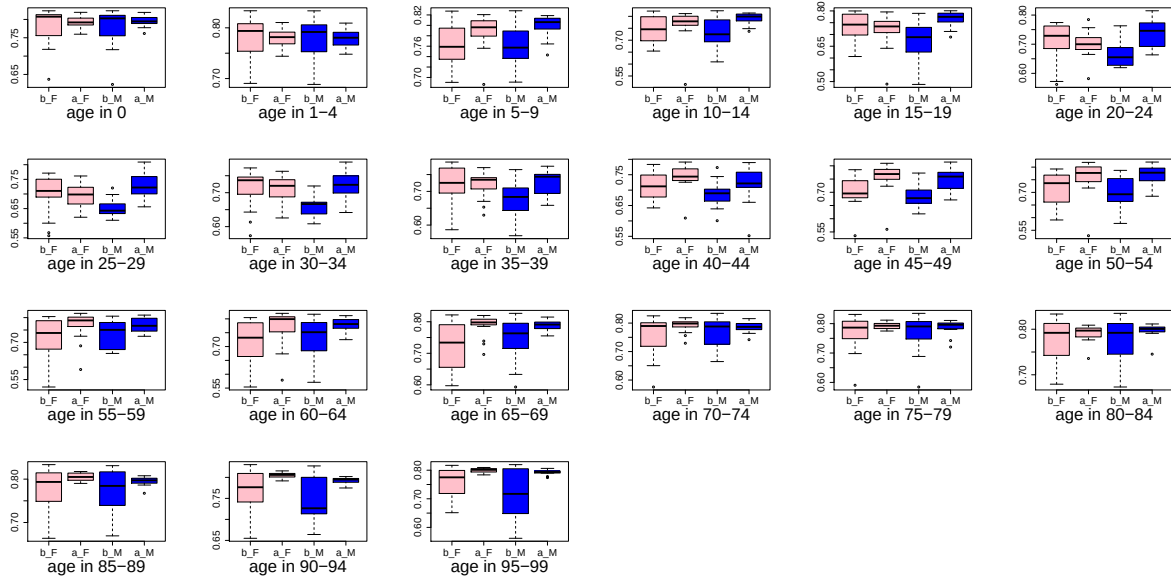


Figure 6.11. Boxplot of estimated H before/after world war two across age groups aggregated over countries with b_F, a_F b_M, and a_M representing before WW2 female (red), after WW2 female, before WW2 male (blue), and after WW2 male

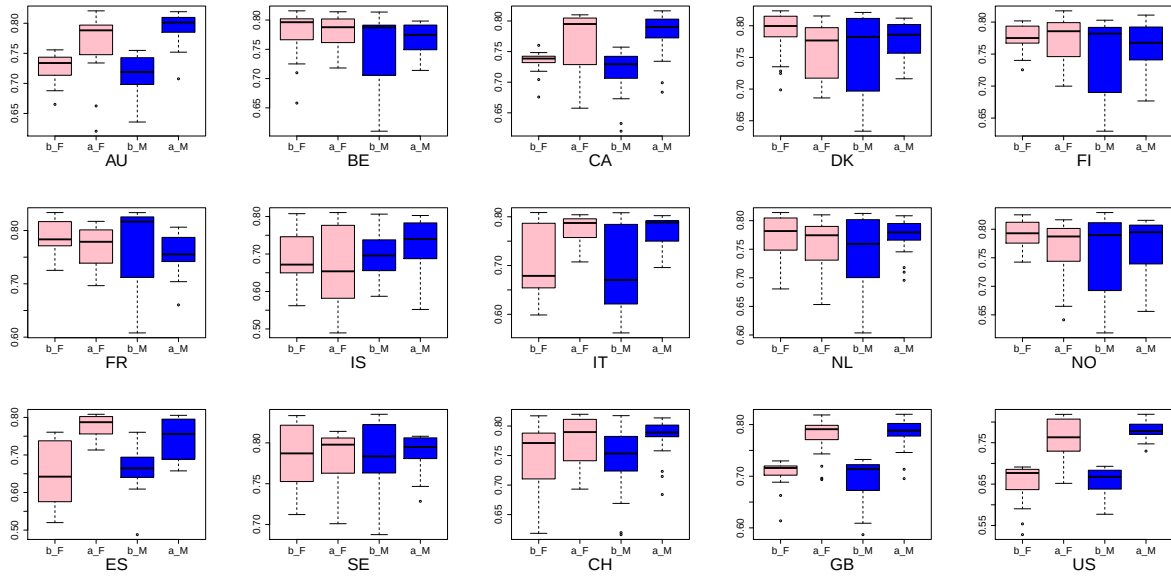


Figure 6.12. Boxplot of estimated H by ages before/after world war two across countries aggregated over age groups for female (red) and male (blue).

6.4 Conclusion

This chapter applies the Hurst exponent to analyse long memory properties in the death count series cross classified by age and gender from 16 countries in HMD. Hurst exponent H is applied to demonstrate the prevalence of long memory feature in death count series and study the pattern of long memory across different countries, age groups, genders and time periods.

Results show that mortality persistence is relatively weaker in male with wider spread, in countries like Iceland and Japan with less data and US with more cultural diversity, in middle age group and in the period before WW2. We further find countries heavily involved in the wars seem to have different structural response in the mortality processes than those less involved. Due to mortality shocks from the wars, there is a creation of cohort effect in European populations. These findings not only provide evidence of long memory in the death count series but also hint the development of LC models for death counts with long memory and possibly cohort effect in the next two chapters.

CHAPTER 7

Mortality models for single age group incorporating long memory

7.1 Background

Forecasting life expectancy and mortality are two important aspects for the study of demography. We demonstrate the existence of long memory using Hurst exponent in Chapter 6 (see Section 6.3). Results provide overwhelming statistical evidence of long memory in the death count series for the 16 selected countries in HMD (Vladimir et al., 2017). In this chapter, we show the existence of long memory motivates us to develop new mortality models which is the single age Gegenbauer long memory Lee-Carter (LC) (Lee and Carter, 1992) model with generalised Poisson distribution that combines LC model and a generalised linear Gegenbauer autoregressive moving average (GLGARMA) model to capture the stochastic mortality and the oscillatory long memory properties in death count series. The proposed mortality models can also incorporate a graduation model structure. This adds flexibility in our proposed model as the spline component in the mean function allows the general mortality level to change across each segment of time. We illustrate how this extended model enhances the mortality forecasts. We argue that the forecasting of mortality rates can be more accurate if such a stochastic feature is incorporated, consequently construction of life tables can also be more precise if the long memory feature in mortality data can be captured.

Bayesian inference is applied and Deviance Information Criterion (DIC) is evaluated to select between different LC model extensions of our proposed models in terms of both

in-sample fits and out-of-sample forecasts performance. Then the models are applied to analyse death count data sets from 16 different countries divided according to genders and age groups. Estimates of mortality rates are applied to calculate life expectancies when constructing life tables. On comparing different life expectancy estimates, results show the LC model without the long memory component may underestimate life expectancy. This underestimation has adverse impact on the old-age support programs in social security and pension and may eventually lead to insufficient funds in a pension scheme. In summary, it is crucial to include long memory feature in mortality models to estimate life expectancies in the construction of life tables.

7.2 Proposed long memory mortality models

We develop new mortality models for death counts by combining GLGARMA model in Equation (2.11) and LC model. Let the random vector $\mathbf{Y}_{x,1:T} = (Y_{x,1}, Y_{x,2}, \dots, Y_{x,T})$ with $Y_{x,t} \in \mathbb{W}$ the set whole number denotes the death count series for age group $\mathcal{S}_x = \{x_1, \dots, x_g\}$ and year $t \in \{1, \dots, T\}$. The natural sigma algebra for the observed data filtration is defined as

$$\mathcal{F}_{x,1:t-1} = \sigma(Y_{x,1}, Y_{x,2}, \dots, Y_{x,t-1}). \quad (7.1)$$

7.2.1 Single age Gegenbauer long memory Lee-Carter model

The Gegenbauer long memory LC model for single age group is given by the following definition:

Definition 7.2.1 (Proposed model of order (p, d, q)). *Condition on the central death rate $\mu_{x,t}(\kappa_{x,t})$ and central exposure $E_{x,t}$, we assume that $Y_{x,t} | \mathcal{F}_{x,1:t-1}, E_{x,t}, \mu_{x,t}(\kappa_{x,t})$ forms an independent Markov process,*

$$Y_{x,t} | \mathcal{F}_{x,1:t-1}, E_{x,t}, \mu_{x,t}(\kappa_t) \sim \text{GP}(E_{x,t} \mu_{x,t}(\kappa_t), \nu_x),$$

$$\Phi_x(B) \ln \mu_{x,t} = \mathbb{B}_x(t; r, l, \boldsymbol{\tau}) + \kappa_{x,t} + (1 - 2u_x B + B^2)^{-d_x} \Theta_x(B)(\varepsilon_{x,t}), \quad (7.2)$$

where $0 < d_x < 1/2$, $|u_x| < 1$, $\Phi(B) = 1 - \phi_1 B - \dots - \phi_p B^p$ and $\Theta(B) = 1 + \theta_1 B + \dots + \theta_q B^q$ with age specific coefficients for each age group x ,

$$\begin{aligned} \kappa_{x,t} &= \eta_x \kappa_{x,t-1} + \varepsilon_{x,t}^{\kappa}, \quad \varepsilon_{x,t}^{\kappa} \stackrel{i.i.d}{\sim} N(0, \sigma_{x,\varepsilon}^2) \quad \text{and} \quad \varepsilon_{x,t}^{\varepsilon} \stackrel{i.i.d}{\sim} N(0, \sigma_{x,\varepsilon^{\kappa}}^2), \\ \mathbb{B}_x(t; r, l, \boldsymbol{\tau}) &= \sum_{i=1}^l \left(a_{x,i}^{(0)} + a_{x,i}^{(1)}(t - \tau_i) + a_{x,i}^{(2)}(t - \tau_i)^2 + a_{x,i}^{(3)}(t - \tau_i)^3 + \dots + a_{x,i}^{(r)}(t - \tau_i)^r \right) \cdot \mathbf{1}_{t \in [\tau_i, \tau_{i+1}]}, \end{aligned}$$

where the B-spline $\mathbb{B}_x(t; r, l, \boldsymbol{\tau})$ is of order r with l knot points $\boldsymbol{\tau} = (\tau_1, \dots, \tau_l)$. This temporal spline represents a smooth graduation (time) effect. Furthermore, the period effect $\kappa_{x,t}$ for each age group is a latent AR(1) process in which we enforce stationarity through $|\eta_x| < 1$.

Remark 7.2.1. We remark that higher order variants of the latent intensity such as AR(p) models can easily be accommodated. In practice, we find it suitable to focus our attention here on the classical LC type model which has an AR(1) structure, except we extend the classical LC model by restricting strictly second order stationary processes.

Considering some special cases of Equation (7.2), the long memory LC model can be further divided into eight sub-models with their mean functions defined in Table 7.1.

Table 7.1. Eight sub-models in which Model 1,4,8 are graduation models, Model 3,4,7,8 are LC type models, Model 1-4 are short memory models and Model 5-8 are long memory models.

Model	mean function
model 1	$\ln \mu_{x,t} = \mathbb{B}_x(t; 0, l, \boldsymbol{\tau})$
model 2	$\ln \mu_{x,t} = \alpha_x + (1 - \phi_x B)^{-1} (1 - \theta_x B) \varepsilon_{x,t}$
model 3	$\ln \mu_{x,t} = \alpha_x + \kappa_{x,t}$
model 4	$\ln \mu_{x,t} = \mathbb{B}_x(t; 0, l, \boldsymbol{\tau}) + \kappa_{x,t}$
model 5	$\ln \mu_{x,t} = \alpha_x + (1 - B)^{-2d_x} \varepsilon_{x,t}$
model 6	$\ln \mu_{x,t} = \alpha_x + (1 - 2u_x B + B^2)^{-d_x} \varepsilon_{x,t}$
model 7	$\ln \mu_{x,t} = \alpha_x + \kappa_{x,t} + (1 - 2u_x B + B^2)^{-d_x} \varepsilon_{x,t}$
model 8	$\ln \mu_{x,t} = \mathbb{B}_x(t; 0, l, \boldsymbol{\tau}) + \kappa_{x,t} + (1 - 2u_x B + B^2)^{-d_x} \varepsilon_{x,t}$

The accuracies of parameter estimates for each model are assessed by a series of simulation studies described in Section 7.3.

7.2.2 Canonical reduced form of latent process long memory LC model

There are several disturbance terms in our proposed models, which come from the latent period effect $\kappa_{x,t}$ specification. These disturbance terms in a linear univariate structural time series model can be combined in an appropriate manner to obtain a reduced form canonical time series, accommodating the fractional difference operator that produces the persistence in the long memory component of the model (Harvey, 1990). For instance, Hyndman et al. (2008) showed that an ARIMA form is obtained by eliminating the state variables from a linear innovation state space model. Such reduced forms when available can be beneficial for interpreting aspects of the state space model such as the ACF and the autocovariance function, which in some cases may then be obtained in closed form. To see why this is possible, we note that such form can be easily represented by Wold representation and hence variance, autocorrelation structure and latent process can be obtained. This simplified form helps us to understand better the impact of long memory features on the period effect. We study the reduced form of the mean function in model 7 as an example since other models are just special cases. Model 7 is the $(0, d, 0)$ order model of has the following general structure

$$\begin{aligned}\Phi_x(B)\log \mu_{x,t} &= \alpha_x + \kappa_{x,t} + (1 - 2u_x B + B^2)^{-d_x} \Theta_x(B) \varepsilon_{x,t}, \\ \kappa_{x,t} &= \eta_x \kappa_{x,t-1} + \varepsilon_{x,t}^\kappa.\end{aligned}\tag{7.3}$$

Since

$$\kappa_{x,t} - \eta_x \kappa_{x,t-1} = (1 - \eta_x B) \kappa_{x,t} = \varepsilon_{x,t}^\kappa \Rightarrow \kappa_{x,t} = \frac{\varepsilon_{x,t}^\kappa}{(1 - \eta_x B)},$$

then we have

$$\kappa_{x,t} - \kappa_{x,t-1} = \frac{(1 - B)}{(1 - \eta_x B)} \varepsilon_{x,t}^\kappa.\tag{7.4}$$

By taking differences of (7.3) with respect to time t and $t - 1$, we obtain

$$\Phi_x(B)(\log \mu_{x,t} - \log \mu_{x,t-1}) = (\kappa_{x,t} - \kappa_{x,t-1}) + (1 - 2u_x B + B^2)^{-d_x} \Theta_x(B)(\varepsilon_{x,t} - \varepsilon_{x,t-1}).$$

By substituting (7.4), we obtain the following reduced form

$$\begin{aligned}\Phi_x(B)(1 - B) \log \mu_{x,t} &= \frac{(1 - B)}{(1 - \eta_x B)} \varepsilon_{x,t}^\kappa + (1 - 2u_x B + B^2)^{-d_x} \Theta_x(B)(1 - B) \varepsilon_{x,t}, \\ \Phi_x(B) \log \mu_{x,t} &= \frac{1}{(1 - \eta_x B)} \varepsilon_{x,t}^\kappa + (1 - 2u_x B + B^2)^{-d_x} \Theta_x(B) \varepsilon_{x,t}.\end{aligned}$$

For the case that $\varepsilon_{x,t}^\kappa$ and $\varepsilon_{x,t}$ are both i.i.d white noise, we have a transformed GARMA model represented as

$$\Phi_x(B) \log \mu_{x,t} = \left[\frac{1}{(1 - \eta_x B)} + (1 - 2u_x B + B^2)^{-d_x} \Theta_x(B) \right] \varepsilon_{x,t}.$$

Based on the generating function $\frac{1}{(1-cx)} = \sum_{i=0}^{\infty} (cx)^i$ for constant c and Equation (2.16), we have

$$\frac{1}{(1 - \eta_x B)} = \sum_{i=0}^{\infty} (\eta_x^i) B^i \quad \text{and} \quad (1 - 2u_x B + B^2)^{-d_x} = \sum_{i=0}^{\infty} \psi_{x,i} B^i.$$

For a transformed GARMA $(0, d, 0)$ model, we have

$$\log \mu_{x,t} = \sum_{i=0}^{\infty} (\eta_x^i + \psi_{x,i}) \varepsilon_{x,t-i}.$$

Figure 7.1 shows the impact of period effect on the ACF and spectral density function of GARMA model. The strength of the long memory is reduced and the spectral density plots show that the long memory pattern becomes less symmetric around the dominant frequency.

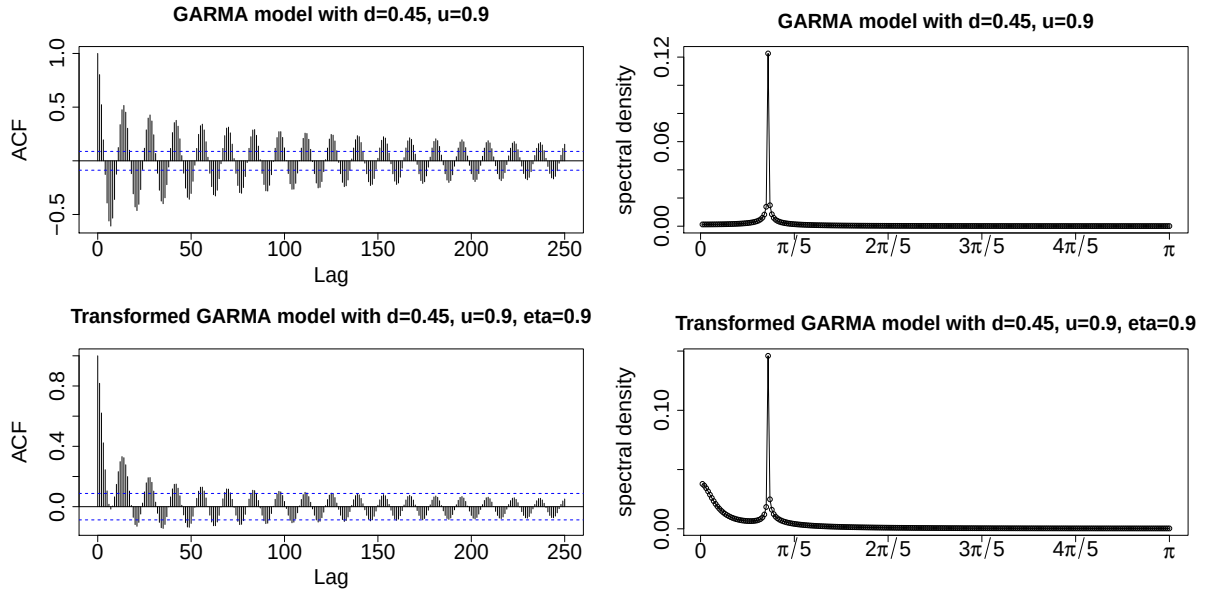


Figure 7.1. The ACF plot and spectral density plot for GARMA model with $d_x = 0.45$ and $u_x = 0.9$ and transformed GARMA model with $d_x = 0.45$, $u_x = 0.9$ and $\eta_x = 0.9$

7.2.3 Bayesian approach

One clear advantage of a Bayesian approach is that it replaces the high dimension integrals in evaluating the marginal likelihood functions in the maximum likelihood (ML) approach with posterior sampling. This advantage particularly applies to our proposed model because the model involves latent variables in both mean and period effect $\kappa_{x,t}$ functions. Moreover, posterior predictive distributions can be obtained to provide distributional forecast summaries. The Bayesian approach also allows the incorporation of prior beliefs. In this section, we introduce the Bayesian approach on proposed model 7.2.1. The basic idea of the Bayesian approach can be described by the following complete likelihood function, under the model structure of model 8, as an example, as

$$\begin{aligned}
f(\mathbf{y}_{x,1:T} | \mathbf{E}_{x,1:T}, \boldsymbol{\vartheta}_x^*) &= \prod_{t=1}^T f(y_{x,t} | E_{x,t}, \boldsymbol{\vartheta}_x, \mu_{x,t}(\kappa_{x,t})) f(\mu_{x,t} | \boldsymbol{\vartheta}_x, \kappa_{x,t}) f(\kappa_{x,t} | \boldsymbol{\vartheta}_x) \\
&= \prod_{t=1}^T \frac{E_{x,t} \exp(\mathbb{B}_x(t; 0, l, \boldsymbol{\tau}) + \kappa_{x,t} + (1 - 2u_x B + B^2)^{-d_x} \varepsilon_{x,t}) (1 - \nu_x)}{\Gamma(y_{x,t} + 1) \exp(y_{x,t} \nu_x)} \\
&\times \frac{[E_{x,t} \exp(\mathbb{B}_x(t; 0, l, \boldsymbol{\tau}) + \kappa_{x,t} + (1 - 2u_x B + B^2)^{-d_x} \varepsilon_{x,t}) (1 - \nu_x) + y_{x,t} \nu_x]^{y_{x,t}-1}}{\exp\{E_{x,t} \exp[\mathbb{B}_x(t; 0, l, \boldsymbol{\tau}) + \kappa_{x,t} + (1 - 2u_x B + B^2)^{-d_x} \varepsilon_{x,t}](1 - \nu_x)\}} \\
&\times \frac{1}{\sqrt{2\pi}\sigma_{x,\varepsilon}^2} \exp\left(-\frac{\varepsilon_{x,t}^2}{2\sigma_{x,\varepsilon}^2}\right) \frac{1}{\sqrt{2\pi}\sigma_{x,\varepsilon^\kappa}^2} \exp\left(-\frac{(\kappa_{x,t} - \eta_x \kappa_{x,t-1})^2}{2\sigma_{x,\varepsilon^\kappa}^2}\right),
\end{aligned}$$

where $\boldsymbol{\mu}_{x,1:t}(\boldsymbol{\kappa}_{x,1:t}) = (\mu_{x,1}(\kappa_{x,1}), \mu_{x,2}(\kappa_{x,2}), \dots, \mu_{x,t}(\kappa_{x,t}))$. The following priors $\pi(\boldsymbol{\vartheta}_x)$ are adopted in this chapter

$$\begin{aligned}
\nu_x &\sim \text{U}(-1, 1), \quad u_x \sim \text{U}(-1, 1), \quad d_x \sim \text{U}(0, 1/2), \quad \alpha_x, a_{x,i}^{(0)}, \dots, a_{x,i}^{(r)} \sim \text{N}(-5, 1), \\
\eta_x &\sim \text{U}(-1, 1), \quad \sigma_{x,\varepsilon}^2 \sim \Gamma(1, 1) \quad \text{and} \quad \sigma_{x,\varepsilon^\kappa}^2 \sim \Gamma(1, 1),
\end{aligned}$$

where $\text{U}(a_u, b_u)$ denotes the uniform prior on the range (a_u, b_u) for the long memory parameters u_x and d_x , $\Gamma(a, b)$ denotes the gamma prior with shape and scale parameters a and b respectively.

Bayesian forecasting is another application of Bayesian inference. The predictive distribution can be derived from the Bayesian filtering equations $f(y_{x,t} | E_{x,t}, \mathcal{F}_{x,1:t})$ which compute the marginal posterior distribution or filtering distribution of the state $y_{x,t}$ at each time t given the history of the measurements up to the time t . The Bayesian

filter for computing the predicted distribution at the time T are given by the following Bayesian filtering equations. Suppose we want m -step forecasts $\mathbf{y}_{x,T+1:T+m}$ with the observed data filtration $\mathcal{F}_{x,1:T}$ for g age groups defined in (7.1) and observed exposures $\mathbf{E}_{x,1:T+m}$. We note that $\mathbf{E}_{x,T+1:T+m}$ can not be directly observed in general. However, there are many methods to estimate $\mathbf{E}_{x,T+1:T+m}$. For example, the rational expectation model defined as a piecewise constant extension $E_{x,T+s} = \mathbb{E}(\mathbf{E}_{x,1:T+s-1})$ for $s = 1, 2, \dots, m$ can be applied to estimate future exposures $\mathbf{E}_{x,T+1:T+m}$. Alternatively, we can assume a stochastic model $f(\mathbf{E}_{x,T+1:T+m} | \mathbf{E}_{x,1:T}, \boldsymbol{\vartheta}, \mathcal{F}_{x,1:T})$ for future exposures. The posterior predictive distribution for $\mathbf{y}_{x,T+1:T+m}$ is defined as

$$\begin{aligned}
& f(\mathbf{y}_{x,T+1:T+m} | \mathbf{E}_{x,1:T+m}, \mathcal{F}_{x,1:T}) \\
&= \int \dots \int f(\mathbf{y}_{x,T+1:T+m} | E_{x,1:T+m}, \boldsymbol{\mu}_{x,1:T+m}(\boldsymbol{\kappa}_{x,1:T+m}), \boldsymbol{\vartheta}_x, \mathcal{F}_{x,1:T}) \\
& \quad \times f(\boldsymbol{\mu}_{x,1:T+m}(\boldsymbol{\kappa}_{x,1:T+m}) | \boldsymbol{\mu}_{x,1:T}(\boldsymbol{\kappa}_{x,1:T}), \boldsymbol{\vartheta}_x, \mathcal{F}_{x,1:T}) \\
& \quad \times f(\boldsymbol{\kappa}_{x,1:T+m} | \boldsymbol{\kappa}_{x,1:T}, \boldsymbol{\vartheta}_x, \mathcal{F}_{x,1:T}) \pi(\boldsymbol{\vartheta}_x | \mathcal{F}_{x,1:T}) d\boldsymbol{\mu}_{x,1:T+m}(\boldsymbol{\kappa}_{x,1:T+m}) d\boldsymbol{\kappa}_{x,1:T+m} d\boldsymbol{\vartheta}_x, \\
&= \int \dots \int \prod_{s=1}^m [f(y_{x,T+s} | E_{x,1:T+m}, \boldsymbol{\mu}_{x,1:T+s}(\boldsymbol{\kappa}_{x,1:T+s}), \boldsymbol{\vartheta}_x, \mathcal{F}_{x,1:T}) \\
& \quad \times f(\boldsymbol{\mu}_{x,T+s}(\boldsymbol{\kappa}_{x,T+s}) | \boldsymbol{\mu}_{x,T+s-1}(\boldsymbol{\kappa}_{x,T+s-1}), \boldsymbol{\vartheta}_x, \mathcal{F}_{x,1:T}) \\
& \quad \times f(\boldsymbol{\kappa}_{x,T+s} | \boldsymbol{\kappa}_{x,T+s-1}, \boldsymbol{\vartheta}_x, \mathcal{F}_{x,1:T})] f(\boldsymbol{\mu}_{x,1:T}(\boldsymbol{\kappa}_{x,1:T}) | \boldsymbol{\vartheta}_x, \mathcal{F}_{x,1:T}) f(\boldsymbol{\kappa}_{x,1:T} | \boldsymbol{\vartheta}_x, \mathcal{F}_{x,1:T}) \\
& \quad \times \pi(\boldsymbol{\vartheta}_x | \mathcal{F}_{x,1:T}) d\boldsymbol{\mu}_{x,1:T+s}(\boldsymbol{\kappa}_{x,1:T+s}) d\boldsymbol{\kappa}_{x,1:T+s} d\boldsymbol{\vartheta}_x,
\end{aligned}$$

and this integral can be approximated by the Monte Carlo estimator, constructed from posterior samples according to:

$$\hat{f}(y_{x,T+s} | E_{x,1:T+s}, \mathcal{F}_{x,1:T}) = \frac{1}{L} \sum_{l=1}^L f(y_{x,T+s} | \mathbf{E}_{x,1:T+s}, \boldsymbol{\mu}_{x,1:T+s}^{(l)}(\boldsymbol{\kappa}_{x,1:T+s}^{(l)}), \boldsymbol{\vartheta}_x^{(l)}, \mathcal{F}_{x,1:T}), \quad (7.5)$$

where L is the number of iterations after burn-in in each Markov chain Monte Carlo (MCMC) sampler run given the current window of information and $\boldsymbol{\mu}_{x,1:T+s}^{(l)}(\boldsymbol{\kappa}_{x,1:T+s}^{(l)})$, $\boldsymbol{\kappa}_{x,1:T+s}^{(l)}$ and $\boldsymbol{\vartheta}_x^{(l)}$ are the l -th draw in the posterior sample of $\boldsymbol{\mu}_{x,1:T+s}(\boldsymbol{\kappa}_{x,1:T+s})$, $\boldsymbol{\kappa}_{x,1:T+s}$ and $\boldsymbol{\vartheta}_x$, respectively.

7.3 Simulation Study

To evaluate the accuracy of parameter estimation for the models in Table 7.1, we first perform synthetic studies using simulated time series with different levels of length

similar to the mortality data. We note that in case when the mortality data are too short, specialised bootstrap procedures can also be used to validate such accuracy. We clarify our synthetic studies according to four case studies. In each study, we simulate data from the eight submodels and all simulated data for each model are generated by the same seed, adopt four different lengths of data, namely, $n_1 = 100$, $n_2 = 150$, $n_3 = 250$ and $n_4 = 550$. We purposely choose a large data size of $n_4 = 550$ to verify if some inaccuracies in parameter estimates are due to insufficient sample size. Moreover, true parameter values are chosen to generate death counts in similar magnitude to the observed rescaled death counts, often in the range of 0 to 250.

Case study 7.3.1. *Parameter accuracy across levels of d_x and u_x .*

In this study, we test the accuracies of parameter estimates across levels of d_x and u_x using model 8 which is the full model consisting of $\mathbb{B}_x(t; 0, l, \boldsymbol{\tau})$ spline, period effect and long memory effect. We compare the effect of varying levels of u_x (indicates length of oscillatory ACF) and d_x (strength of long memory) on the parameter accuracy. We also consider the parameter accuracy when u_x is negative which gives instantaneous oscillatory ACF.

Table 7.2. True values, estimates and creditable intervals for parameters d_x and u_x of model 8

d_x	$\hat{d}_x$	2.5%	97.5%	u_x	$\hat{u}_x$	2.5%	97.5%	η_x	$\hat{\eta}_x$	2.5%	97.5%	$a_{x,1}$	$\hat{a}_{x,1}$	2.5%	97.5%
Data time series of length $n_3 = 250$															
0.45	0.4160	0.3246	0.4940	0.9	0.9010	0.8888	0.9158	0.5	0.1062	-0.7038	0.8268	-5	-5.0212	-5.7903	-4.2737
0.3	0.3486	0.2321	0.4860	0.9	0.8805	0.8512	0.9052	0.5	0.0232	-0.7041	0.7396	-5	-4.9543	-5.6825	-4.2269
0.1	0.1835	0.0123	0.4545	0.9	0.4071	-0.9141	0.9591	0.5	0.2688	-0.7240	0.7825	-5	-5.2163	-5.9206	-4.5155
0.45	0.3851	0.2548	0.4928	0.3	0.3083	0.2578	0.3447	0.5	0.4374	-0.1994	0.8121	-5	-5.3220	-6.0108	-4.6299
0.45	0.4186	0.2928	0.4965	0.1	0.1179	0.0970	0.1348	0.5	0.4307	0.0359	0.8180	-5	-5.2912	-5.9568	-4.6437
0.45	0.3732	0.3644	0.3795	-0.5	0.3307	0.2673	0.4166	0.5	1.0051	0.9768	1.0662	-5	-4.0509	-4.7846	-3.5191
0.45	0.1233	0.1233	0.1233	-0.9	0.6947	0.6947	0.6947	0.5	1.0742	1.0742	1.0742	-5	-3.5114	-3.5114	-3.5114
Data time series of length $n_4 = 550$															
0.45	0.4298	0.3116	0.4750	0.9	0.9074	0.8588	0.9357	0.5	0.8041	0.7167	0.8786	-5	-4.8555	-6.0511	-3.1534
0.3	0.2374	0.1814	0.2973	0.9	0.8923	0.8714	0.9073	0.5	0.9943	0.9729	1.0021	-5	-5.1098	-6.0621	-4.0510
0.1	0.1458	0.0883	0.2033	0.9	0.8234	0.7020	0.9858	0.5	0.9956	0.9769	1.0026	-5	-5.0087	-5.7866	-4.0176
0.45	0.3816	0.3390	0.4587	0.3	0.3010	0.2977	0.3073	0.5	0.9797	0.9333	0.9947	-5	-4.8524	-5.4817	-4.1618
0.45	0.3755	0.3038	0.4591	0.1	0.1020	0.0952	0.1085	0.5	0.9824	0.9500	0.9986	-5	-5.0171	-5.7749	-4.2370
0.45	0.3763	0.2663	0.4855	-0.5	-0.4962	-0.5062	-0.4862	0.5	0.8900	0.5309	0.9710	-5	-4.5965	-5.4897	-3.7135
0.45	0.4211	0.3406	0.4929	-0.9	-0.9006	-0.9048	-0.8969	0.5	0.8014	0.5826	0.9209	-5	-4.3557	-5.1766	-3.4797

According to the results in Table 7.2, there are two main findings. Firstly, the accuracies of parameter estimates are affected by d_x , the strength of long memory in a given time series. If the strength is weak (d_x is close to 0), the oscillatory features in the ACF, as indicated by u_x , are very difficult to distinguish from the white noise. Hence the accuracies will get worse. However, as the majority of mortality data we study show strong long memory features, we believe the problem of estimating u_x under a weak long memory signal will not be an issue.

Secondly, when a time series has instantaneously oscillating ACF with negative u_x , the accuracy of parameter estimates are usually worse with a moderate data size ($n_3 = 250$). However, this problem can be improved when the data length increases to $n_4 = 550$. Results suggest that negative u_x is harder to estimate and so larger sample size is needed to obtain reasonable estimates of u_x and the other parameters. However, as all u_x are estimated to be positive for the mortality data we study, the problem with estimating a negative u_x is again not an issue in our studies.

Case study 7.3.2. *Accuracy of AR parameter in period effect*

Period effect is another important component in our proposed model. We compares the accuracy of parameter estimate η_x for the AR term in the period effect in models 3 and 7. The period effect describes a stationary process if $|\eta_x| < 1$. Results show that the accuracy will decrease with increasing model complexity (model 7). For model 3, η_x can be estimated to a reasonable accuracy with the shortest data length of $n_1 = 100$. However, after adding the long memory structure in model 7, it can be properly estimated only with a longer data length of $n_2 = 150$ and $n_3 = 250$. Hence, this issue can again be solved by increasing the sample size. Although, the accuracy for negative η_x is not as promising as those of the positives, the η_x in our data analysis are again consistently positive. Ne refers to the studies has not been implemented because we mainly focus on positive η_x .

Table 7.3. True values, estimates and credible intervals for parameters η_x model 3 and model 7 with data length $n_1 = 100$, $n_2 = 150$ and $n_3 = 250$ when $d = 0.45$, $u = 0.9$ for both model 3 and model 7

True value	model 3, n_1			model 7, n_1			model 7, n_2			model 7, n_3		
η_x	$\hat{\eta}_x$	2.5%	97.5%	$\hat{\eta}_x$	2.5%	97.5%	$\hat{\eta}_x$	2.5%	97.5%	$\hat{\eta}_x$	2.5%	97.5%
0.1	0.2839	-0.7045	0.9191	0.4529	-0.6796	0.9114	0.1544	-0.7470	0.8627	0.1163	-0.6999	0.8433
0.3	0.5315	0.1661	0.9096	0.4218	-0.7246	0.9317	0.1740	-0.7937	0.8898	0.2776	-0.7504	0.8779
0.7	0.8708	0.7280	0.9891	0.8654	0.7095	0.9791	0.7470	0.4811	0.9393	0.6826	-0.4203	0.9474
0.9	0.9561	0.8993	0.9958	0.9295	0.8433	0.9924	0.8711	0.7165	0.9851	0.9064	0.8273	0.9839
-0.1	0.0619	-0.7987	0.8922	0.3984	-0.7426	0.8900	NA	NA	NA	-0.1835	-0.8032	0.7808
-0.3	-0.2508	-0.8699	0.8676	0.1567	-0.8650	0.8877	NA	NA	NA	-0.3171	-0.7160	0.4520
-0.7	-0.5768	-0.6916	-0.2517	-0.6223	-0.9880	0.7263	NA	NA	NA	-0.7610	-0.8841	-0.6135
-0.9	-0.7570	-0.8996	-0.6026	-0.9351	-0.9949	-0.8395	NA	NA	NA	-0.9326	-0.9776	-0.8777

Case study 7.3.3. Accuracy of spline function parameters

The spline function $\mathbb{B}_x(t; 0, l, \boldsymbol{\tau})$ allows the intercepts $a_{x,i}^{(0)}$ in $\mu_{x,t}$ to remain fixed in each stage i of S data points marked by knot points $\boldsymbol{\tau}$ and vary across stages. Hence it provides level information for the death rate in each stage of the data set. Table 7.4 reports the spline parameter estimates $a_{x,1}^{(0)}$ across length S for models 1, 4 and 8 which contain the spline components $\mathbb{B}_x(t; 0, l, \boldsymbol{\tau})$ in the mean function. Results show that, model 1, as a simple form of graduation model, provides reasonable estimates with small sample size ($n_1 = 100$) and moderate number of knot points such that each segment contains approximately 10 data points. Again, the accuracy of $a_{x,1}^{(0)}$ can be improved by increasing S rather than data size in general because the longer stage will provide more data information to estimate the parameters in the spline function better. As the simplest model, model 1 in Table 7.4 shows that it provides accurate parameter estimates for the shortest data of size 100 with each stage containing only $S = 10$ data points. However, with more complicated model structures, $a_{x,i}^{(0)}$ are highly sensitive to initial values and prior distributions. This estimating issue can be solved by increasing S . Taking model 4 with data length n_1 in Table 7.4 as an example, the estimate of $a_{x,1}^{(0)}$ is reasonable when $S = 30$. However, if we only increase the data length to n_2 but use a shorter stage length of $S = 10$, the estimate of $a_{x,1}^{(0)}$ is around -3 (not reported in the table) which is very inaccurate. For model 8, $S = 40$ data points in each stage will be sufficient to estimate $a_{x,1}^{(0)}$ accurately.

Table 7.4. True values, estimates and credible intervals for parameters $a_{x,1}^{(0)}$ of model 4 with data length $n_1 = 100$, of model 8 with data length $n_3 = 250$, $d = 0.45$ and $u = 0.9$ and of model 1 with data length n_1 and $S = 10$.

S	Model 4 with n_1				Model 8 with n_3			Model 1 with n_1 and $S = 10$			
	$a_{x,1}^{(1)}$	$\hat{a}_{x,1}^{(1)}$	2.5%	97.5%	$\hat{a}_{x,1}^{(1)}$	2.5%	97.5%	$a_{x,1}^{(1)}$	$\hat{a}_{x,1}^{(1)}$	2.5%	97.5%
20	-5	-2.4595	-3.9549	-1.0197	-4.0704	-4.8834	-3.1895	-4	-4.0568	-4.2295	-3.6367
30	-5	-5.5521	-5.8005	-4.8135	-3.9243	-3.9243	-2.9678	-5	-5.0111	-5.1235	-4.9334
40	-5	-5.0877	-5.7601	-4.2712	-4.2558	-5.0473	-3.4081	-6	-6.0033	-6.1089	-5.9005
50	-5	-5.0102	-5.3397	-4.6504	-4.9001	-5.3397	-4.3623	-7	-7.0000	-7.0558	-6.9450

Case study 7.3.4. Accuracy of forecasts with model mis-specification

Model specification is also an important factor in model selection. Given an ARMA(1,1) model,

$$Y_t = c + \phi Y_{t-1} + \varepsilon_t - \theta \varepsilon_{t-1},$$

the 1-step ahead forecast $\hat{y}_{t+1}$ can be represented as

$$\hat{Y}_{t+1} = c + \phi Y_t + 0 - \theta \varepsilon_t, \quad \text{since } \hat{\varepsilon}_{t+1} = 0$$

with the variance of forecast error

$$\text{Var}(Y_{t+1} - \hat{Y}_{t+1}) = \text{Var}(\varepsilon_{t+1}) = \sigma^2.$$

For ARFIMA(1,d,1) model,

$$Y'_t = c + \phi Y'_{t-1} + (1 - B)^{-d} (\varepsilon_t - \theta \varepsilon_{t-1}),$$

and the 1-step ahead forecast $\hat{y}_{t+1}$ can be represented as

$$\hat{Y}'_{t+1} = c + \phi Y'_t - (1 - B)^{-d} \theta \varepsilon_t,$$

where the long memory component $(1 - B)^{-d}$ with range $|\varphi_j| < 1$ is

$$(1 - B)^{-d} = \sum_{j=0}^{\infty} \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)} B^j = \sum_{j=0}^{\infty} \varphi_j B^j.$$

Hence, the variance of forecast error is given by

$$\text{Var}(Y'_{t+1} - \hat{Y}'_{t+1}) = \text{Var}((1 - B)^{-d} \varepsilon_{t+1}) = \left(1 + \sum_{j=1}^{\infty} \varphi_j^2\right) \sigma^2.$$

where $\left(1 + \sum_{j=1}^{\infty} \varphi_j^2\right)$ is a positive number greater than 1. Consequently, the variance of forecast error $\text{Var}(Y'_{t+1} - \hat{Y}'_{t+1})$ is much greater than $\text{Var}(Y_{t+1} - \hat{Y}_{t+1})$. We have demonstrated the presence of long memory in most mortality data. This shows that for model with mis-specification by dropping the long memory model, the variance of model parameters will be underestimated. Further, we aim to investigate how this model mis-specification affects the forecast accuracy. We first simulate data from model 7 with period effect and long memory and forecast 20-step ahead for $m = 20$ time points using models 7, 4 and 3 with different data length n_1 , n_3 and n_4 . We note that models 3 and 4 contain period effects but with no long memory. To facilitate comparison, we vary the levels of d_x which indicate different strengths of long memory. The forecasts $\hat{y}_{x;T+s}$ are estimated using the posterior mean of the posterior sample described in Equation (7.5). The four selection criteria, namely, MAE, RMSE, MAPE and MASE in Section 5.3.3 are used to assess the forecast accuracy. Table 7.5 shows that the forecast performance of model 3 and 4 gets worse with higher level of d and larger data in general as the discrepancy between the true and fitted models is larger size and clearer respectively. Model 7 with long memory component is much better than the other two models with small sample size $n_1 = 100$ and the performance of model 3 improves with larger sample size. Overall, model 7 with long memory is preferred when the data show long memory features.

Table 7.5. Four selection criteria to measure the accuracy of the 20 steps ahead forecasts using models 7, 4 and 3 with different data length when $u = 0.9$, $\eta = 0.5$ and $\mathbb{B}_x(t; 0, l, \tau) = -5$.

d	n_1			n_3			n_4		
	Model 7	Model 4	Model 3	Model 7	Model 4	Model 3	Model 7	Model 4	Model 3
MAE									
0.45	2.250	5.900	3.900	1.750	30.050	1.900	2.250	37.450	2.350
0.3	2.050	4.100	3.150	1.950	26.700	2.000	1.850	20.950	1.900
0.1	1.000	5.150	3.000	1.800	6.800	1.750	2.000	9.950	1.800
RMSE									
0.45	5.500	11.162	7.510	2.377	33.259	2.387	3.585	50.697	3.931
0.3	3.041	9.252	4.489	2.291	29.641	3.017	3.186	28.521	3.286
0.1	1.483	10.313	3.937	2.236	8.526	2.179	3.768	15.101	3.647
MAPE									
0.45	0.455	1.192	0.788	0.455	7.805	0.494	0.600	9.987	0.627
0.3	0.532	1.065	0.818	0.609	8.344	0.625	0.661	7.482	0.679
0.1	0.274	1.411	0.822	0.632	2.386	0.614	0.690	3.431	0.621
MASE									
0.45	0.415	1.088	0.719	0.639	10.980	0.694	0.855	14.231	0.893
0.3	0.609	1.217	0.935	1.029	14.092	1.056	1.034	11.707	1.062
0.1	0.365	1.882	1.096	1.036	3.915	1.008	1.152	5.729	1.036

Note: the best models among models 7, 4 and 3 are shaded by light grey colour.

7.4 Mortality studies and life table construction for univariate mortality models

We perform three analyses in this section. The first analysis performs in-sample fitting using all years whereas the second analysis performs out-of-sample forecast using all except the last 20 years to train models and forecast 20 years ahead. These two analyses consider the five selected countries, namely Australia, Iceland, Japan, U.K. and U.S., to guide the model selection based on performance of both in-sample fitting and out-of-sample forecast. The last analysis is our aim to provide life tables for the future 20 years of all 16 countries. This life table analysis uses all data to train models and forecast mortality rates 20 years ahead using traditional LC model (model 3) and the best long memory model (model 8) selected by DIC. We construct the life tables based on mortality rate forecasts using model 3 and 8, without and with long memory features and compare their performances.

7.4.1 In-sample fitting for mortality rates

We study three important components, namely long memory, LC type and graduation, in our proposed model and demonstrate these features for the five selected countries. To study each component independently of the others, we consider model 6, model 3 and model 1 which contain parameters (d_x, u_x) , η_x and $\mathbb{B}_x(t; 0, l, \tau)$ alone respectively apart from the general level parameter α_x . We also compare the goodness of fit for the eight models and use DIC to select the best fitted model for each mortality data set. Furthermore, the difference between the mortality rates smoothed by each model and the mortality rates from HMD are evaluated by MAEs, RMSEs, MAPEs and MASEs (Section 5.3.3) to provide additional guidelines in model selection. Then the best model is used to provide estimates of mortality rates and then life expectancies for the five selected countries.

Long memory features using model 6:

The estimated d_x and u_x are obtained from model 6 which is a GLGARMA model containing only long memory component. The left hand panel in Figure 7.2 describes the estimated values of d_x and u_x for females and right hand panel for male. These in-sample fitting results agree with the empirical study of long memory in Section 6.2. For female, Iceland data shows much less pronounced memory properties (lower d_x), especially for older age groups. All the other four countries illustrate long memory because the estimates of d_x are close to 0.5. The estimates of u_x for these five countries are all close to 1 except the older age group in Iceland. These results suggest that the majority of mortality data follow ARFIMA type long memory rather than Gegenbauer type long memory. For males, the strength of long memory in Iceland again is weaker, especially in older age groups. For both female and male, the estimated values of d_x in last age group of Iceland are very low, which may give imprecise estimates of u_x because the long memory features of a weak long memory time series are difficult to distinguish from white noise.

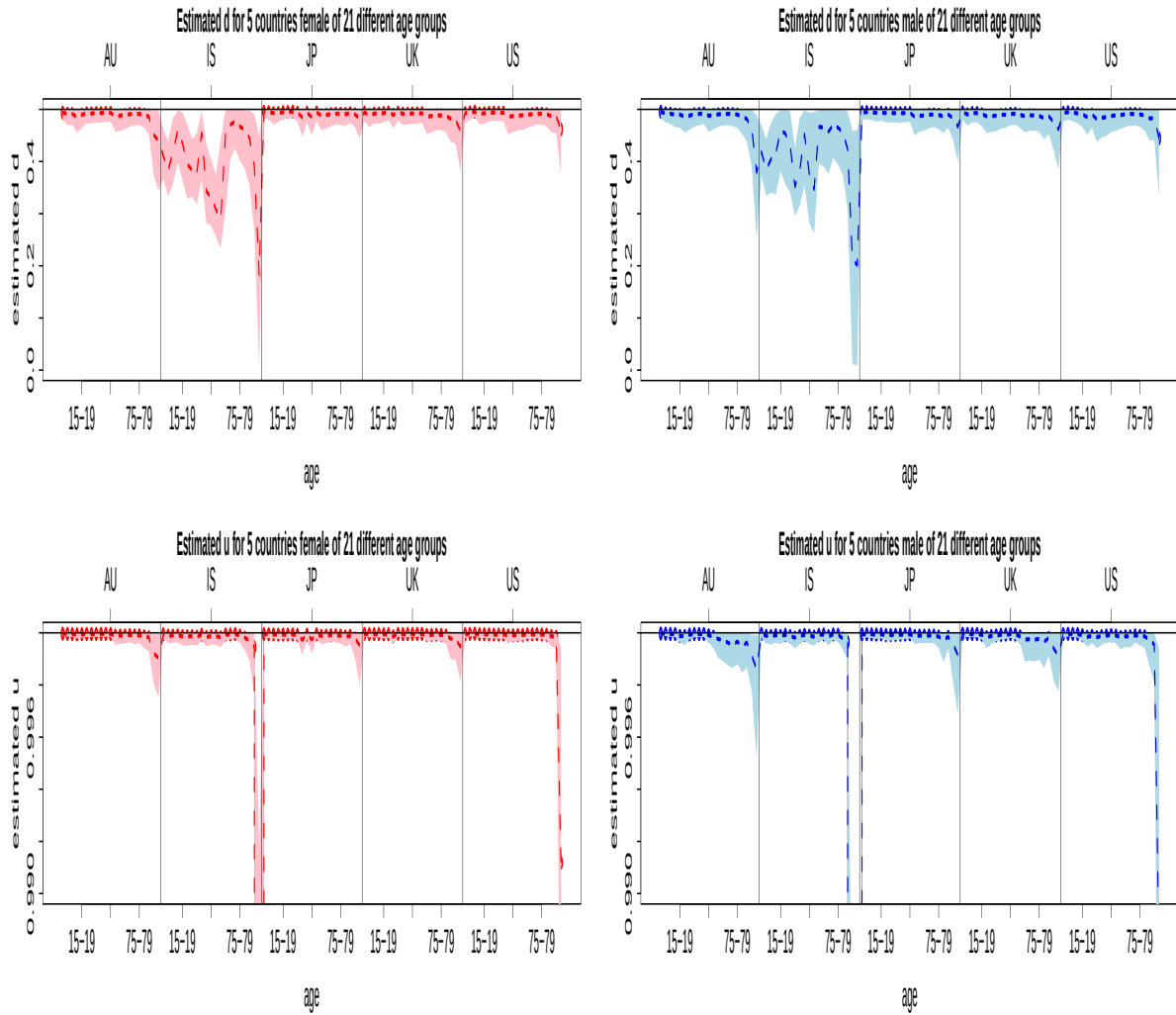


Figure 7.2. Estimated d_x (first row) and u_x (second row) using female (left panel) and male (right panel) death counts fitted by model 6 for five selected countries. The estimated value of u_x for Iceland female in the last age group is 0.08313 and for Iceland male in the last two age group, they are -0.03799 and -0.07389. For US male, the last reading is 0.9875.

Period effect using model 3:

The estimated parameter η_x of the AR term describes the stationarity condition in period effect. Model 3 as a basic period effect model is selected to explore the features of η_x . Figure 7.3 shows that the estimated values of η_x for female and male are close to 1, which means the period effect tends to be non-stationary to capture the trend. For both

female and male, the estimated values of η_x in last age group are much lower, especially for Iceland. This suggests the mortality rates tend to be stable with very marginal improvement in longevity despite the substantial advancement in medical health over the years.

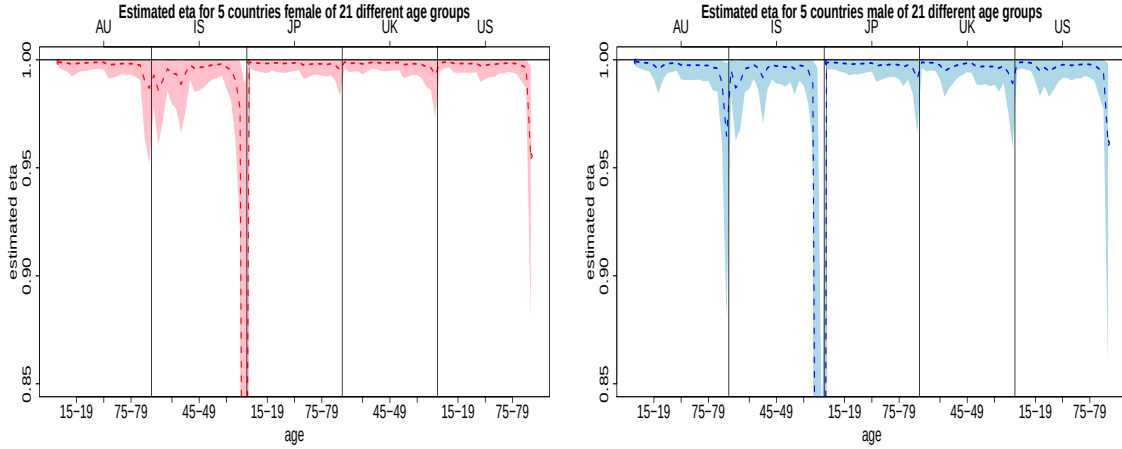


Figure 7.3. Estimated η_x for period effect when death counts of five selected countries are fitted using model 3 for female (left panel) and male (right panel). The estimated value of η_x for Iceland female in the last age group is 0.5351 and for male, the last two values are 0.5975 and -0.03941.

Pattern of graduation effects using model 1: The estimated coefficients $a_{x,i}^{(0)}$ for the $\mathbb{B}_x(t; 0, l, \boldsymbol{\tau})$ spline in model 1 are plotted in Figure 7.4 across the 21 age groups x in each of the l stages as marked by the l knot points $\boldsymbol{\tau} = (\tau_1, \dots, \tau_l)$ across year. Each stage represents a consistent mortality level for a period of 20 years and this length of 20 years is chosen to ensure enough information to estimate the coefficient in each stage to a desirable level of accuracy. Due to the difference in data length, the five countries have different number of stages. For Australia, U.K. and U.S., there are four stages. For Japan and Iceland, there are three and eight stages respectively. All of these graphs illustrate the general pattern across age groups that in each stage, the estimated $a_{x,i}^{(0)}$ declines from age 0 to the lowest level around age 10-14. After reaching the lowest level, $a_{x,i}^{(0)}$ increases slowly in general but there is a sharper increase and then a level-off giving a platform pattern between age 15-34. This phenomenon is more obvious in male population and in recent years because young males are more keen to take up higher risk tasks or activities than young females or people in the past. This pattern of changes of $a_{x,i}^{(0)}$ during a life cycle follows a decreasing trend over years and confirms

the improvements in living and medical conditions for all the five selected countries. Apart from these parameter estimates, we remark that the dispersion parameter ν are consistently estimated to be close to -1 for all age groups.

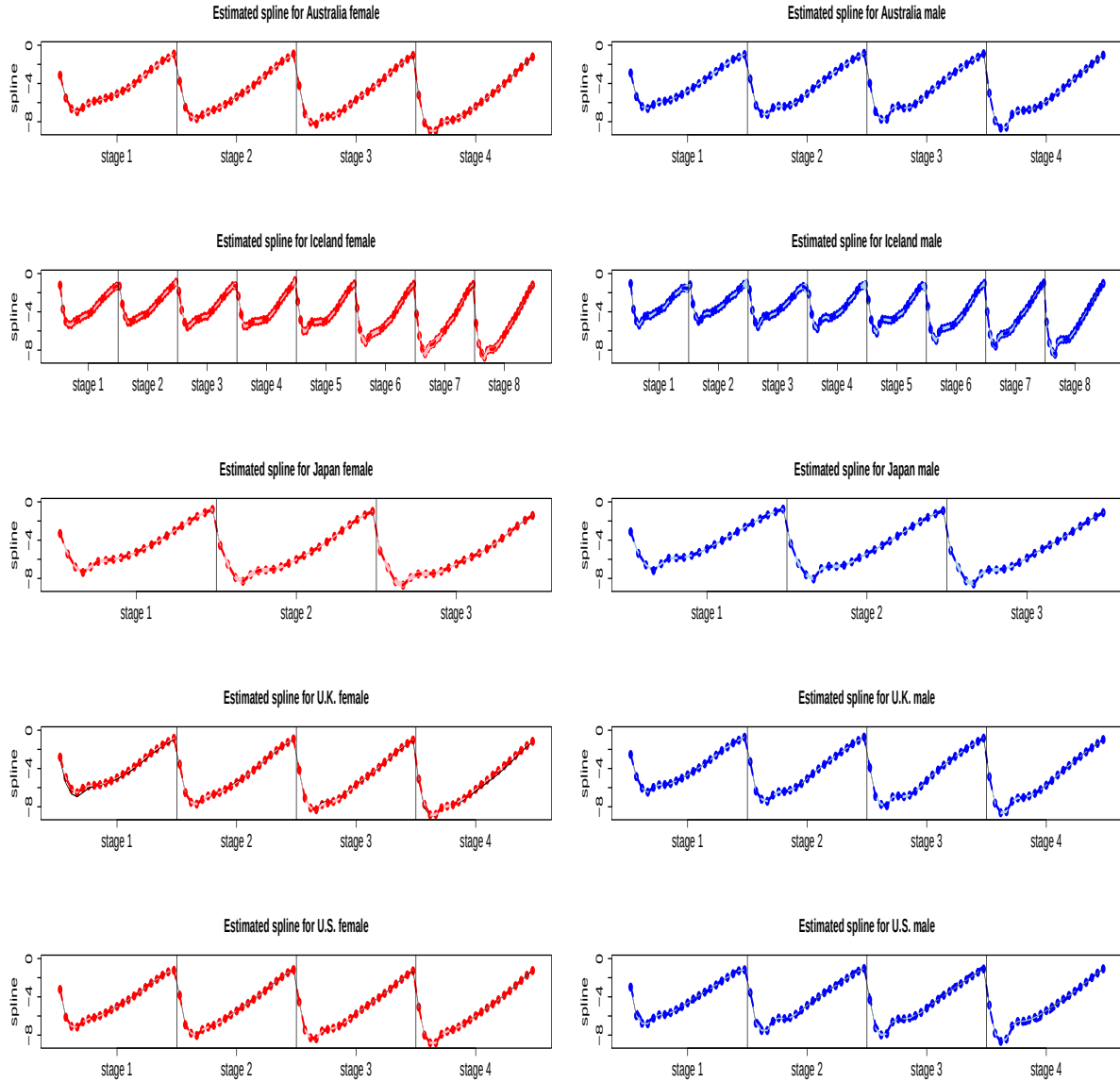


Figure 7.4. Estimated spline coefficients $a_{x,i}^{(0)}$ for 21 age groups in each of the l stages by fitting five countries death counts for female (left) and male (right) using model 1. Each block in each plot represents a stage of 20 years in which the coefficient is assumed to be consistent. The first row is for Australia, followed by Iceland, Japan, U.K. and U.S..

Model fit for all models using DIC :

Figure 7.5 depicts DIC values across age groups for each of the eight models in Table 7.1 cross-classified by gender and the five selected countries. For the countries with long memory features, such as Australia, Japan, U.K. and U.S., the performance of models incorporating a long memory component is significantly enhanced (lower DIC) compared to short memory models (models 1-4; without long memory structures). For Iceland data with short memory, the short memory models are better than long memory models.

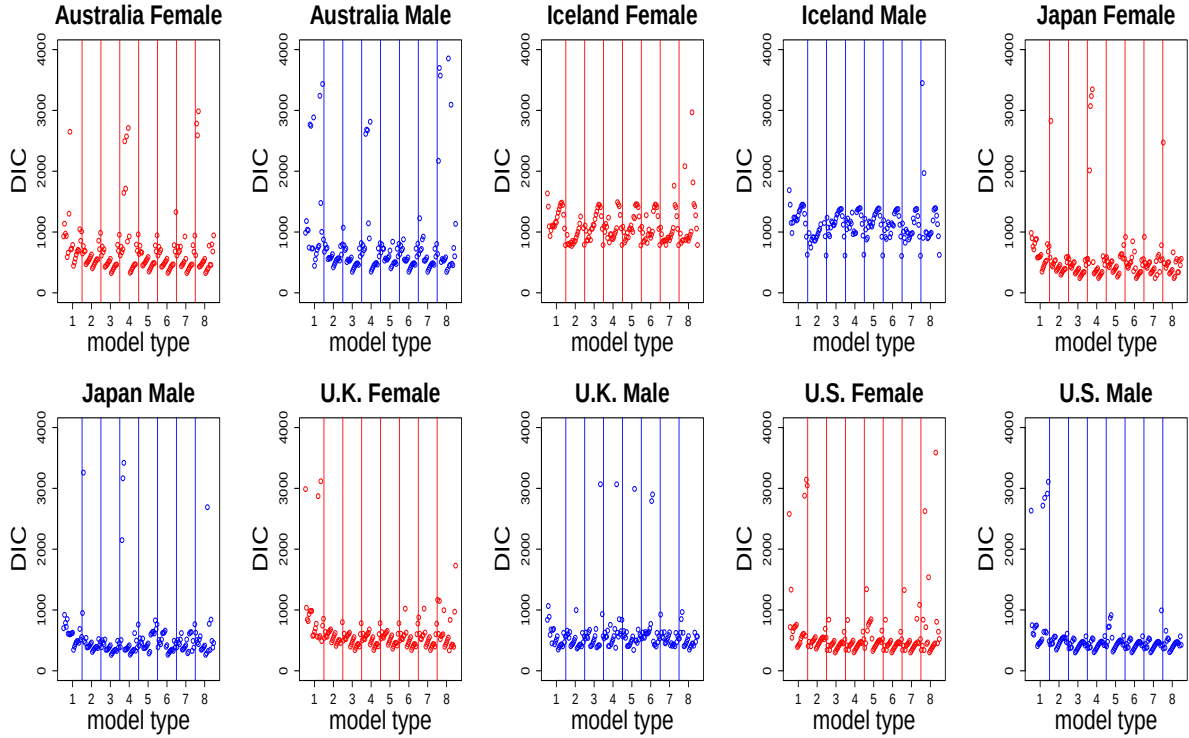


Figure 7.5. DIC of the eight models fitted to death counts of the five countries by gender. Each model type is represented by dots in the order of 21 age groups.

Table 7.6 shows the performance of the eight models using four selection criteria, namely, MAEs, RMSEs, MAPEs and MASEs defined in Section 5.3.3 by comparing the estimated mortality rates $\hat{\mu}_{x,t}$ with the observed $\hat{\mu}_{x,t}$ from HMD. Generally speaking, long memory models (model 5 to 8) perform better than short memory models. Model 8 is the best fitted model as it has the greatest number of highlights (the lowest criteria across a row) in Table 7.6 as well as the overall lowest median DIC of 476.56.

Table 7.6. Four selection criteria to measure the accuracy of estimated mortality rates $\hat{\mu}_{x,1:T}$ for five selected countries.

Country	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8
MAEs								
AU f	0.00360	0.00159	0.00167	0.00171	0.00155	0.00158	0.00153	0.00175
AU m	0.00462	0.00152	0.00201	0.00197	0.00194	0.00199	0.00205	0.00222
IS f	0.01782	0.01233	0.01536	0.01395	0.01454	0.01485	0.01440	0.01458
IS m	0.02506	0.02258	0.02248	0.02148	0.02215	0.02161	0.02124	0.02084
JP f	0.00512	0.00129	0.00114	0.00143	0.00110	0.00122	0.00119	0.00149
JP m	0.00535	0.00211	0.00191	0.00136	0.00193	0.00205	0.00225	0.00217
UK f	0.02011	0.01995	0.01996	0.01997	0.02008	0.02101	0.02100	0.01515
UK m	0.02430	0.02390	0.02391	0.02392	0.03587	0.02472	0.02469	0.01164
US f	0.00229	0.00357	0.00065	0.00074	0.00081	0.00064	0.00065	0.00086
US m	0.00269	0.00111	0.00086	0.00094	0.00100	0.00087	0.00085	0.00106
RMSEs								
AU f	0.00905	0.00878	0.00653	0.00659	0.00617	0.00628	0.00609	0.00649
AU m	0.01104	0.00606	0.00705	0.00677	0.00679	0.00683	0.00690	0.00715
IS f	0.06251	0.06455	0.06151	0.05711	0.06051	0.06211	0.06238	0.05897
IS m	0.08520	0.09091	0.08242	0.08114	0.08267	0.08120	0.08159	0.08074
JP f	0.01178	0.00880	0.00438	0.00517	0.00519	0.00461	0.00462	0.00506
JP m	0.01207	0.01254	0.00769	0.00469	0.00875	0.00813	0.00867	0.00849
UK f	0.03949	0.03906	0.03897	0.03895	0.03929	0.03999	0.03998	0.03037
UK m	0.04577	0.04463	0.04451	0.04451	0.10556	0.04568	0.04561	0.02030
US f	0.00488	0.05927	0.00227	0.00242	0.00271	0.00227	0.00230	0.00274
US m	0.00585	0.00472	0.002731	0.00285	0.00331	0.00308	0.002725	0.00305
MAPEs								
AU f	0.14667	0.01859	0.02672	0.02743	0.02222	0.02561	0.02534	0.02959
AU m	0.13965	0.01594	0.02282	0.02374	0.02044	0.02154	0.02229	0.06776
IS f	0.28994	0.25005	0.24985	0.22698	0.23653	0.24156	0.23214	0.24677
IS m	0.35822	0.36739	0.32135	0.30709	0.31671	0.30898	0.30360	0.29795
JP f	0.40740	0.01242	0.01484	0.01571	0.01375	0.01479	0.01423	0.01680
JP m	0.26781	0.01483	0.01630	0.01648	0.01594	0.01644	0.01615	0.01794
UK f	1.44200	1.28745	1.28715	1.28745	1.29081	1.34336	1.34344	0.40952
UK m	1.50761	1.34537	1.34591	1.34585	2.29490	1.39665	1.39666	1.70815
US f	0.10651	1.99711	0.01274	0.01518	0.01482	0.01237	0.01271	0.01602
US m	0.09495	0.01424	0.01391	0.01587	0.01588	0.01354	0.01372	0.01704
MASEs								
AU f	0.11136	0.04912	0.05164	0.05293	0.04776	0.04893	0.04733	0.05366
AU m	0.15361	0.05054	0.06685	0.06545	0.06383	0.06548	0.06206	0.06721
IS f	0.36625	0.29662	0.31561	0.28673	0.29878	0.30514	0.28834	0.38781
IS m	0.44227	0.45528	0.39676	0.37914	0.39103	0.38147	0.37484	0.36785
JP f	0.12760	0.03211	0.02837	0.02749	0.02735	0.03024	0.02930	0.03205
JP m	0.12300	0.04860	0.04391	0.03055	0.04446	0.04688	0.03812	0.04338
UK f	0.53562	0.53138	0.53155	0.53185	0.53487	0.53997	0.53976	0.45698
UK m	0.59478	0.58497	0.58511	0.58540	0.71037	0.58133	0.58062	0.50538
US f	0.09378	0.14635	0.02681	0.03045	0.03314	0.02632	0.02262	0.02416
US m	0.10577	0.04382	0.03365	0.03684	0.03936	0.03430	0.03333	0.03572
DIC								
Median	725.10	551.35	477.40	716.81	509.96	506.42	510.83	476.56

Note: the best models among models 1-8 are shaded by light grey colour.

In-sample mortality rate estimates using models 3 and 8:

In order to visualise the enhanced performance of model 8 as compared with the traditional LC model (model 3), Figures 7.6, 7.7, 7.8, 7.9 and 7.10 plot the time series of observed and fitted mortality rates each superimposed with their 95% credible intervals for three age groups (10-14, 40-45 and 65-69) representing young, mid and senior age groups for female and male respectively in Australia, Iceland, Japan, U.K. and U.S.. The performance of model 8 with long memory structure is much better than model 3 as model 8 can fit the trend and capture the peaks of the time series better in general. As an example, it is interestingly to see that model 8 can capture the small peak of increased mortality for U.K. during the time of WW2 better. Model 8 also gives narrower confidence intervals indicating more precise estimation. In comparing mortality rate across countries, Iceland has more variation over years and wider confidence intervals possibly due to its small population size. All mortality rates show a decreasing trend over years. The drop seems to be faster before 1950 (WW2) and slower after with leveling off thereafter for the youth and middle-aged groups but it shows an opposite trend with a convex (instead of concave) shape for senior age group. This shows that the improvement of mortality is faster for the youth and middle-aged groups in earlier years but after 1950, the improvement in the senior group is more obvious which possibly benefits from the further advancement in medical services. The trend of mortality rates also displays more variation for male and for U.S..

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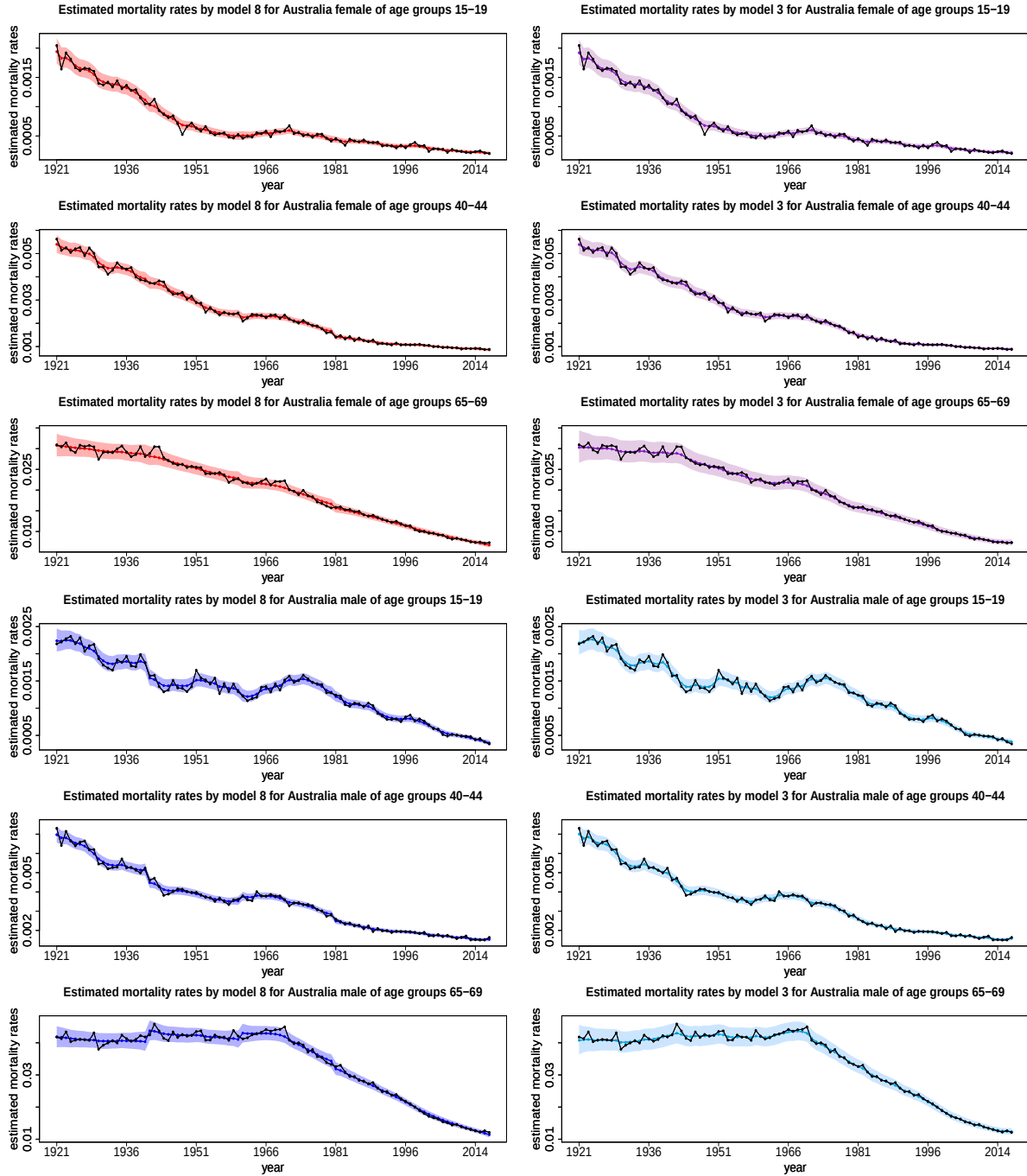


Figure 7.6. Observed mortality rate $\mu_{x,t}$ (black line) from HMD and estimated mortality rate $\hat{\mu}_{x,t}$ model 8 (left) and model 3 (right) for female (red) and male (blue) for the three age groups (10-14, 40-45 and 65-69) in Australia.

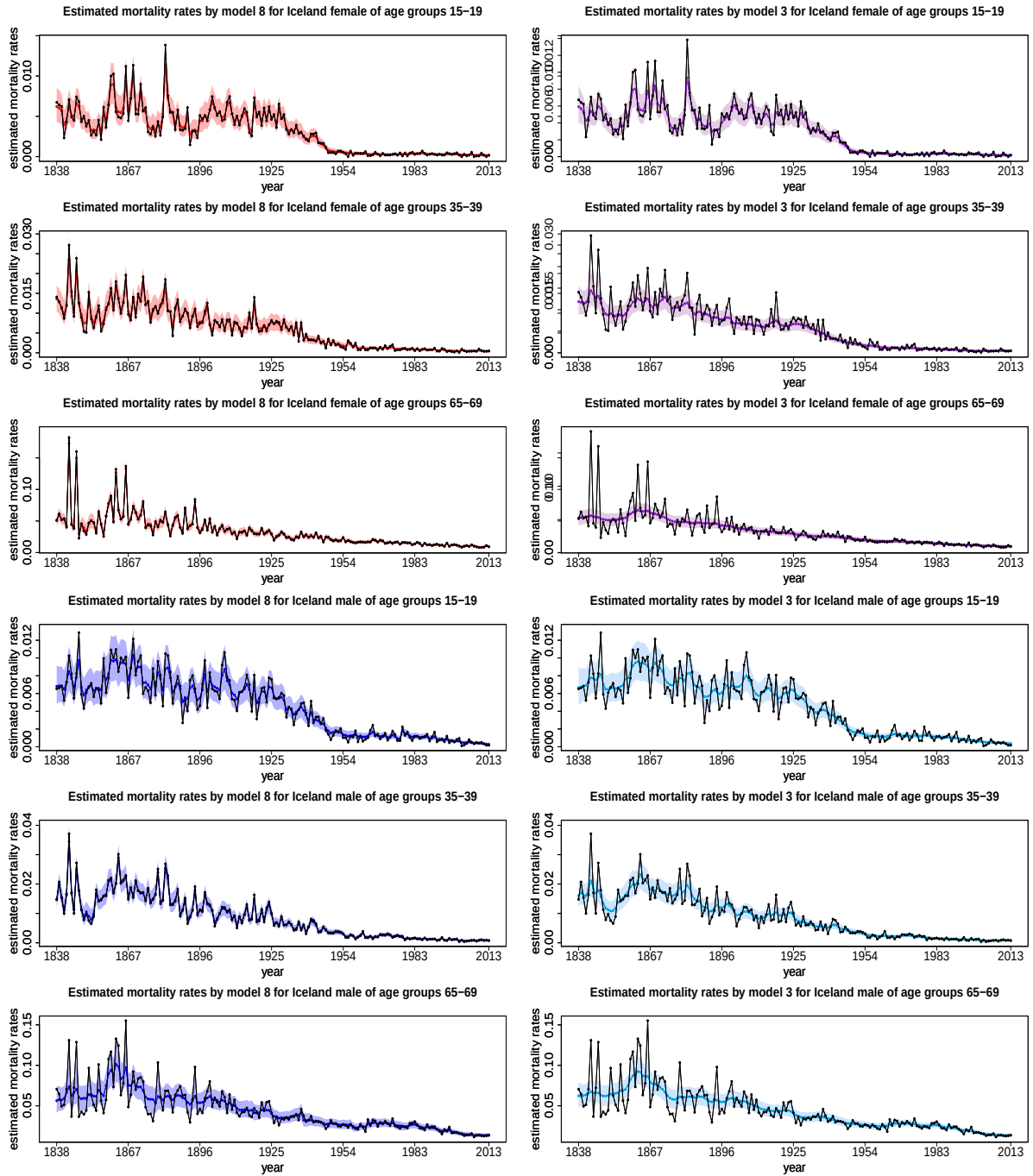


Figure 7.7. Observed mortality rate $\mu_{x,t}$ (black line) from HMD and estimated mortality rate $\hat{\mu}_{x,t}$ using model 8 (left) and model 3 (right) for female (red) and male (blue) for the three age groups (young, middle and senior) in Iceland.

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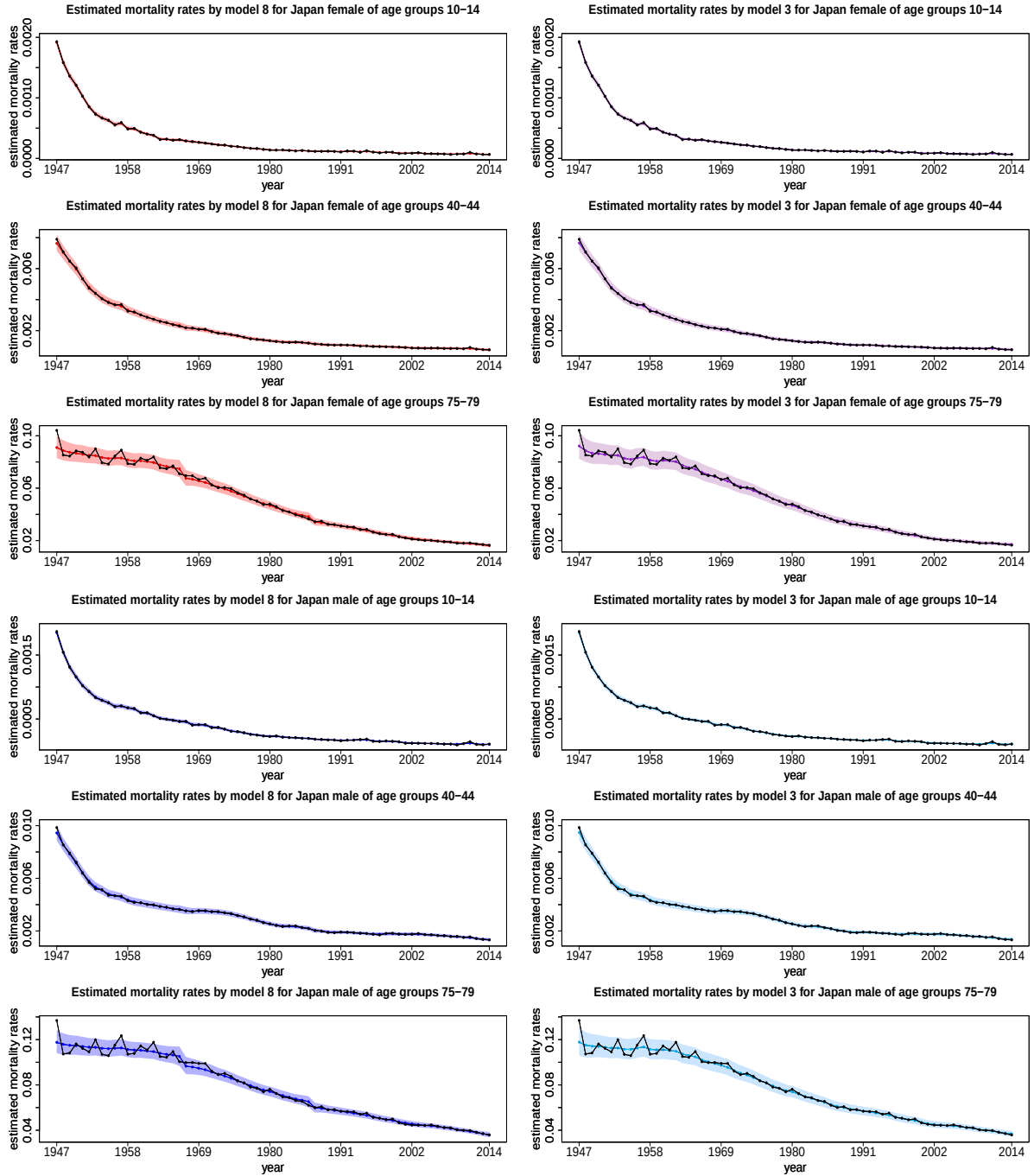


Figure 7.8. Observed mortality rate $\mu_{x,t}$ (black line) from HMD and estimated mortality rate $\hat{\mu}_{x,t}$ using model 8 (left) and model 3 (right) for female (red) and male (blue) for the three age groups (young, middle and senior) in Japan.

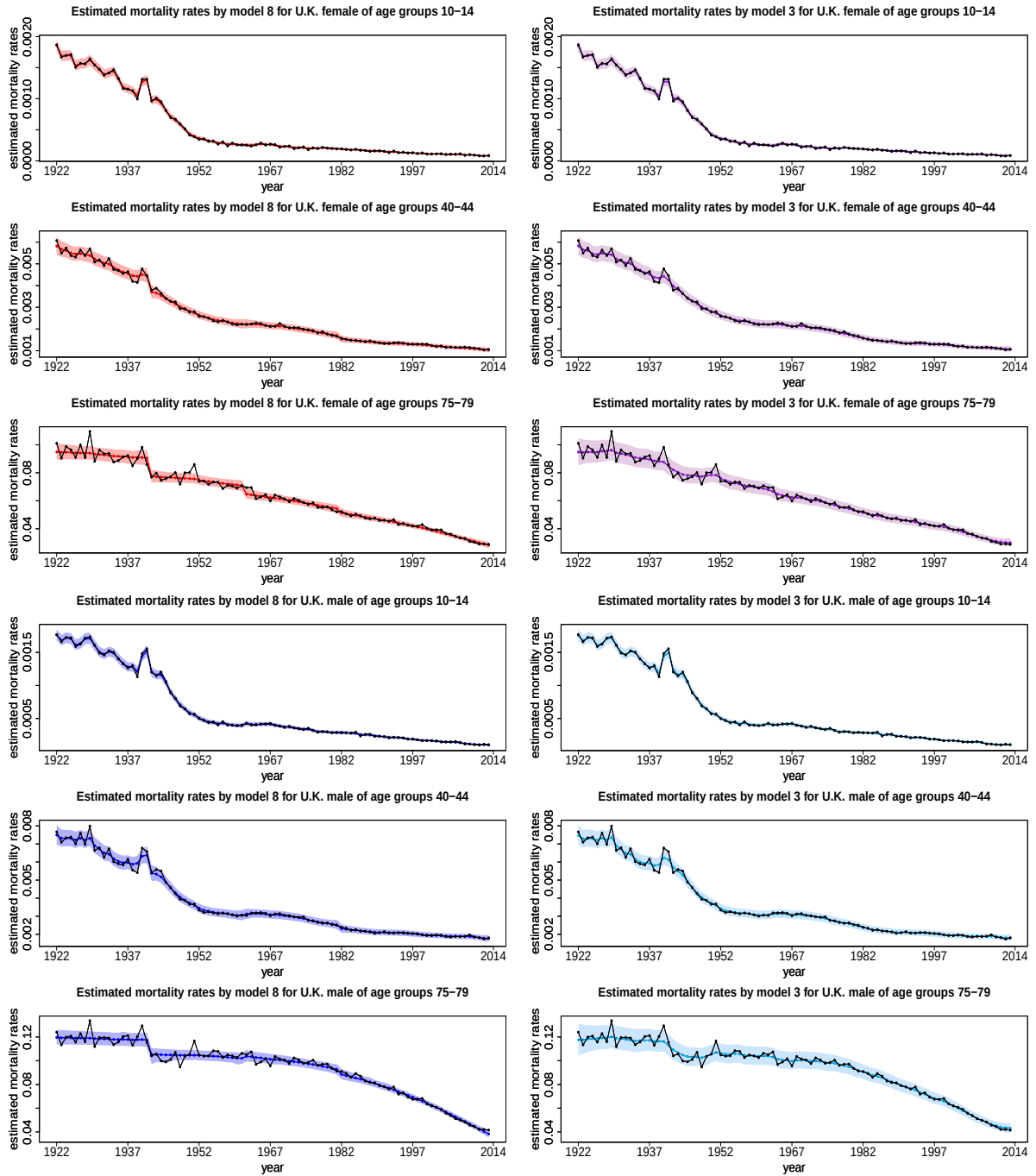


Figure 7.9. Observed mortality rate $\mu_{x,t}$ (black line) from HMD and estimated mortality rate $\hat{\mu}_{x,t}$ using model 8 (left) and model 3 (right) for female (red) and male (blue) for the three age groups (young, middle and senior) in U.K..

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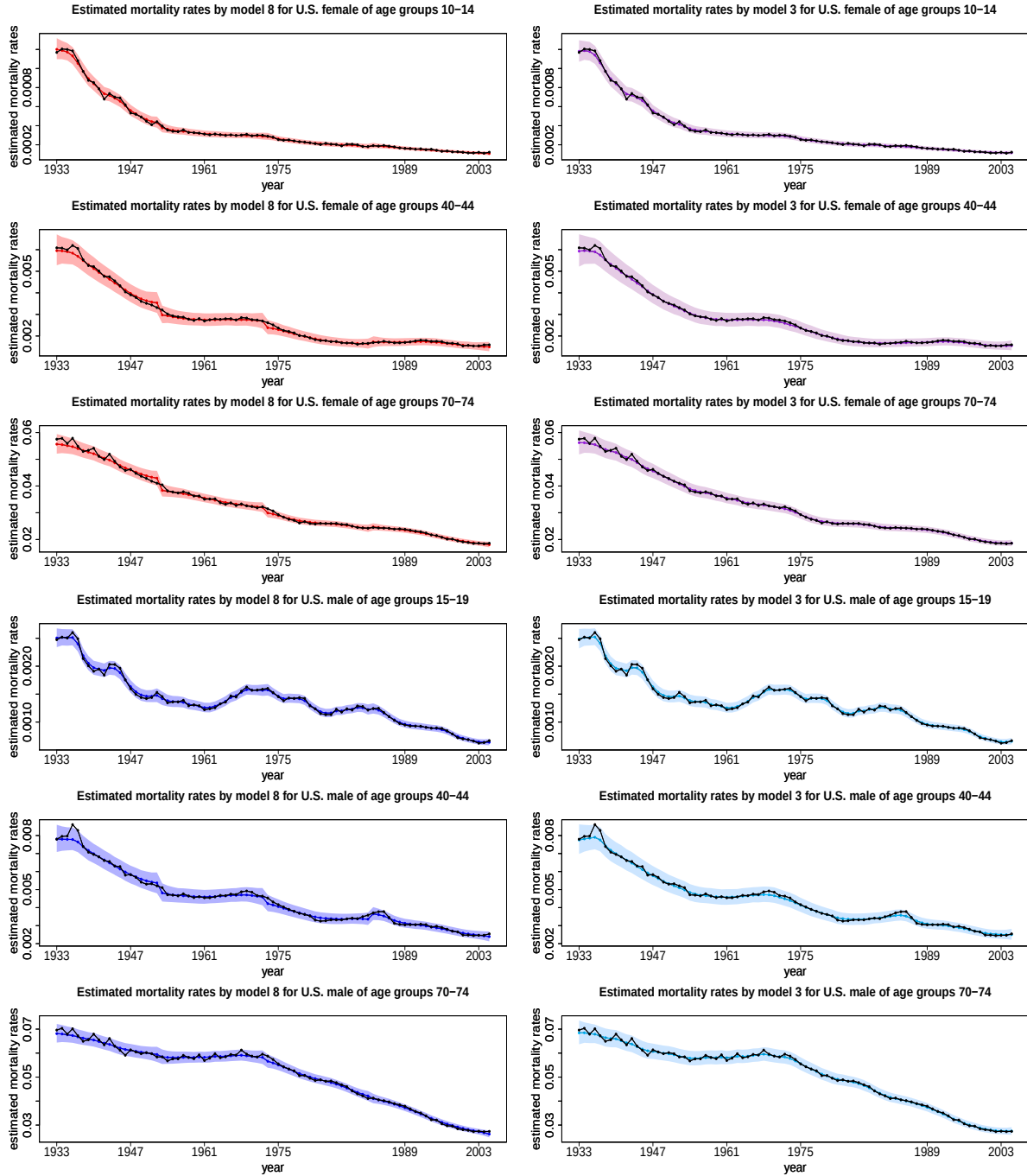


Figure 7.10. Observed mortality rate $\mu_{x,t}$ (black line) from HMD and estimated mortality rate $\hat{\mu}_{x,t}$ using model 8 (left) and model 3 (right) for female (red) and male (blue) for the three age groups (young, mid and senior) in U.S..

7.4.2 In sample fitting for life expectancy

We first illustrate the life expectancy calculation in this section.

7.4.2.1 Definition of life expectancy

Using the posterior mean of the forecasted mortality rates $\mu_{x,t}$, we can evaluate the period life expectancy at different ages and construct life tables (Koissi et al., 2006). We consider the same age groups $x \in \{0, 1-4, 5-9, \dots, 95-99\}$ and let $\tilde{x}$ to represent the initial age of each age group, that is $\tilde{x} \in \{0, 1, 5, \dots, 90, 95\}$ with group center $\dot{x} = \{0.5, 3, 7.5, 12.5, \dots, 97.5\}$. Defining further $n_{\tilde{x}}$ as the length of the interval of age group x (corresponds to $\tilde{x}$), we have $n_0 = 1, n_1 = 4, n_5 = 5, \dots, n_{95} = 5$. We then calculate the probability that a person aged $\tilde{x}$ in year t will die in the next $n_{\tilde{x}}$ years as

$$q_{\tilde{x},t}^{n_{\tilde{x}}} = \frac{n_{\tilde{x}}\mu_{x,t}}{1 + n_{\tilde{x}}(1 - a(\tilde{x}, n_{\tilde{x}}))\mu_{x,t}},$$

where $a(\tilde{x}, n_{\tilde{x}})$ is the average fraction of the $n_{\tilde{x}}$ years lived by the people who aged initially at $\tilde{x}$ will die during that age interval. Using the assumption that deaths are distributed uniformly in the interval, we set $a(\tilde{x}, n_{\tilde{x}}) = 0.5$ for all $\tilde{x}$. The hypothetical number of people $l_{\tilde{x}+n_{\tilde{x}},t}$ alive at age $\tilde{x} + n_{\tilde{x}}$ is determined by

$$l_{\tilde{x}+n_{\tilde{x}},t} = l_{\tilde{x},t}(1 - q_{\tilde{x},t}^{n_{\tilde{x}}}) = l_{0,t} \prod_{k=0}^{\tilde{x}} (1 - q_{k,t}^{n_k}),$$

where $l_{0,t} = 100,000$. We can then calculate the number of deaths

$$d_{\tilde{x},t}^{n_{\tilde{x}}} = l_{\tilde{x},t} - l_{\tilde{x}+n_{\tilde{x}},t} = l_{\tilde{x},t}q_{\tilde{x},t}^{n_{\tilde{x}}},$$

and the person-years lived

$$L_{\tilde{x},t}^{n_{\tilde{x}}} = n_{\tilde{x}}(l_{\tilde{x}+n_{\tilde{x}},t} + a(\tilde{x}, n_{\tilde{x}}) \times d_{\tilde{x},t}^{n_{\tilde{x}}}) = n_{\tilde{x}}l_{\tilde{x},t}(1 - 0.5q_{\tilde{x},t}^{n_{\tilde{x}}}).$$

The total future lifetime of the $l_{\tilde{x},t}$ people who attain age $\tilde{x}$ is $T_{\tilde{x},t} = \sum_{j>\tilde{x}} L_{j,t}^{n_{\tilde{x}}}$, where $j \in \{0, 5, \dots, 90, 95\}$. Hence, a period life expectancy at age $\tilde{x}$ can be derived by

$$e_{\tilde{x},t} = \frac{T_{\tilde{x},t}}{l_{\tilde{x},t}}. \quad (7.6)$$

7.4.2.2 Bayesian approach to life expectancy calculation

According to the definition of life expectancy $e_{\tilde{x},t}$ in Equation (7.6), we can now obtain a relationship directly between $e_{\tilde{x},t}$ and mortality rates $\mu_{x,t}$ as

$$\begin{aligned} e_{\tilde{x},t} &= \frac{\sum_{j>\tilde{x}} n_j l_{j,t} (1 - 0.5q_{j,t}^{n_j})}{l_{0,t} \prod_{k=0}^{\tilde{x}-n_{\tilde{x}}} (1 - q_{k,t}^{n_k})} = \frac{\sum_{j>\tilde{x}} n_j (1 - 0.5q_{j,t}^{n_j}) l_{0,t} \prod_{k=0}^{j-n_j} (1 - q_{k,t}^{n_k})}{l_{0,t} \prod_{k=0}^{\tilde{x}-n_{\tilde{x}}} (1 - q_{k,t}^{n_k})} \\ &= \frac{\sum_{j>\tilde{x}} n_j (1 - 0.5 \frac{n_j \mu_{j,t}}{1+0.5n_j \mu_{j,t}}) l_{0,t} \prod_{k=0}^{j-n_j} (1 - \frac{n_k \mu_{k,t}}{1+0.5n_k \mu_{k,t}})}{l_{0,t} \prod_{k=0}^{\tilde{x}-n_{\tilde{x}}} (1 - \frac{n_k \mu_{k,t}}{1+0.5n_k \mu_{k,t}})}. \end{aligned} \quad (7.7)$$

In order to calculate life expectancy, mortality rates $\mu_{x,t}$ for all age groups are needed. For calculation convenience, we denote $\boldsymbol{\mu}_t \in \mathbb{R}^g$ to be a vector of all mortality rates at year t , $\mathbf{e}_t \in \mathbb{R}^g$ the corresponding vector for life expectancy and $\mathcal{F}_{1:T}$ to be the filtration for all age groups. Therefore, in Equation (7.7) this shows that we can represent the random vector, $\mathbf{e}_t \in \mathbb{R}^g$, by a vector value transformation of the random mortality rates $\boldsymbol{\mu}_t$, which we denote by $\mathbf{e}_t = \varrho(\boldsymbol{\mu}_t)$ with $\varrho: \mathbb{R}^g \rightarrow \mathbb{R}^g$, each element of the vector function ϱ corresponding to a mapping for age group x is given by Equation (7.7). We are interest in obtaining posterior predictive distribution of $\mathbf{e}_{T+1:T+m}$ given $\mathcal{F}_{1:T}$. We first need to obtain posterior predictive distribution of $\boldsymbol{\mu}_{T+1:T+m}$ given $\mathcal{F}_{1:T}$ which is defined as follows:

$$\begin{aligned} &f(\boldsymbol{\mu}_{T+1:T+m} | \mathcal{F}_{1:T}) \\ &= \int \dots \int f(\boldsymbol{\mu}_{T+1:T+m}(\boldsymbol{\kappa}_{T+1:T+m}) | \boldsymbol{\mu}_{1:T}(\boldsymbol{\kappa}_{1:T}), \boldsymbol{\vartheta}, \mathcal{F}_{1:T}) f(\boldsymbol{\mu}_{1:T}(\boldsymbol{\kappa}_{1:T}) | \boldsymbol{\vartheta}, \mathcal{F}_{1:T}) \\ &\quad \times f(\boldsymbol{\kappa}_{1:T+m} | \boldsymbol{\vartheta}, \mathcal{F}_{1:T}) \pi(\boldsymbol{\vartheta} | \mathcal{F}_{1:T}) d\boldsymbol{\mu}_{1:T+m}(\boldsymbol{\kappa}_{1:T+m}) d\boldsymbol{\kappa}_{1:T+m} d\boldsymbol{\vartheta}, \\ &= \int \dots \int \prod_{s=1}^m [f(\boldsymbol{\mu}_{T+s}(\boldsymbol{\kappa}_{T+s}) | \boldsymbol{\mu}_{T+s-1}(\boldsymbol{\kappa}_{T+s-1}), \boldsymbol{\vartheta}, \mathcal{F}_{1:T}) f(\boldsymbol{\kappa}_{T+s} | \boldsymbol{\kappa}_{T+s-1}, \boldsymbol{\vartheta}, \mathcal{F}_{1:T})] \\ &\quad \times f(\boldsymbol{\mu}_{1:T}(\boldsymbol{\kappa}_{1:T}) | \boldsymbol{\vartheta}, \mathcal{F}_{1:T}) f(\boldsymbol{\kappa}_{1:T} | \boldsymbol{\vartheta}, \mathcal{F}_{1:T}) \pi(\boldsymbol{\vartheta} | \mathcal{F}_{1:T}) d\boldsymbol{\mu}_{1:T+m}(\boldsymbol{\kappa}_{1:T+m}) d\boldsymbol{\kappa}_{1:T+m} d\boldsymbol{\vartheta}. \end{aligned} \quad (7.8)$$

Remark 7.4.1. Equation (7.8) is an example for autoregressive order 1, which means that $\boldsymbol{\mu}_{T+s}$ only depends on $\boldsymbol{\mu}_{T+s-1}$.

Our interest is in obtaining the posterior predictive distribution of the life expectancy $\mathbf{e}_{T+1:T+m}$ given $\mathcal{F}_{1:T}$. Equation (7.7) shows that we can represent $\mathbf{e}_t$ by a vector value transformation of the random mortality rates $\boldsymbol{\mu}_t$, which we denote by $\varrho(\mathbf{e}_t)$. We note the distribution of $\mathbf{e}_t$ condition on $\mathcal{F}_{1:T}$ is then obtained via the following generic expression

$$f_{\mathbf{e}_t}(\mathbf{e}_t|\mathcal{F}_{1:t}) = f_{\boldsymbol{\mu}_t}(\varrho^{-1}(\mathbf{e}_t)|\mathcal{F}_{1:t})|J(\varrho(\boldsymbol{\mu}_t))|,$$

where $f_{\mathbf{e}_t}$ represents the multivariate probability density function (pdf) of life expectancies $\mathbf{e}_t$, and $f_{\boldsymbol{\mu}_t}$ is the multivariate pdf of mortality rates $\boldsymbol{\mu}_t$. In addition, J is the Jacobian of the transformation $\varrho(\cdot)$ which maps mortality rates to life expectancies according to Equation (7.6). The marginal mean and variance which summarise the predictive distribution of life expectancy for each age group can be calculated as below:

$$\begin{aligned}\mathbb{E}(\mathbf{e}_t) &= \int \dots \int \mathbf{e}_t f_{\mathbf{e}_t}(\mathbf{e}_t) d\mathbf{e}_{x_1,t} \dots d\mathbf{e}_{x_g,t}, \\ \mathbb{V}\text{ar}(\mathbf{e}_t) &= \int \dots \int (\mathbf{e}_t - \mathbb{E}(\mathbf{e}_t))^2 f_{\mathbf{e}_t}(\mathbf{e}_t) d\mathbf{e}_{x_1,t} \dots d\mathbf{e}_{x_g,t}.\end{aligned}$$

In practice, this procedure can be approximated by the Monte Carlo estimator, constructed from posterior samples according to Algorithm 1. Furthermore, unlike the

Algorithm 1 Compute posterior predictive distribution for life expectancy $f(\mathbf{e}_{T+1:T+m}|\mathcal{F}_{1:T})$

- 1: Draw L posterior samples $\boldsymbol{\mu}_{T+1:T+m}$ under the filtration $\mathcal{F}_{1:T}$.
- 2: Calculate L samples of $\mathbf{e}_{T+1:T+m}^{(l)}$ based on $\boldsymbol{\mu}_{T+1:T+m}^{(l)}$ according to Equation (7.7) as the approximated predictive distribution.
- 3: Construct the kernel density estimation based on posterior predictive distribution of the L samples $\mathbf{e}_{T+1:T+m}^{(l)}$.
- 4: Calculate posterior summaries based on the approximated predictive distribution of $\mathbf{e}_{T+1:T+m}^{(l)}$

$$\begin{aligned}\widehat{\mathbb{E}}(\mathbf{e}_{T+1:T+m}) &= \frac{1}{L} \sum_{l=1}^L \mathbf{e}_{T+1:T+m}^{(l)}, \\ \widehat{\mathbb{V}\text{ar}}(\mathbf{e}_{T+1:T+m}) &= \frac{1}{L-1} \left[\sum_{l=1}^L \left(\mathbf{e}_{T+1:T+m}^{(l)} \right)^2 - [\widehat{\mathbb{E}}(\mathbf{e}_{T+1:T+m})]^2 \right].\end{aligned}$$

where L is the number of iterations after burn-in in each MCMC sampler run given the window of information $\mathcal{F}_{1:T}$ and $\boldsymbol{\mu}_{T+1:T+m}^{(l)}$ is the l -th draw in the posterior sample of $\boldsymbol{\mu}_{T+1:T+m}$.

Bayesian filtering which is based on filtration $\mathcal{F}_{x,1:T}$, the Bayesian smoothing uses also the future measurements for computation. The estimator of states $y_{x,t}$ at each time $t < T$ conditional on all the measurements up to a time T is $f(y_{x,t}|\mathcal{F}_{x,1:T})$ (Särkkä,

2013). The theorem proposed by Kitagawa (1987) provides the marginal posterior distributions for Bayesian smoothing.

Theorem 7.4.1 (The Bayesian smoothing). *The Bayesian smoother for computing the smoothed mortality rates distributions $f(\boldsymbol{\mu}_t|\boldsymbol{\vartheta}, \mathcal{F}_{1:T})$ for any $t < T$ are given by the following Bayesian smoothing equations*

$$f(\boldsymbol{\mu}_t|\boldsymbol{\vartheta}, \mathcal{F}_{1:T}) = f(\boldsymbol{\mu}_t|\boldsymbol{\vartheta}, \mathcal{F}_{1:t}) \int \left[\frac{f(\boldsymbol{\mu}_{t+1:T}(\boldsymbol{\kappa}_{t+1:T})|\boldsymbol{\mu}_t(\boldsymbol{\kappa}_t), \boldsymbol{\vartheta})f(\boldsymbol{\mu}_{t+1:T}(\boldsymbol{\kappa}_{t+1:T})|\boldsymbol{\vartheta}, \mathcal{F}_{1:T})}{f(\boldsymbol{\mu}_{t+1:T}(\boldsymbol{\kappa}_{t+1:T})|\boldsymbol{\vartheta}, \mathcal{F}_{1:t})} \right] \\ \times f(\boldsymbol{\kappa}_{1:T}|\boldsymbol{\vartheta}, \mathcal{F}_{1:T})d\boldsymbol{\mu}_{t+1:T}(\boldsymbol{\kappa}_{t+1:T})d\boldsymbol{\kappa}_{1:T}.$$

where $f(\boldsymbol{\mu}_{t+1:T}(\boldsymbol{\kappa}_{t+1:T})|\boldsymbol{\vartheta}, \mathcal{F}_{1:t})$ is the predictive distribution defined in Equation (7.8).

Since the Bayesian smoothing of life expectancy $f(e_t|\mathcal{F}_{x,1:T})$ can be calculated by transformation of mortality rates $\mu_{x,t}$, the smoothed posterior distribution of life expectancy $e_{\tilde{x},t}$ is given by Algorithm 1.

In-sample fitting for life expectancies:

As model 8 clearly provides more accurate mortality rate estimates, it is chosen to estimate life expectancy for the five selected countries. Figure 7.11 graphs the life expectancy of Australia calculated using the mortality rate estimates from model 8 for both genders and different age groups. Figure 7.12, 7.13, 7.14 and 7.15 display the estimated life expectancies for Iceland, Japan, U.K. and U.S.. We note that the life expectancy for the last age group is always 2.5 and hence is omitted. The plots display a linear increasing trend across year for female at younger age but a sharper increase after 1966 for senior age group. This sharper increase pattern is more noticeable for male particularly at senior age. The plots also show close agreement with the life expectancies calculated using the observed mortality rates from HMD. So we can conclude that the performance of in-sample life expectancy estimates are satisfactory.

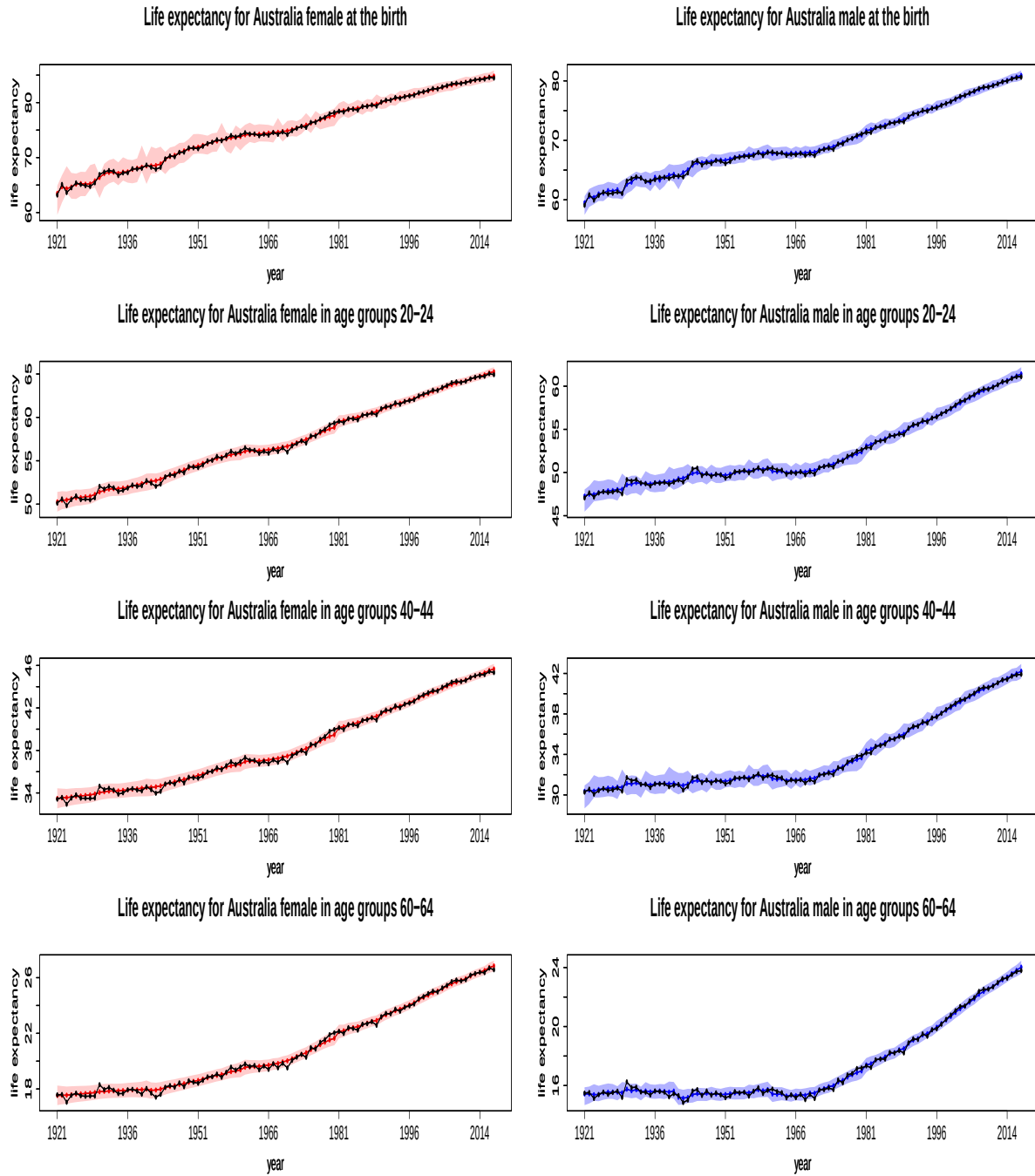


Figure 7.11. Observed life expectancy $e_{\tilde{x},t}$ (black line) from HMD and life expectancy calculated by the estimated mortality rates from model 8 (red in female and blue in male) for both female (left column) and male (right column) and four age groups (0, 20-24, 40-44 and 60-64) in Australia.

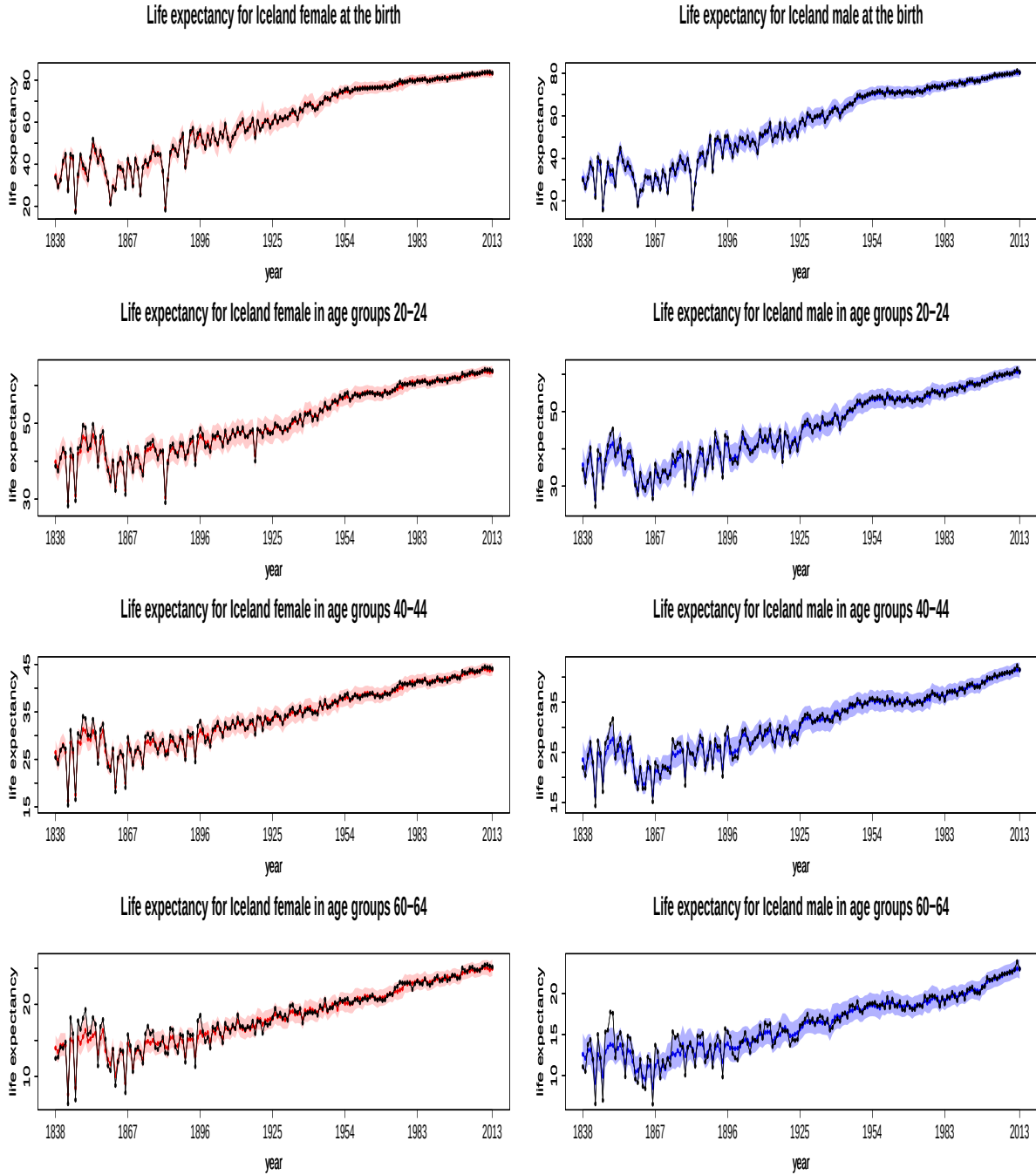


Figure 7.12. Observed life expectancy $e_{\tilde{x},t}$ (black line) from HMD and life expectancy calculated by the estimated mortality rates from model 8 (red in female and blue in male) for both female (left column) and male (right column) and four age groups (0, 20-24, 40-44 and 60-64) in Iceland.

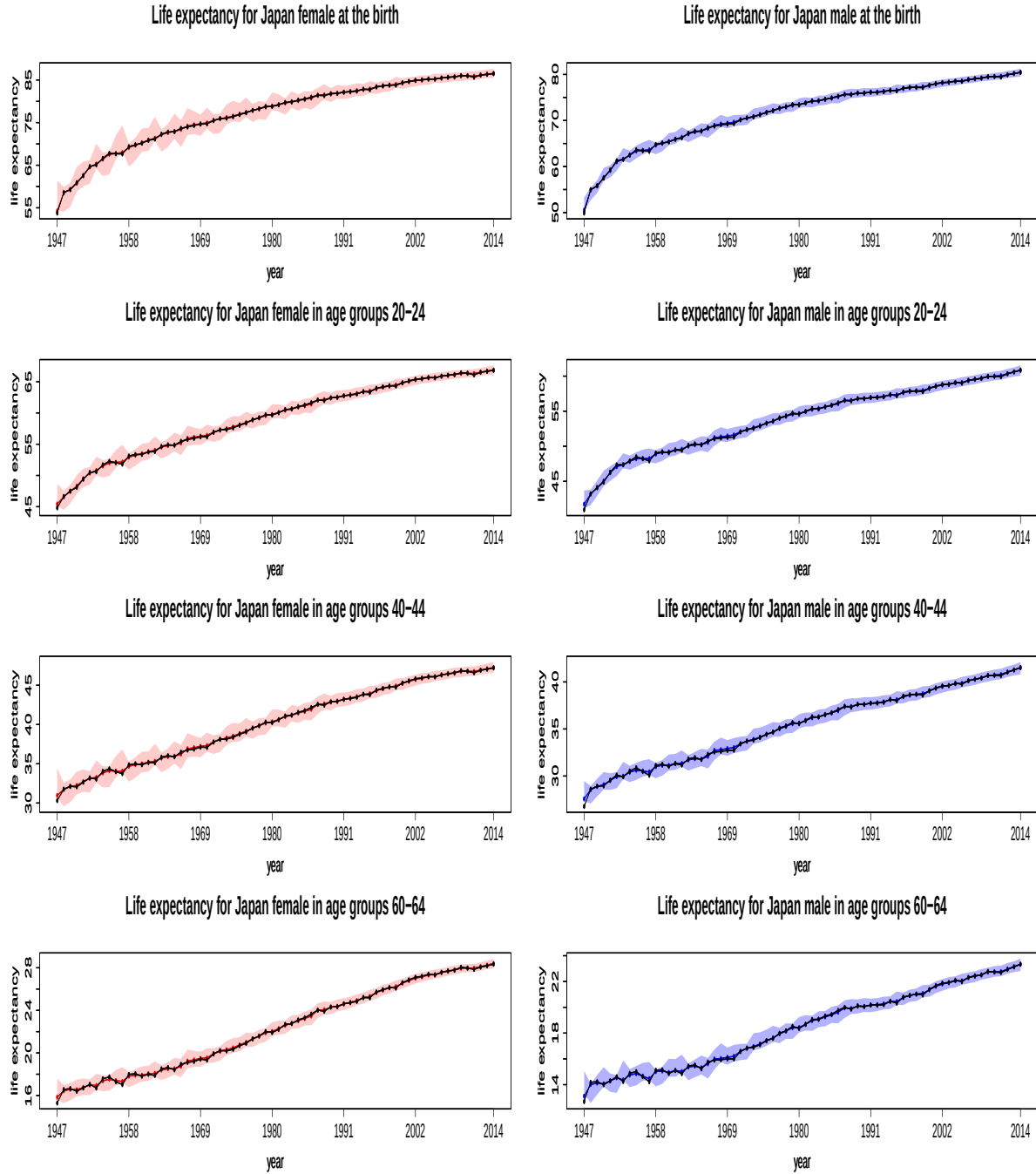


Figure 7.13. Observed life expectancy $e_{\tilde{x},t}$ (black line) from HMD and life expectancy calculated by the estimated mortality rates from model 8 (red in female and blue in male) for both female (left column) and male (right column) and four age groups (0, 20-24, 40-44 and 60-64) in Japan.

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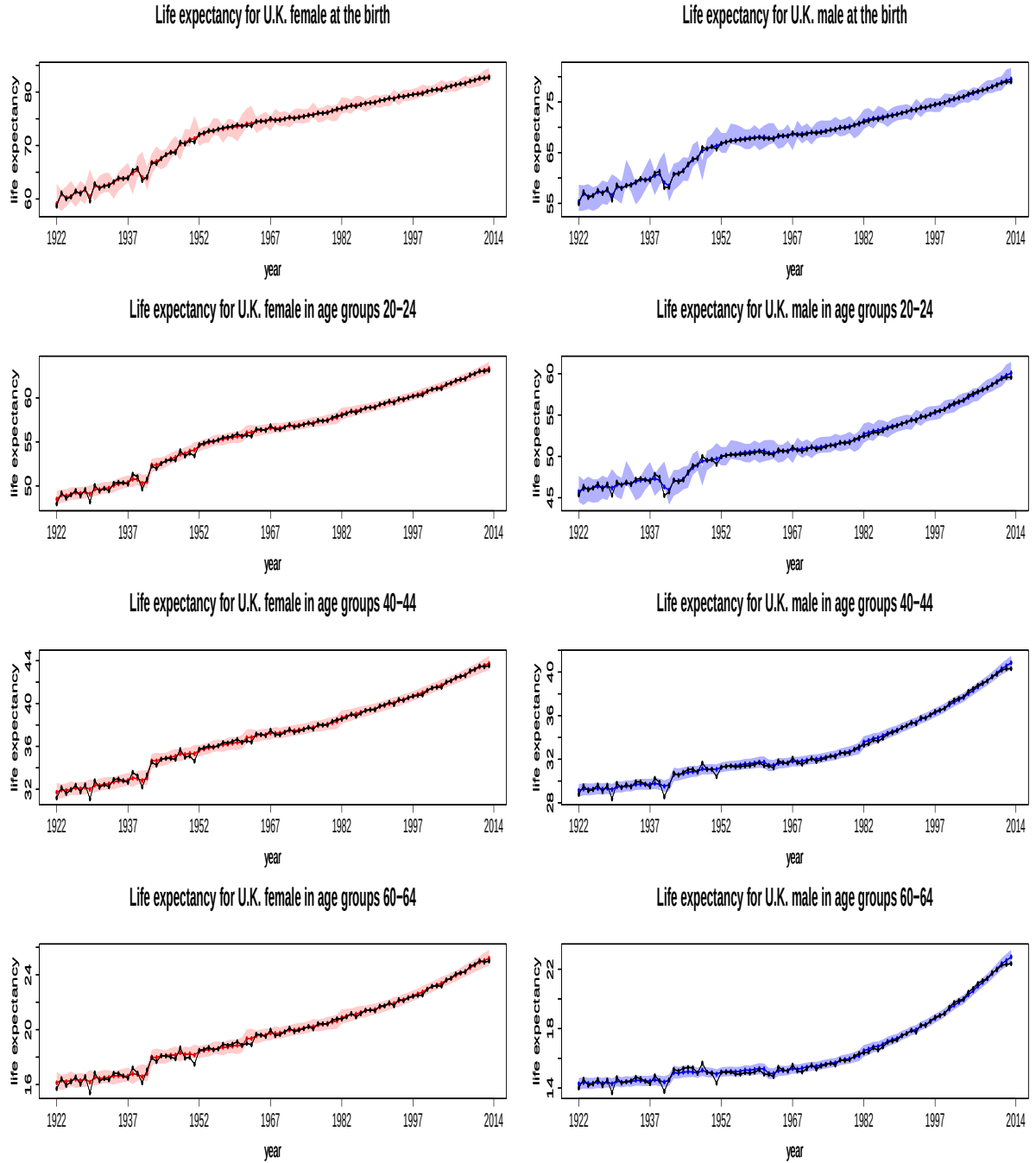


Figure 7.14. Observed life expectancy $e_{\tilde{x},t}$ (black line) from HMD and life expectancy calculated by the estimated mortality rates from model 8 (red in female and blue in male) for both female (left column) and male (right column) and four age groups (0, 20-24, 40-44 and 60-64) in U.K..

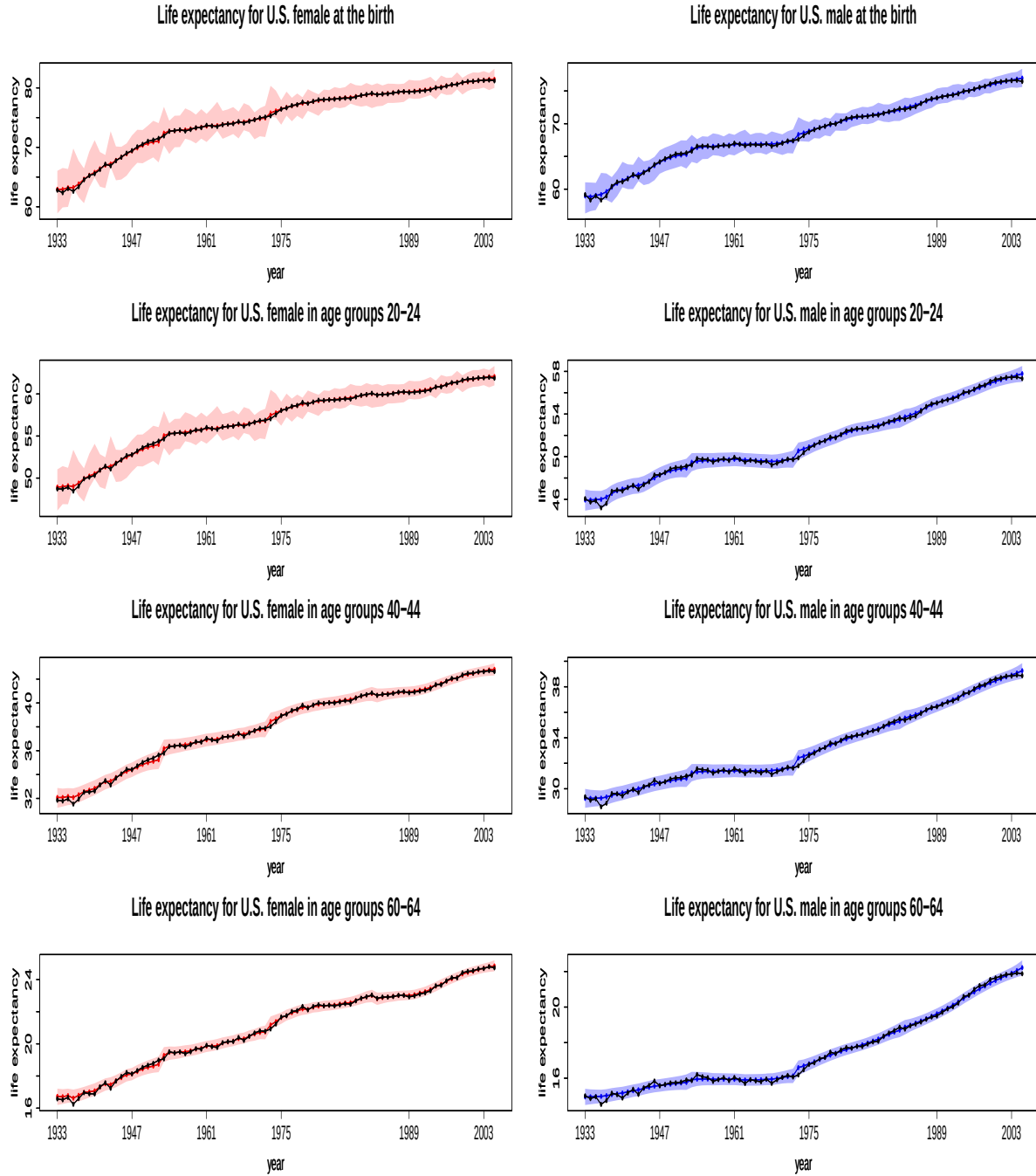


Figure 7.15. Observed life expectancy $e_{\tilde{x},t}$ (black line) from HMD and life expectancy calculated by the estimated mortality rates from model 8 (red in female and blue in male) for both female (left column) and male (right column) and four age groups (0, 20-24, 40-44 and 60-64) in U.S..

7.4.3 Out-of-sample forecasting for mortality rates

In order to measure the forecast performance of the eight models, the number of death counts $\mathbf{Y}_{x,1:T}$ by 21 age groups and both genders of the five selected countries are each divided into two parts, the training $\mathbf{Y}_{x,1:(T-20)}$ and forecast $\mathbf{Y}_{x,(T-19):T}$ for years 1994 to 2014. Similar to Table 7.6, Table 7.7 reports the four criteria that compare the forecasted mortality rates $\hat{\boldsymbol{\mu}}_{x,(T-19):T}$ with the observed for the eight models. Model 8 again provides the best forecasts in all expect five cases in model 3 for Iceland which is a short memory data set. To view the better performance of model 8 compared with model 3, Figure 7.16 plots the time series of observed $\boldsymbol{\mu}_{x,T-19:T}$ from HMD, the forecasted $\hat{\boldsymbol{\mu}}_{x,T-19:T}$ using model 3 and the forecasted $\hat{\boldsymbol{\mu}}_{x,T-19:T}$ using model 8 with 95% credible interval for the young, middle and old age groups, respectively. The forecasted mortality rates for Iceland, Japan, U.K. and U.S. are graphed in Figure 7.17, 7.18, 7.19 and 7.20. The forecast performance of model 8 with long memory component is much better than the traditional LC model (model 3) because the forecasted mortality rates are closer to the observed and have sharper credible intervals which enclose the observed except the senior male group in 2014. Furthermore, the four criteria for the forecasted death counts $\hat{\mathbf{Y}}_{x,(T-19):T}$ using all the eight models are reported in Table 7.8. Table 7.8 reported the four criteria for the eight models evaluated. Again we verify that model 8 is the best forecast model in all expect two cases from Iceland female using model 3.

To view the enhanced forecast performance of model 8 compared with model 3, Figure 7.21 plots the time series of observed $\mathbf{Y}_{x,T-19:T}$ (in black colour) from HMD, the forecasted $\hat{\mathbf{Y}}_{x,T-19:T}$ using model 3 (female in purple and male in skyblue) and the forecasted $\hat{\mathbf{Y}}_{x,T-19:T}$ using model 8 with 95% credible interval (female in red and male in blue) for the young, mid and old age groups, respectively). The forecast performance of model 8 with long memory component is much better than the traditional LC model (model 3) because the forecasted values have a sharper credible interval.

Table 7.7. Four selection criteria to measure the accuracy of forecasted mortality rates $\hat{\mu}_{x,(T-19):T}$ for five selected countries.

Country	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8
MAE								
AU f	0.0089	0.0167	0.0059	0.0194	0.0152	0.0061	0.0047	0.0018
AU m	0.0139	0.0191	0.0099	0.0237	0.0181	0.0121	0.0101	0.0046
IS f	0.0065	0.0152	0.0072	0.0349	0.0192	0.0095	0.0073	0.0345
IS m	0.0094	0.0153	0.0067	0.0245	0.0204	0.0084	0.0079	0.0148
JP f	0.0154	0.0225	0.0101	0.0153	0.0287	0.0135	0.0102	0.0055
JP m	0.0124	0.0200	0.0093	0.0236	0.0276	0.0140	0.0083	0.0037
UK f	0.0116	0.0162	0.0059	0.0222	0.0221	0.0079	0.0068	0.0049
UK m	0.0152	0.0215	0.0111	0.0265	0.0255	0.0134	0.0112	0.0018
US f	0.0037	0.0097	0.0033	0.0209	0.0149	0.0044	0.0033	0.0027
US m	0.0075	0.0124	0.0065	0.0140	0.0155	0.0079	0.0065	0.0042
RMSE								
AU f	0.0155	0.0291	0.0123	0.0416	0.0288	0.0137	0.0102	0.0042
AU m	0.0233	0.0307	0.0187	0.0468	0.0313	0.0231	0.0193	0.0116
IS f	0.0125	0.0262	0.0153	0.0857	0.0389	0.0185	0.0158	0.0868
IS m	0.0218	0.0248	0.0127	0.0652	0.0430	0.0152	0.0143	0.0237
JP f	0.0322	0.0451	0.0226	0.0322	0.0575	0.0314	0.0229	0.0143
JP m	0.0230	0.0361	0.0187	0.0453	0.0503	0.0290	0.0174	0.0083
UK f	0.0184	0.0284	0.0123	0.0468	0.0397	0.0171	0.0132	0.0107
UK m	0.0253	0.0355	0.0211	0.0528	0.0438	0.0256	0.0212	0.0034
US f	0.0069	0.0163	0.0067	0.0395	0.0260	0.0093	0.0067	0.0058
US m	0.0131	0.0201	0.0127	0.0231	0.0259	0.0157	0.0127	0.0101
MAPE								
AU f	0.847	1.906	0.384	0.900	3.554	0.488	0.396	0.195
AU m	0.965	1.696	0.483	0.841	2.690	0.550	0.466	0.206
IS f	0.184	0.402	0.190	0.921	0.506	0.237	0.183	0.451
IS m	0.205	0.503	0.220	0.804	0.670	0.261	0.203	0.218
JP f	1.463	1.839	0.403	0.879	2.816	0.578	0.410	0.323
JP m	1.065	1.441	0.394	0.862	2.233	0.556	0.392	0.226
UK f	0.619	1.816	0.343	0.872	4.664	0.444	0.332	0.172
UK m	0.747	1.389	0.410	0.832	3.097	0.528	0.411	0.145
US f	0.311	0.903	0.196	0.901	1.347	0.269	0.198	0.110
US m	0.474	0.957	0.341	0.814	1.242	0.423	0.343	0.186
MASE								
AU f	0.866	1.620	0.573	1.188	0.759	0.306	0.234	0.091
AU m	1.067	1.039	0.538	0.650	0.495	0.330	0.276	0.125
IS f	0.653	0.454	0.215	1.040	0.572	0.281	0.217	0.494
IS m	0.918	0.759	0.332	1.214	1.012	0.416	0.389	0.211
JP f	0.507	2.258	0.333	0.505	0.950	0.447	0.338	0.181
JP m	0.637	1.477	0.241	0.611	0.713	0.361	0.308	0.139
UK f	0.349	1.489	0.539	2.046	2.037	0.725	0.205	0.150
UK m	0.734	1.040	0.538	1.282	1.233	0.649	0.541	0.095
US f	0.239	0.633	0.109	0.695	0.496	0.145	0.109	0.090
US m	0.576	0.952	0.344	0.742	0.823	0.420	0.346	0.114

Note: the best models among models 1-8 are shaded by light grey colour.

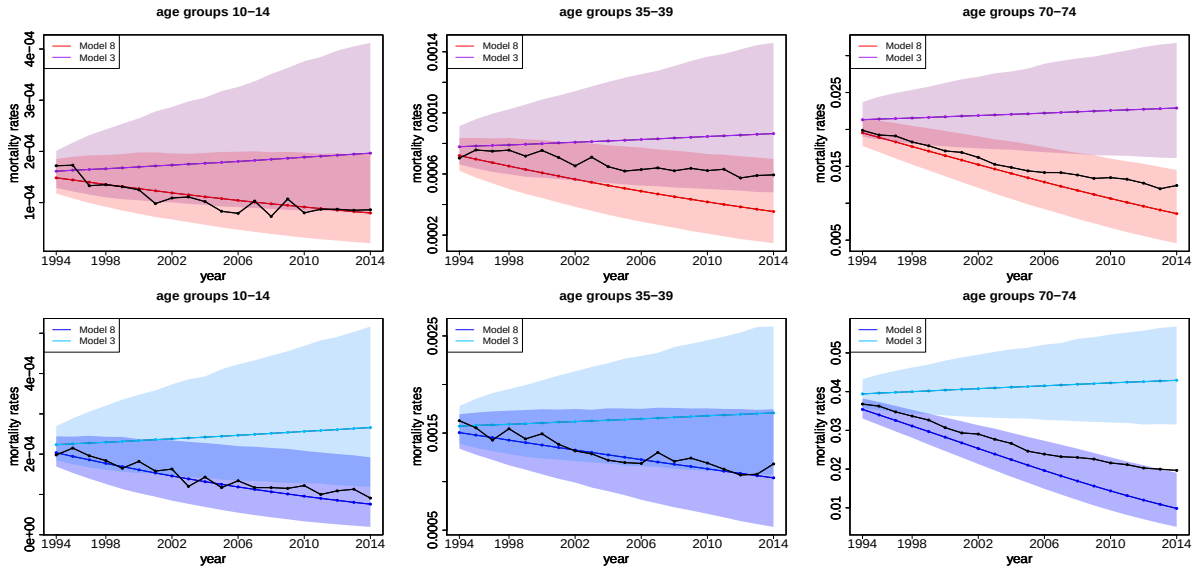


Figure 7.16. Observed mortality rates $\mu_{x,T-20:T}$ (black line) from HMD and forecasted $\hat{\mu}_{x,T-20:T}$ by model 3 (purple in female and skyblue in male) and model 8 (red in female and blue in male) for both female (first row) and male (second row) and three age groups (10-14, 35-39 and 70-74) in Australia.

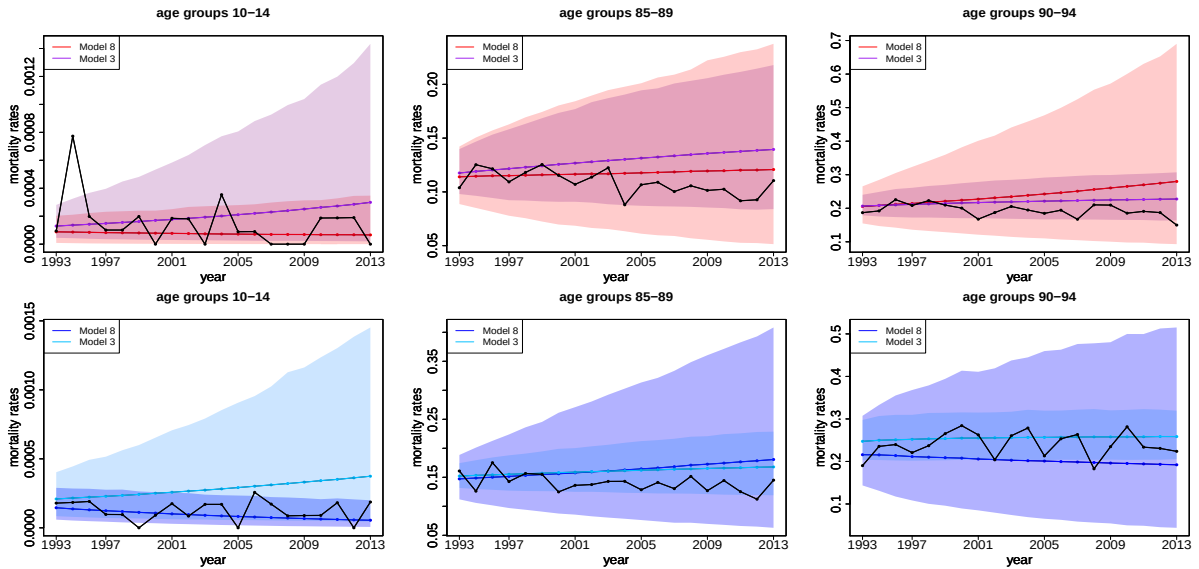


Figure 7.17. Observed mortality rates $\mu_{x,T-20:T}$ (black line) from HMD and forecasted $\hat{\mu}_{x,T-20:T}$ by model 3 (purple in female and skyblue in male) and model 8 (red in female and blue in male) for both female (first row) and male (second row) and three age groups (10-14, 35-39 and 70-74) in Iceland.

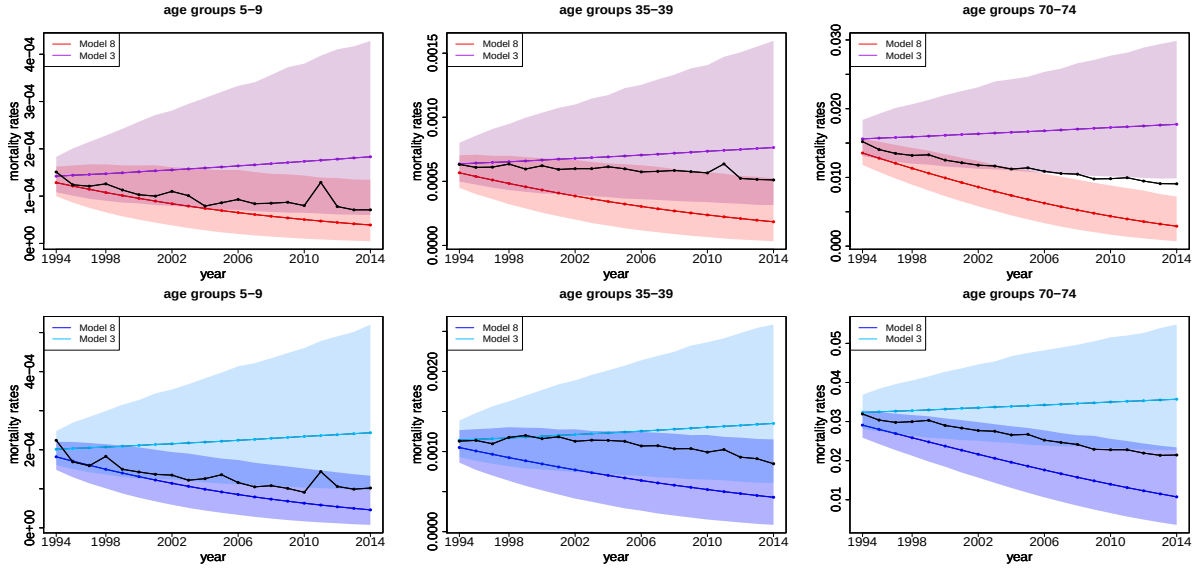


Figure 7.18. Observed mortality rates $\mu_{x,T-20:T}$ (black line) from HMD and forecasted $\hat{\mu}_{x,T-20:T}$ by model 3 (purple in female and skyblue in male) and model 8 (red in female and blue in male) for both female (first row) and male (second row) and three age groups (10-14, 35-39 and 70-74) in Japan.

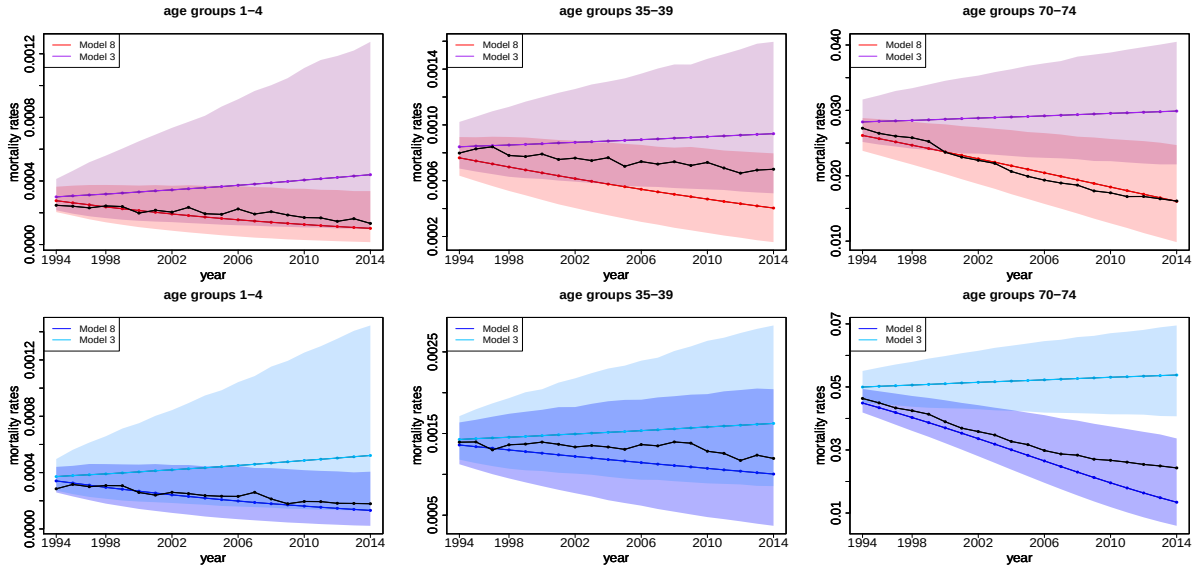


Figure 7.19. Observed mortality rates $\mu_{x,T-20:T}$ (black line) from HMD and forecasted $\hat{\mu}_{x,T-20:T}$ by model 3 (purple in female and skyblue in male) and model 8 (red in female and blue in male) for both female (first row) and male (second row) and three age groups (10-14, 35-39 and 70-74) in U.K..

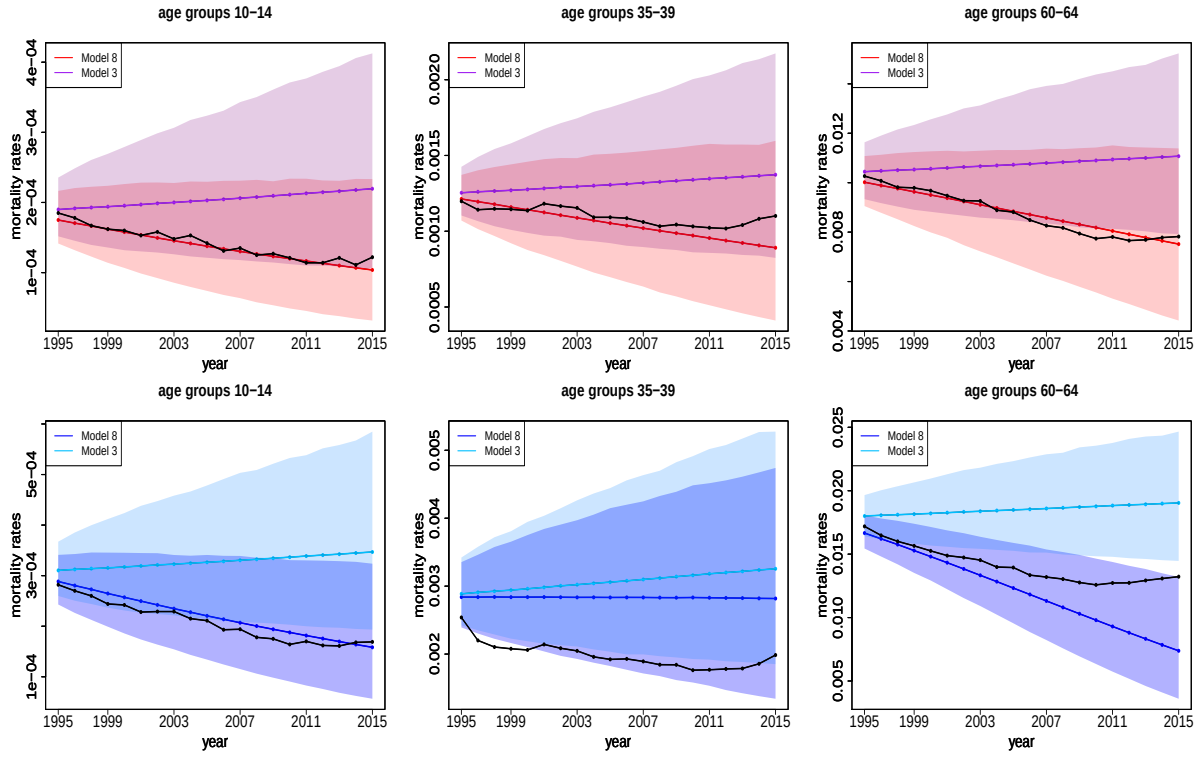


Figure 7.20. Observed mortality rates $\mu_{x,T-20:T}$ (black line) from HMD and forecasted $\hat{\mu}_{x,T-20:T}$ by model 3 (purple in female and skyblue in male) and model 8 (red in female and blue in male) for both female (first row) and male (second row) and three age groups (10-14, 35-39 and 70-74) in U.S..

Table 7.8. Four selection criteria to measure the accuracy of forecasted death counts $\hat{\mathbf{Y}}_{x,(T-19):T}$ for the five selected countries.

Country	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8
MAE								
AU f	97.7	213.8	63.6	192.2	323.2	128.9	63.9	21.7
AU m	99.4	179.3	60.0	123.0	270.3	75.1	62.3	25.4
IS f	16.8	48.8	15.5	32.9	73.5	23.3	14.8	13.2
IS m	22.4	61.1	19.3	27.8	86.7	26.2	20.0	24.8
JP f	209.7	288.9	124.6	161.0	421.8	174.3	126.0	83.7
JP m	82.5	126.7	45.0	101.3	167.9	66.2	37.5	20.2
UK f	40.9	90.9	22.5	84.8	187.0	31.5	23.7	18.1
UK m	38.0	65.5	22.9	52.5	118.8	28.5	23.0	6.1
US f	22.6	67.2	20.6	128.4	106.0	28.5	20.8	12.9
US m	34.9	60.5	28.6	67.7	76.3	35.5	28.8	15.9
RMSE								
AU f	177.1	368.9	155.6	424.2	604.5	385.4	153.6	47.7
AU m	149.1	266.4	111.2	174.0	538.0	140.5	114.1	52.3
IS f	25.8	72.3	25.6	60.9	126.5	36.3	25.8	25.0
IS m	31.4	79.5	29.5	46.2	147.3	39.1	31.7	34.1
JP f	448.4	654.2	371.4	314.7	944.6	543.8	373.6	293.2
JP m	127.6	185.6	86.8	156.5	268.4	133.5	77.6	42.8
UK f	62.8	131.4	53.4	185.5	307.2	78.5	56.4	51.1
UK m	55.7	91.0	40.3	75.6	236.6	49.8	40.4	9.5
US f	40.6	110.6	45.6	276.3	180.2	66.4	45.5	27.4
US m	48.4	77.0	46.6	83.3	100.7	58.9	46.7	34.0
MAPE								
AU f	0.846	1.905	0.384	0.898	3.390	0.480	0.380	0.195
AU m	0.965	1.696	0.482	0.840	2.689	0.550	0.466	0.206
IS f	0.386	1.189	0.379	0.802	1.790	0.539	0.342	0.280
IS m	0.486	1.391	0.440	0.633	1.975	0.571	0.374	0.408
JP f	1.394	1.838	0.403	0.878	2.813	0.577	0.410	0.323
JP m	1.058	1.501	0.393	0.861	2.232	0.555	0.391	0.226
UK f	0.708	1.812	0.341	0.870	4.653	0.442	0.330	0.172
UK m	0.746	1.388	0.410	0.830	3.091	0.528	0.411	0.145
US f	0.312	0.904	0.197	0.899	1.350	0.270	0.199	0.109
US m	0.473	0.956	0.341	0.812	1.241	0.422	0.343	0.186
MASE								
AU f	1.041	2.277	0.677	1.034	1.740	0.694	0.344	0.162
AU m	1.192	1.806	0.604	0.934	2.052	0.571	0.473	0.193
IS f	1.686	3.275	1.043	2.210	4.932	1.532	0.972	0.408
IS m	1.839	4.114	1.301	1.872	5.841	1.743	1.274	0.666
JP f	2.397	3.302	0.624	0.806	2.113	0.873	0.631	0.402
JP m	0.967	0.991	0.352	0.792	1.313	0.518	0.443	0.264
UK f	1.041	2.311	0.573	2.158	4.756	0.800	0.163	0.125
UK m	0.750	1.292	0.451	1.036	2.344	0.562	0.453	0.119
US f	0.183	0.546	0.137	0.849	0.701	0.189	0.138	0.085
US m	0.716	1.242	0.419	0.991	1.117	0.520	0.422	0.209

Note: the best models among models 1-7 are shaded by light grey colour.

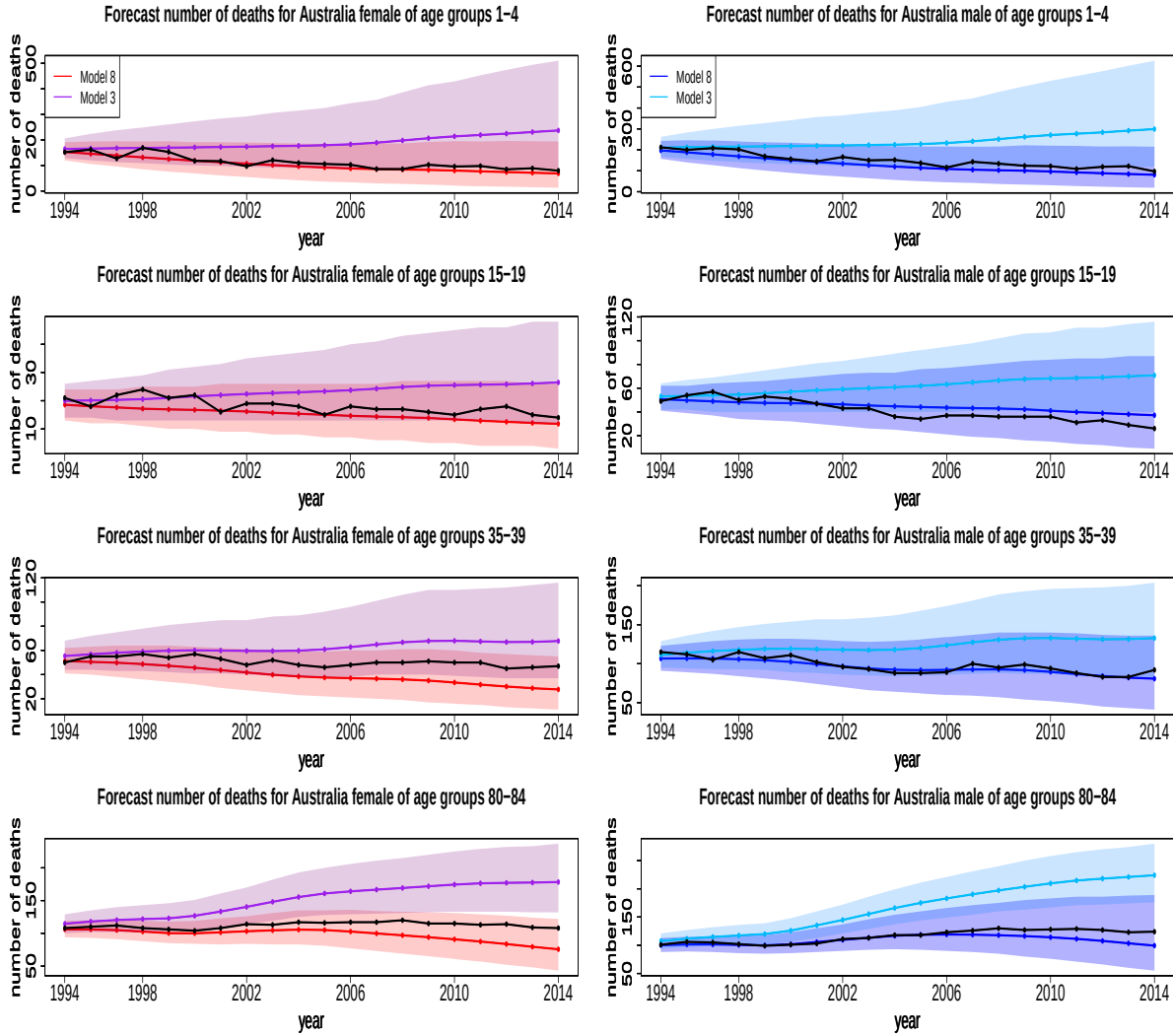


Figure 7.21. Observed death counts $Y_{x,T-19:T}$ (black line) from HMD and forecasted $\hat{Y}_{x,T-19:T}$ by model 3 (purple in female and skyblue in male) and model 8 (red in female and blue in male) for both female (left column) and male (right column) and four age groups (1-4, 15-19, 35-39 and 80-84) in Australia.

7.4.4 Out-of-sample forecasting for life expectancies

Life expectancy forecasts 20 years ahead are similarly calculated using mortality forecasts. Figure 7.22 displays the 20-step ahead forecasts of Australia life expectancy by model 8 for both genders and different age groups (0, 20-24, 40-44 and 60-64). The predicted values are close to the observed life expectancy. The forecasted life expectancy for Iceland, Japan, U.K. and U.S. are provided in Figures 7.23, 7.24, 7.25 and 7.26. The performance of out-of-sample life expectancy forecasts is also reasonable as the

forecasted values are close to the life expectancies calculated using observed mortality rates from HMD and these “observed” life expectancies are contained within the credible intervals.

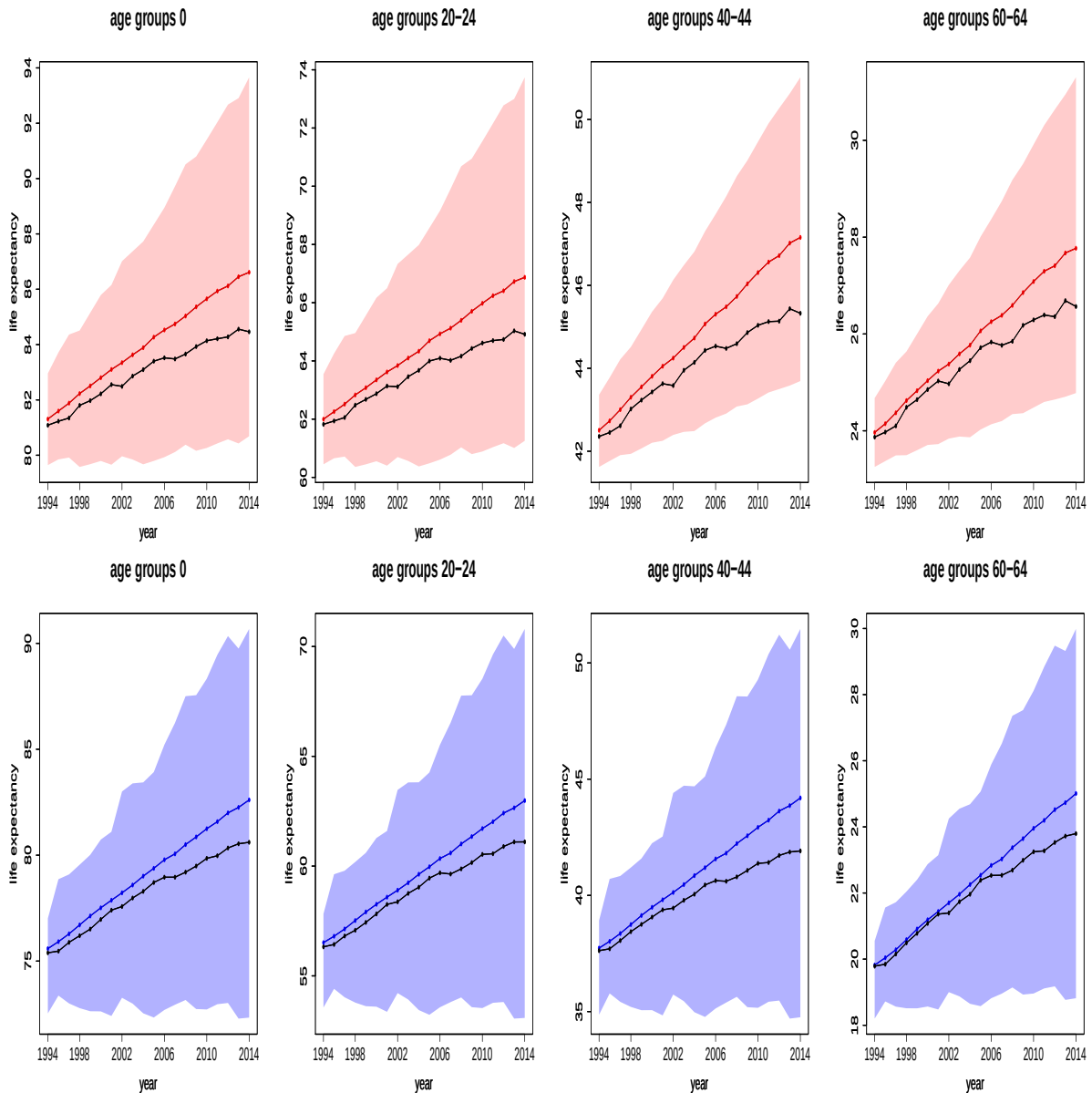


Figure 7.22. Observed life expectancy $e_{\hat{x},t}$ from HMD (black line) and life expectancy calculated by the forecasted mortality rates from model 8 (red in female and blue in male) for both female (first row) and male (second row) and four age groups (0, 20-24, 40-44 and 60-64) in Australia.

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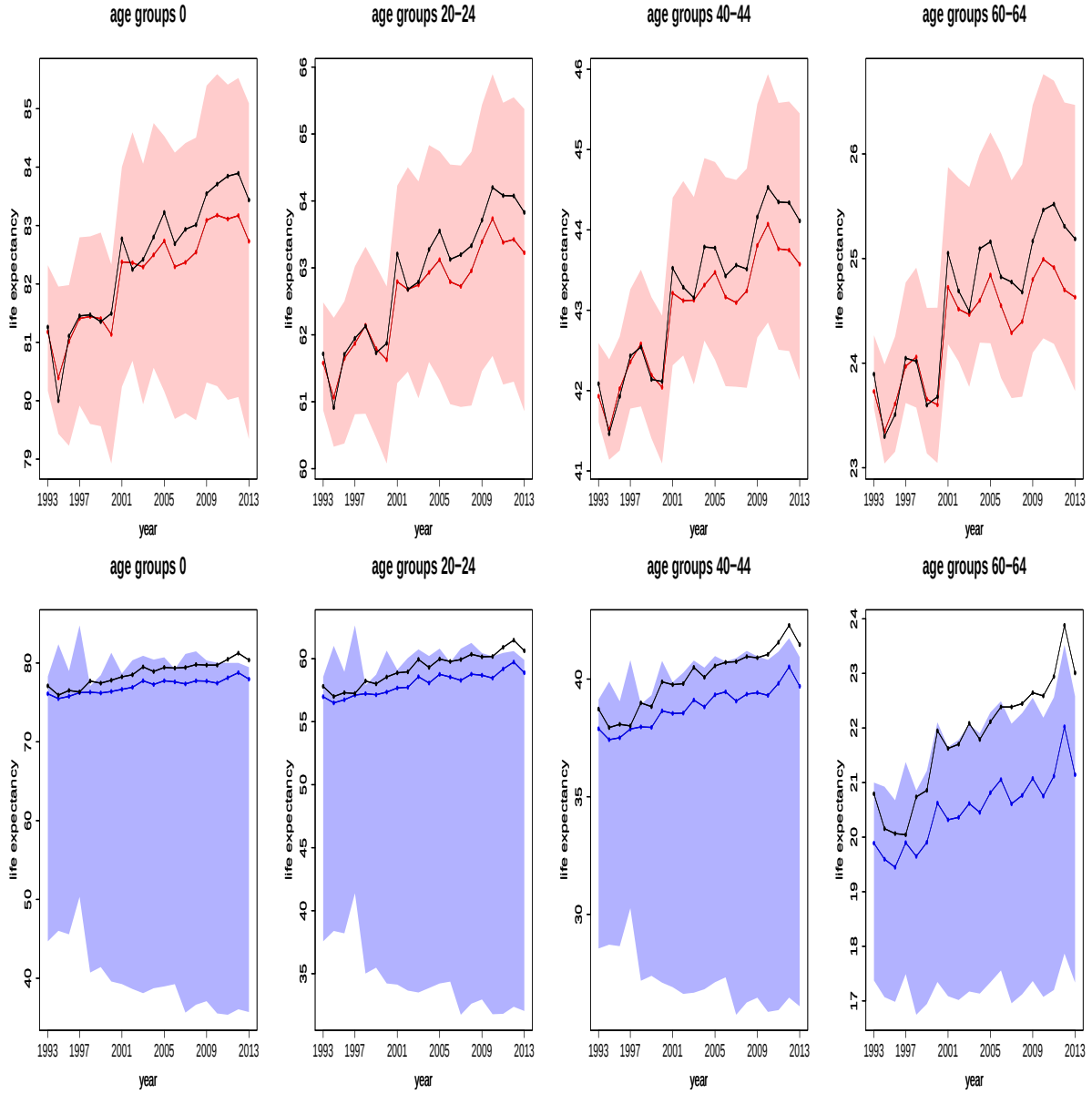


Figure 7.23. Observed life expectancy $e_{\bar{x},t}$ from HMD (black line) and life expectancy calculated by the forecasted mortality rates from model 8 (red in female and blue in male) for both female (first row) and male (second row) and four age groups (0, 20-24, 40-44 and 60-64) in Iceland.

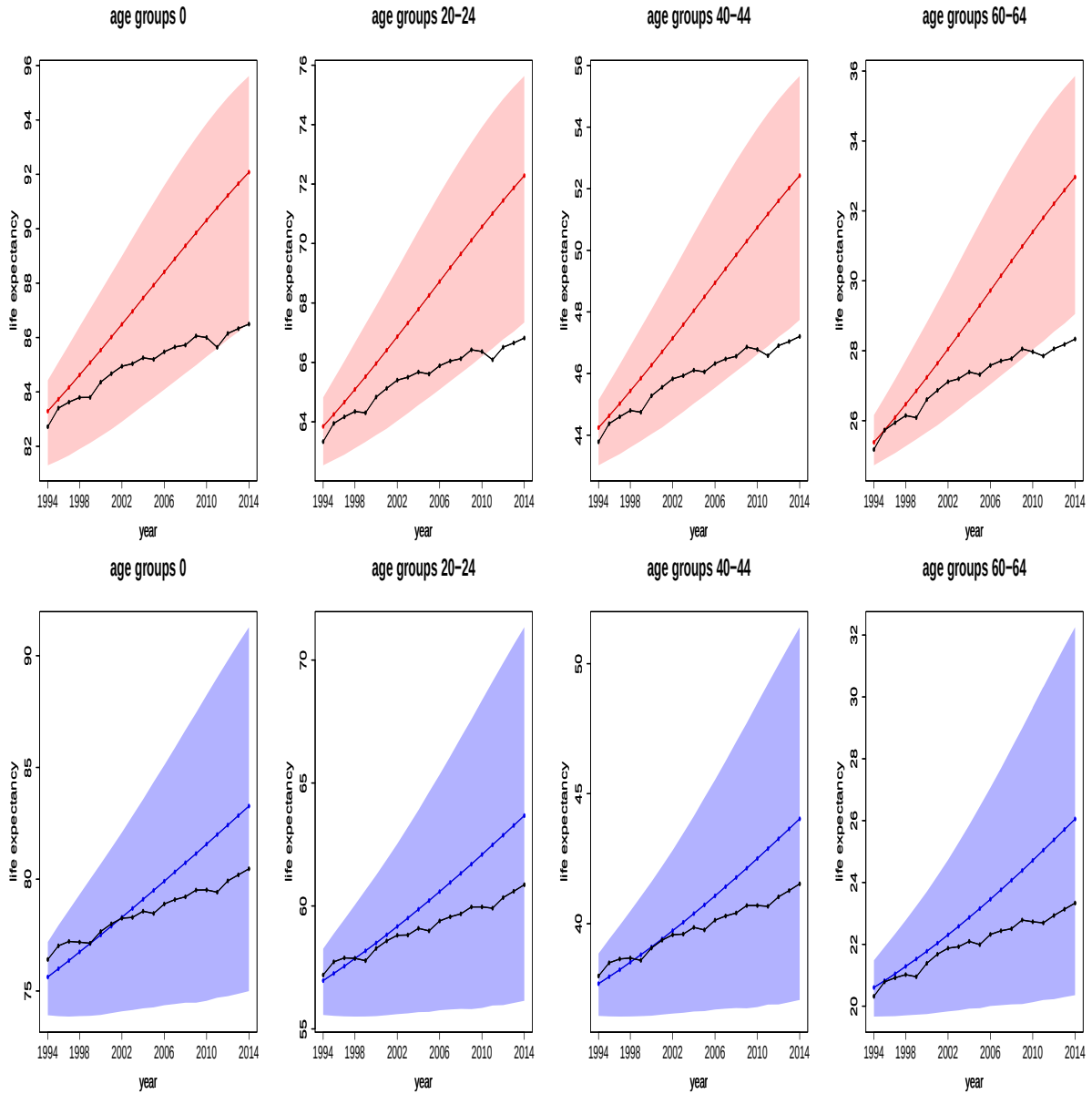


Figure 7.24. Observed life expectancy $e_{\tilde{x},t}$ from HMD (black line) and life expectancy calculated by the forecasted mortality rates from model 8 (red in female and blue in male) for both female (first row) and male (second row) and four age groups (0, 20-24, 40-44 and 60-64) in Japan.

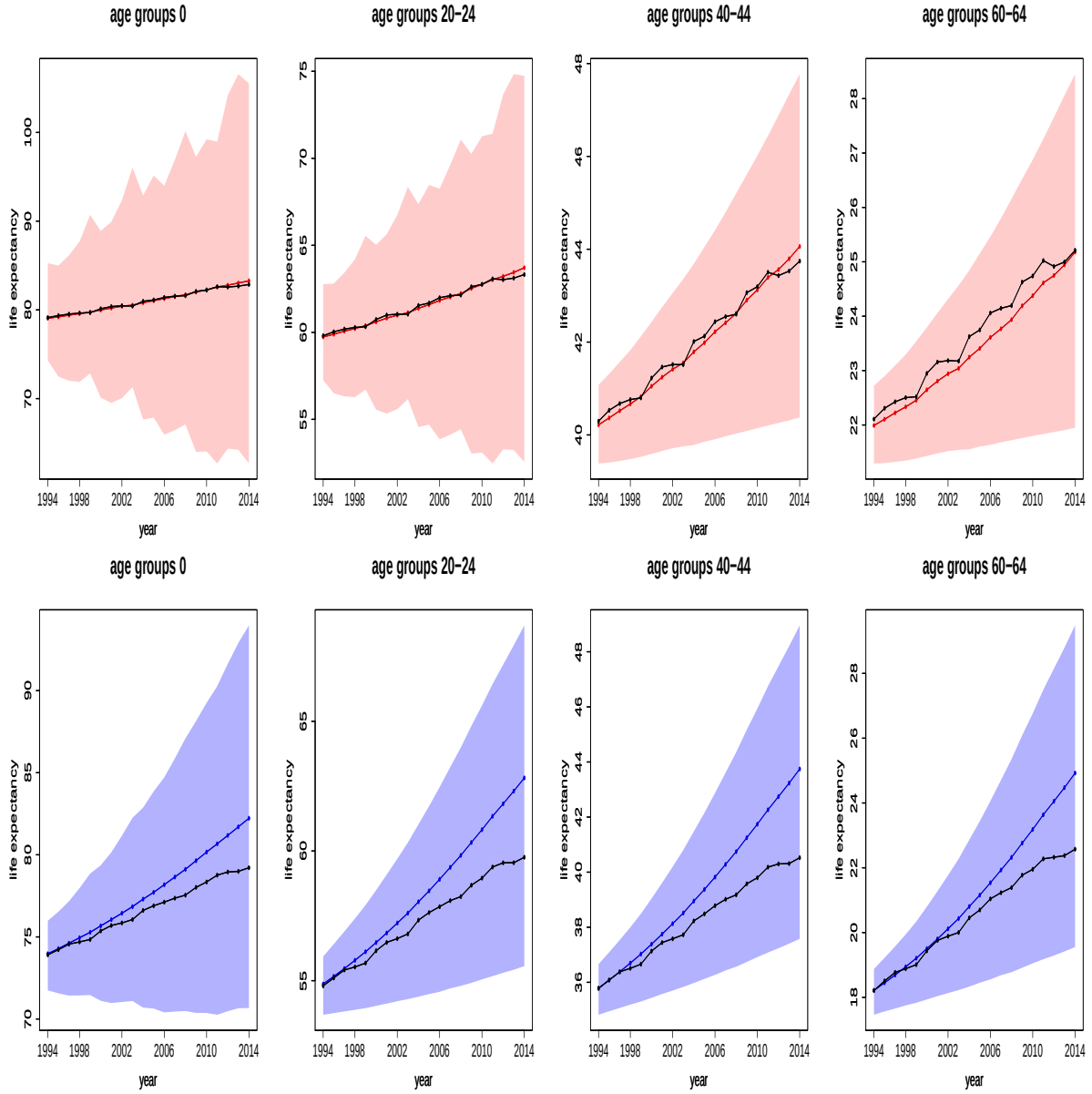


Figure 7.25. Observed life expectancy $e_{\bar{x},t}$ from HMD (black line) and life expectancy calculated by the forecasted mortality rates from model 8 (red in female and blue in male) for both female (first row) and male (second row) and four age groups (0, 20-24, 40-44 and 60-64) in U.K..

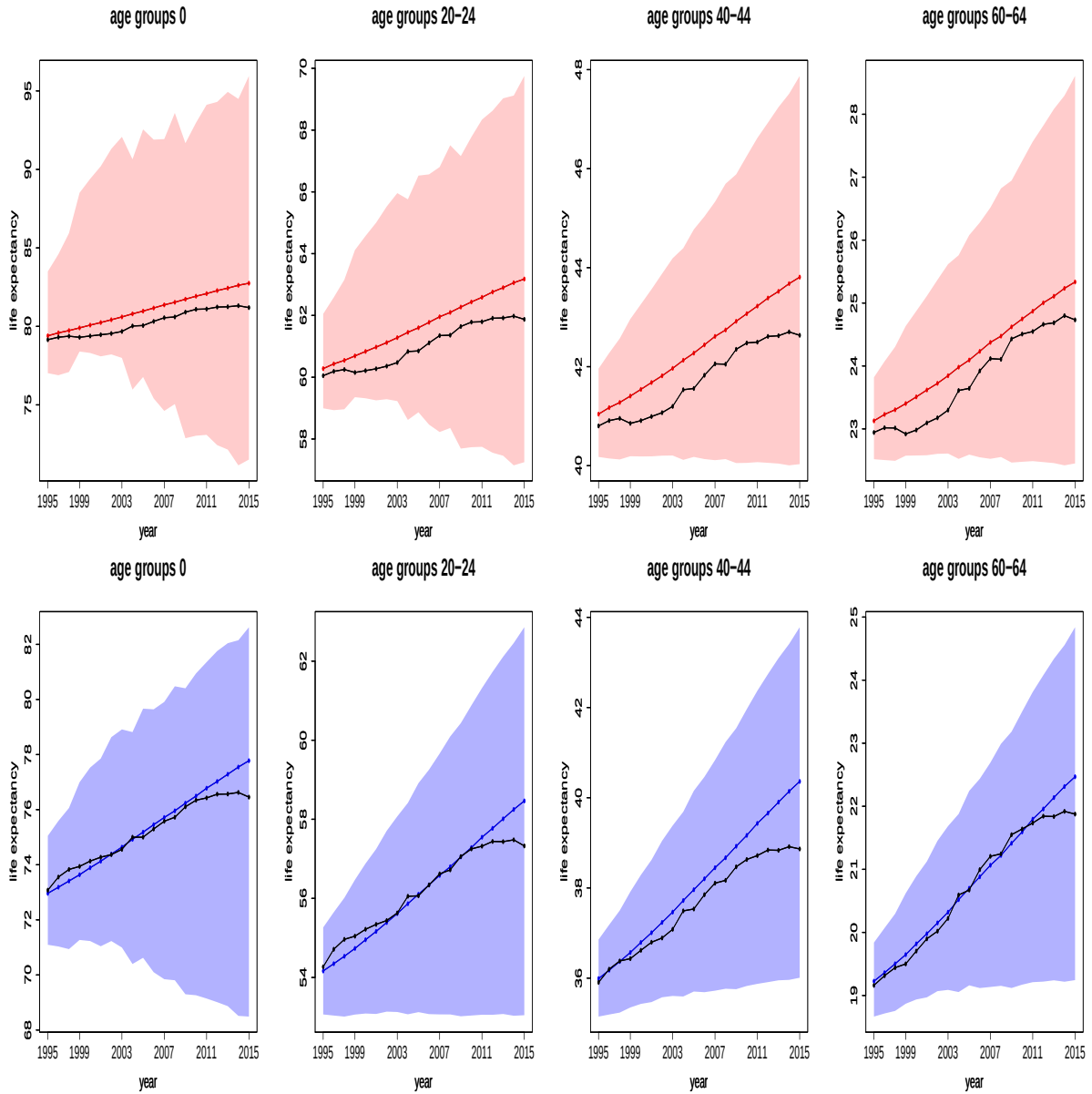


Figure 7.26. Observed life expectancy $e_{\bar{x},t}$ from HMD (black line) and life expectancy calculated by the forecasted mortality rates from model 8 (red in female and blue in male) for both female (first row) and male (second row) and four age groups (0, 20-24, 40-44 and 60-64) in U.S..

7.4.5 Out-of-sample forecasting for life expectancies 20 years ahead

In this analysis, we calculate life expectancy based on the 20 years ahead forecasts of mortality rates for 16 countries using model 8 that is the best performing model in the previous two analyses and model 3 that is the LC model for comparison. Table 7.9 reports the life table forecast for selected age groups $\{0, 15 - 19, 60 - 69\}$ in Australia using model 3 and Table 7.10 for model 8. The life expectancy in the abridged life tables for all 16 countries and all age groups are given in Appendix A. The trend of life expectancy forecast is slowly increasing over the forecast period for all age groups, gender and models. Compared with model 3, the life expectancy forecast using model 8 is higher consistently for all age groups, gender and years.

Table 7.9. The forecasted Australia life expectancy by model 3

age		2015	2016	2017	2018	2019	2020	2021	2022	2023	2024	2025	2026	2027	2028	2029	2030	2031	2032	2033	2034
Model 3 female																					
0	2.5%	85.07	85.29	85.49	85.64	85.77	85.85	85.89	85.97	86.06	86.11	86.23	86.27	86.34	86.38	86.47	86.46	86.54	86.58	86.61	86.68
	mean	84.38	84.35	84.33	84.31	84.29	84.26	84.23	84.21	84.18	84.16	84.13	84.11	84.08	84.05	84.02	83.99	83.97	83.94	83.92	83.89
	2.5%	83.67	83.33	83.07	82.88	82.69	82.53	82.33	82.20	82.02	81.90	81.72	81.60	81.50	81.37	81.25	81.13	81.00	80.84	80.72	80.63
15-19	97.5%	70.41	70.62	70.80	70.95	71.07	71.14	71.18	71.25	71.33	71.38	71.49	71.52	71.59	71.63	71.70	71.70	71.77	71.81	71.85	71.91
	mean	69.77	69.75	69.73	69.71	69.69	69.66	69.64	69.62	69.59	69.57	69.55	69.52	69.50	69.47	69.45	69.42	69.40	69.38	69.35	69.33
	2.5%	69.12	68.81	68.58	68.41	68.23	68.09	67.91	67.79	67.63	67.52	67.36	67.26	67.17	67.05	66.93	66.83	66.72	66.58	66.50	66.42
60-64	97.5%	27.02	27.17	27.30	27.42	27.51	27.55	27.59	27.65	27.71	27.75	27.84	27.87	27.90	27.93	27.98	28.01	28.06	28.09	28.12	28.16
	mean	26.51	26.49	26.48	26.46	26.45	26.43	26.41	26.39	26.38	26.36	26.34	26.32	26.31	26.29	26.27	26.25	26.23	26.22	26.20	26.18
	2.5%	26.01	25.79	25.62	25.49	25.35	25.24	25.12	25.03	24.92	24.84	24.73	24.65	24.57	24.51	24.42	24.32	24.26	24.17	24.12	24.06
65-69	97.5%	22.56	22.70	22.82	22.93	23.01	23.06	23.10	23.14	23.20	23.24	23.32	23.34	23.39	23.40	23.45	23.48	23.51	23.55	23.57	23.60
	mean	22.09	22.07	22.06	22.05	22.04	22.02	22.00	21.99	21.97	21.96	21.94	21.92	21.91	21.89	21.87	21.85	21.84	21.82	21.81	21.79
	2.5%	21.62	21.43	21.26	21.15	21.03	20.93	20.82	20.73	20.63	20.56	20.47	20.40	20.32	20.25	20.18	20.10	20.03	19.97	19.92	19.85
Model 3 male																					
0	97.5%	81.25	81.53	81.78	81.93	82.13	82.20	82.27	82.37	82.46	82.54	82.66	82.73	82.75	82.79	82.88	82.95	82.96	83.05	83.05	83.13
	mean	80.43	80.40	80.37	80.34	80.31	80.28	80.24	80.21	80.17	80.14	80.11	80.07	80.04	80.00	79.96	79.92	79.89	79.86	79.82	79.79
	2.5%	79.57	79.17	78.85	78.65	78.39	78.17	77.95	77.83	77.64	77.44	77.26	77.08	76.95	76.80	76.62	76.46	76.37	76.17	76.05	75.93
15-19	97.5%	66.61	66.87	67.11	67.26	67.43	67.50	67.57	67.66	67.74	67.81	67.94	68.00	68.03	68.06	68.15	68.19	68.22	68.31	68.32	68.38
	mean	65.84	65.81	65.78	65.76	65.73	65.69	65.65	65.63	65.59	65.56	65.53	65.50	65.47	65.43	65.40	65.36	65.33	65.30	65.26	65.23
	2.5%	65.03	64.66	64.37	64.17	63.94	63.73	63.53	63.43	63.24	63.04	62.88	62.73	62.59	62.44	62.31	62.14	62.07	61.89	61.78	61.69
60-64	97.5%	24.22	24.39	24.54	24.67	24.79	24.83	24.89	24.94	25.00	25.05	25.15	25.18	25.20	25.22	25.29	25.32	25.34	25.42	25.43	25.47
	mean	23.65	23.63	23.61	23.59	23.57	23.55	23.52	23.50	23.48	23.46	23.44	23.41	23.39	23.37	23.34	23.32	23.30	23.28	23.26	23.23
	2.5%	23.08	22.83	22.63	22.51	22.35	22.22	22.08	21.99	21.86	21.75	21.65	21.56	21.48	21.37	21.29	21.19	21.10	20.99	20.93	20.86
65-69	97.5%	20.01	20.17	20.31	20.41	20.52	20.56	20.61	20.66	20.72	20.76	20.84	20.88	20.90	20.92	20.97	21.00	21.02	21.10	21.10	21.14
	mean	19.50	19.48	19.46	19.45	19.43	19.41	19.38	19.37	19.35	19.33	19.31	19.29	19.27	19.24	19.22	19.20	19.18	19.16	19.14	19.12
	2.5%	18.99	18.77	18.58	18.48	18.33	18.20	18.08	18.00	17.89	17.80	17.72	17.61	17.56	17.46	17.39	17.29	17.23	17.11	17.08	17.01

Table 7.10. The forecasted Australia life expectancy by model 8

age		2015	2016	2017	2018	2019	2020	2021	2022	2023	2024	2025	2026	2027	2028	2029	2030	2031	2032	2033	2034
Model 8 female																					
0	97.5%	85.02	85.36	85.70	86.01	86.33	86.64	86.92	87.23	87.52	87.82	88.14	88.40	88.74	89.00	89.29	89.56	89.84	90.13	90.39	90.62
	mean	84.38	84.63	84.87	85.13	85.37	85.61	85.87	86.12	86.37	86.62	86.87	87.12	87.37	87.62	87.86	88.10	88.35	88.59	88.83	89.07
	2.5%	83.72	83.86	83.99	84.15	84.30	84.52	84.69	84.87	85.08	85.25	85.45	85.65	85.86	86.06	86.28	86.46	86.70	86.89	87.12	87.31
15-19	97.5%	70.37	70.69	71.00	71.30	71.60	71.89	72.16	72.45	72.73	73.01	73.31	73.57	73.89	74.14	74.42	74.68	74.95	75.24	75.49	75.72
	mean	69.77	70.00	70.23	70.47	70.70	70.93	71.17	71.41	71.65	71.88	72.12	72.36	72.60	72.84	73.07	73.30	73.54	73.77	74.01	74.24
	2.5%	69.14	69.28	69.42	69.58	69.71	69.92	70.07	70.27	70.46	70.61	70.81	71.00	71.21	71.38	71.60	71.79	72.02	72.21	72.42	72.59
60-64	97.5%	27.07	27.31	27.55	27.78	28.03	28.27	28.48	28.72	28.97	29.18	29.45	29.66	29.93	30.14	30.38	30.61	30.85	31.09	31.31	31.51
	mean	26.56	26.74	26.93	27.13	27.32	27.52	27.72	27.92	28.12	28.31	28.51	28.72	28.92	29.12	29.32	29.51	29.71	29.91	30.11	30.30
	2.5%	26.01	26.14	26.30	26.45	26.58	26.73	26.88	27.05	27.21	27.36	27.52	27.67	27.83	28.02	28.18	28.34	28.53	28.69	28.84	29.01
65-69	97.5%	22.62	22.84	23.06	23.28	23.51	23.73	23.92	24.15	24.37	24.58	24.81	25.01	25.26	25.47	25.67	25.90	26.11	26.34	26.55	26.75
	mean	22.13	22.30	22.48	22.66	22.83	23.01	23.20	23.38	23.56	23.75	23.93	24.12	24.30	24.49	24.67	24.85	25.04	25.22	25.41	25.59
	2.5%	21.60	21.73	21.87	22.00	22.13	22.27	22.42	22.56	22.71	22.83	22.98	23.13	23.29	23.45	23.58	23.72	23.91	24.05	24.20	24.34
Model 8 male																					
0	97.5%	81.10	81.61	82.08	82.56	83.00	83.45	83.95	84.41	84.86	85.33	85.80	86.24	86.75	87.18	87.66	88.09	88.53	88.96	89.43	89.84
	mean	80.41	80.79	81.17	81.56	81.95	82.36	82.76	83.17	83.58	84.00	84.42	84.83	85.26	85.68	86.09	86.52	86.93	87.35	87.76	88.17
	2.5%	79.65	79.90	80.19	80.48	80.83	81.13	81.46	81.81	82.14	82.50	82.85	83.17	83.57	83.97	84.34	84.74	85.11	85.50	85.89	86.27
15-19	97.5%	66.48	66.95	67.38	67.84	68.27	68.69	69.16	69.61	70.05	70.51	70.96	71.40	71.89	72.31	72.78	73.20	73.64	74.06	74.51	74.93
	mean	65.81	66.16	66.53	66.90	67.28	67.67	68.05	68.45	68.84	69.25	69.65	70.06	70.47	70.88	71.28	71.69	72.10	72.50	72.91	73.31
	2.5%	65.06	65.33	65.61	65.89	66.21	66.52	66.82	67.16	67.46	67.85	68.16	68.50	68.87	69.25	69.61	69.98	70.35	70.74	71.11	71.48
60-64	97.5%	24.32	24.68	25.01	25.37	25.72	26.08	26.44	26.80	27.18	27.56	27.92	28.32	28.70	29.08	29.47	29.84	30.20	30.56	30.93	31.30
	mean	23.82	24.12	24.42	24.73	25.05	25.37	25.70	26.03	26.36	26.70	27.04	27.39	27.74	28.08	28.43	28.78	29.13	29.47	29.82	30.16
	2.5%	23.29	23.53	23.79	24.06	24.35	24.62	24.91	25.20	25.49	25.79	26.08	26.37	26.70	27.00	27.32	27.64	27.94	28.28	28.58	28.90
65-69	97.5%	20.14	20.44	20.74	21.07	21.39	21.73	22.05	22.39	22.74	23.09	23.44	23.78	24.17	24.50	24.87	25.21	25.57	25.90	26.25	26.58
	mean	19.67	19.94	20.22	20.50	20.79	21.09	21.38	21.69	22.00	22.31	22.62	22.94	23.26	23.58	23.91	24.23	24.55	24.88	25.20	25.51
	2.5%	19.17	19.39	19.63	19.89	20.14	20.38	20.67	20.94	21.19	21.46	21.74	22.00	22.30	22.61	22.88	23.18	23.46	23.74	24.04	24.33

7.5 Conclusion

This chapter proposes univariate mortality models by extending the LC model to death counts and incorporating a long memory model structure. Then the models with eight submodels are applied to analyse death count data sets from 16 countries divided according to genders and age groups. Bayesian inference is adopted and predictive posterior distribution is provided for life expectancy forecasts. In-sample fitting is applied to both mortality rate and life expectancy to verify that model 8 with long memory structure provides better estimates than model 3 which is the basic LC model for both types of estimates. Comparison is also made with all 8 models using four selection criteria and DIC. Then out-of-sample forecast is performed 20 years ahead to provide mortality rate estimates for the calculation of life expectancy forecasts. Extensive plots

are provided to visualise the better performance of model 8 relative to model 3 in both cases of in-sample fitting or out-of-sample forecast and in both mortality rate and life expectancy estimates. Finally, life tables are constructed from all 16 countries using both models 3 and 8 allowing the comparison between models in a more comprehensive way. However, there are some limitations to our current models which model one age group at a time and hence fail to capture the cohort effect across age groups. The multivariate extensions will be consider in the next chapter.

CHAPTER 8

Mortality model for multiple age groups incorporating long memory cohort effect

8.1 Background

Increasing the accuracy of mortality rate forecasts and improving the projection of life expectancy is an important consideration for insurance companies and governments since misleading predictions may result in insufficient funds for retirement and pension plans. Chapter 7 demonstrates that the existence of long memory in mortality data improves the modelling and forecasting of mortality data. However, this single age Gegenbauer long memory Lee-Carter (LC) model fails to account for the cohort effect which describes the dependence between mortalities of different age groups. In this chapter we consider an extension from the single age model to model for all age groups simultaneously. This corresponds to the multivariate LC model with Gegenbauer long memory in both mean function and cohort effect which is able to capture the cohort effect found in Chapter 6. This model is called multivariate long memory with cohort long memory (LMLM) model. Alternatively, we can also view our proposed LMLM model as an extension of the LC model to death counts incorporating long memory model structure. Before this model development, we give a brief review of various LC models so developed mainly for modelling mortality rates. With these model structures, we address the model identification issue and explain how our proposed long memory models solve the issue.

LC model is the bench mark mortality model proposed by [Lee and Carter \(1992\)](#) to model simultaneously mortality rates across all age groups. However, since the mortality

rate estimates by LC model are consistently underestimated, attempts have been made to adopt different structures and to model various responses. See Table 8.1 for a list of these models.

Table 8.1. Extensions to the LC model

Model type	Dynamics
Lee and Carter (1992)	$\ln \mu_{x,t} = \alpha_x + \beta_x \kappa_t + \varepsilon_{x,t}$
Renshaw and Haberman (2003)	$\ln \mu_{x,t} = \alpha_x + \sum_{i=1}^k \beta_x^{(i)} \kappa_t^{(i)} + \varepsilon_{x,t}$
Renshaw and Haberman (2006)	$\ln \mu_{x,t} = \alpha_x + \beta_x^{(1)} \kappa_t + \beta_x^{(2)} \zeta_{t-x} + \varepsilon_{x,t}$
Currie (2006)	$\ln \mu_{x,t} = \alpha_x + \kappa_t + \zeta_{t-x} + \varepsilon_{x,t}$
Cairns et al. (2006b)	$\text{logit}(q_{x,t}) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \varepsilon_{x,t}$
Cairns et al. (2009)	$\text{logit}(q_{x,t}) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \zeta_{t-x} + \varepsilon_{x,t}$
Plat (2009)	$\text{logit}(q_{x,t}) = \kappa_t^{(1)} + \kappa_t^{(2)}(\bar{x} - x) + \kappa_t^{(3)}(\bar{x} - x)^+ + \zeta_{t-x} + \varepsilon_{x,t}$

Note: α_x is constant for age group x , $\kappa_t^{(i)}$ is the i -th component period effect, $\beta_x^{(i)}$ is the corresponding coefficient for age group x , ζ_{t-x} is the cohort effect, $\bar{x}$ is the average age in the sample range and $q_{x,t} = 1 - \exp(-\mu_{x,t})$ is the probability that an individual aged exactly x at exact time t will die between t and $t + 1$.

Firstly, Renshaw and Haberman (2003) introduced a multi-period $\sum_{i=1}^k \beta_x^{(i)} \kappa_t^{(i)}$ extension, where $\beta_x^{(i)}$ represents how rapidly or slowly mortality at each age varies when the general level of mortality α_x is fixed at a certain level. Renshaw and Haberman (2006) extended the LC model to extrapolate age-specific cohort effects ζ_{t-x} and age-specific period effects κ_t . Currie (2006) considered a simplified version of the model adopted by Renshaw and Haberman (2006). Cairns et al. (2006b) proposed a new mortality model for $\text{logit}(q_{x,t}) = \ln \frac{q_{x,t}}{1-q_{x,t}}$ using two factors. The first factor models mortality at all age groups in an equal manner and the second factor models mortality being proportional to a age-specific cohort. Furthermore, the addition of a cohort factor was introduced by Cairns et al. (2009). Plat (2009) introduced an additional term

$(\bar{x} - x)^+ := \max(\bar{x} - x, 0)$ to capture the feature of young age mortality, while adopting the desirable features of the existing models.

However, the parameters of these LC models are not uniquely identified. In order to solve the identifiability problem, [Lee and Carter \(1992\)](#) adopted two additional constraints $\sum_t \kappa_t = 0$ and $\sum_x \beta_x = 1$. Moreover, [Cairns et al. \(2009\)](#) claimed that the identifiability issue caused slow convergence to optimal solution in model fitting and high sensitivity to individual data in changing data. The identification constraint issue does not only apply to maximum likelihood estimation. In the Bayesian approach adopted by both [Pedroza \(2006\)](#) and [Kogure and Kurachi \(2010\)](#), such identification constraints increased the difficulties in building an efficient Markov chain Monte Carlo (MCMC) procedure and deriving proper posterior distributions. In fact, this unresolved identifiability issue not only increases the model complexities but also raises computational difficulties. The work of [Fung et al. \(2015\)](#) is among the few literature which also studies the identification constraint issue in the Bayesian context. They pointed out that these constraints are not obvious to use when designing efficient MCMC procedures. It is very difficult to design the proposal of the MCMC to achieve suitable acceptance rates, resulting in high variance in the estimates of mortality rates because these constraints are not unique. For different extensions of LC model, a groups of constraints need to be developed to overcome the identifiability issue. [Delwarde et al. \(2007a\)](#) pointed out that the estimates of age-specific component β_x exhibit an irregular life expectancy projection pattern in most cases, which in term can produce irregular projected life tables and they suggested to improve the estimates by smoothing β_x independently from other parameters. [Hunt and Villegas \(2015\)](#) investigated systematically the convergence problem of the parameter estimates and found that the problem was originated from parameter identifiability of LC models listed in [Table 8.1](#). The specification of identification constraints was studied to improve the model robustness and stability. Alternatively, [Czado et al. \(2005\)](#) implemented a Bayesian approach incorporating appropriate choices of prior distributions to improve the robustness and convergence of parameter estimates. [Hunt and Villegas \(2015\)](#) improved the robustness and convergence rate of the model by imposing additional constraints on the parameters. [Fung et al. \(2015\)](#) improved these issues by providing an alternative formulation of the identification constraints and derived the posterior distributions of the parameters and state process of the LC model. We provide an alternative approach by modifying

the model structure, particularly in the period effect by replacing the non-stationary AR(1) process with a long memory process.

This chapter is structured as follow. The multivariate LC model proposed by [Renshaw and Haberman \(2006\)](#) is extended to multivariate extended LC cohort (MELCC) model for death count with GP distribution. This extended LC model serves as a benchmark death count model for subsequent model comparison as it is a commonly used LC model which corresponds to the third model in Table 8.1 and it nests the first and fourth LC models as special cases. We do not consider the second LC model as there is no clear motivation for setting more than one period effect. This popular MELCC model is compared to our proposed long memory model which incorporates long memory features in both the mean function and cohort effects in the death counts data. The long memory component in the mean function is also designed to replace the period effect. We explain why the identification issues is solved in this long memory model. Bayesian approach to perform the parameter estimation and mortality rate forecasting is provided. Then we apply MELCC and LMLM models to analyse death count data sets from three countries, namely, Australia, U.K. and U.S. separated by genders. Finally, our proposed LMLM is proved to outperform MELCC model and hence is selected to provide mortality rate forecasts for life expectancy calculation in life table construction.

8.2 Multivariate long memory mortality model

In this section, we introduce multivariate extended LC cohort (MELCC) and stationary multivariate long memory mortality model with long memory cohort effect (LMLM) models and adopt GP distribution to both models. The connection between these two models is explored.

8.2.1 Multivariate extended Lee Carter cohort model with discrete distribution

Definition 8.2.1 (MELCC model). *Conditional on the central mortality rate $\mu_t = (\mu_{x_1,t}, \mu_{x_2,t}, \dots, \mu_{x_g,t})$, period effect κ_t , cohort effect $\zeta_t = (\zeta_{x_1,t}, \zeta_{x_2,t}, \dots, \zeta_{x_g,t})$ and central exposure $\mathbf{E}_t = (E_{x_1,t}, E_{x_2,t}, \dots, E_{x_g,t})$, we assume that death counts $\mathbf{Y}_t | \mathcal{F}_{1:t-1}, \mathbf{E}_t, \mu_t(\kappa_t, \zeta_t)$ form an independent Markov process and the MELCC model is defined as*

$$Y_{x,t} | \mathcal{F}_{1:t-1}, E_{x,t}, \mu_{x,t}, \kappa_t, \zeta_{x,t} \sim D(E_{x,t} \mu_{x,t}, \nu_x),$$

$$\ln \mu_{x,t} = \alpha_x + \beta_x \kappa_t + \beta_x^{\zeta} \zeta_{x,t} + \varepsilon_{x,t}, \quad \varepsilon_{x,t} \stackrel{i.i.d}{\sim} N(0, \sigma_{\varepsilon}^2), \quad (8.1)$$

$$\kappa_t = \eta \kappa_{t-1} + \varsigma^{\kappa} + \varepsilon_t^{\kappa}, \quad \varepsilon_t^{\kappa} \stackrel{i.i.d}{\sim} N(0, \sigma_{\kappa}^2), \quad (8.2)$$

$$\zeta_{x_1,t} = \lambda \zeta_{x_1,t-1} + \varsigma + \varepsilon_t^{\zeta}, \quad \varepsilon_t^{\zeta} \stackrel{i.i.d}{\sim} N(0, \sigma_{\zeta}^2), \quad (8.3)$$

$$\zeta_{x_i,t} = \zeta_{x_{i-1},t-1}, \quad i = 2, 3, \dots, g$$

where the vector $\alpha = [\alpha_{x_1}, \dots, \alpha_{x_g}] \in \mathbb{R}^g$ represents the profile of age groups on the log mortality rates and $\beta = [\beta_{x_1}, \dots, \beta_{x_g}] \in \mathbb{R}^g$ measures the interaction of age group and time effect on the log mortality rates, $D(\cdot, \cdot)$ represents the Poisson, NB, GP or DP distributions with mean function $E_{x,t} \mu_{x,t}$, the dispersion parameter $\nu_x \in (-1, 1)$ for GP distribution and $\nu_x > 0$ for NB and DP distributions and ς^{κ} , ς and $|\lambda| < 1$ are all parameters. Furthermore, it is a standard practice to use random walk (i.e. $\eta = 1$). Here we assume κ_t is a random walk with drift process (ARIMA(0, 1, 0)) and the dynamics of $\zeta_{x_1,t}$ is described by a stationary AR(1) process (ARIMA(1, 0, 0)).

We can write the state equation in matrix notation as follows:

$$\begin{pmatrix} \kappa_t \\ \zeta_{x_1,t} \\ \zeta_{x_2,t} \\ \vdots \\ \zeta_{x_{g-1},t} \\ \zeta_{x_g,t} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \eta \kappa_{t-1} \\ \zeta_{x_1,t-1} \\ \zeta_{x_2,t-1} \\ \vdots \\ \zeta_{x_{g-2},t-1} \\ \zeta_{x_{g-1},t-1} \end{pmatrix} + \begin{pmatrix} \varsigma_{\kappa} \\ \varsigma \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \varepsilon_t^{\kappa} \\ \varepsilon_t^{\zeta} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

However, Fung et al. (2017a) stated that the model in Equation (8.1) is not identifiable because the model is invariant to the following transformations of the parameters

$$(\alpha_x, \beta_x, \kappa_t, \beta_{x_i}^\zeta, \zeta_{x,t}) \rightarrow \left(\alpha_x + c_1\beta_x + c_2\beta_x^\zeta, \frac{\beta_x}{c_3}, c_3(\kappa_t - c_1), \frac{\beta_x^\zeta}{c_4}, c_4(\beta_x^\zeta - c_2) \right),$$

where $c_3 \neq 0$ and $c_4 \neq 0$. To overcome this identification issue, four additional constraints

$$\sum_{x \in \mathcal{S}_x} \beta_x = 1, \quad \sum_{x \in \mathcal{S}_x} \beta_x^\zeta = 1, \quad \sum_{t=1}^T \kappa_t = 0 \quad \text{and} \quad \sum_{t=1-x_g}^{T-x_1} \zeta_{x,t} = 0$$

where $\mathcal{S}_x = \{x_1, \dots, x_g\}$ are introduced.

8.2.2 Long memory mortality model with long memory cohort effect model

In order to capture trends in mortality data, the period effect component in MELCC model is designed to be a non-stationary process adopting the constraints in LC model. In fact, the identification constraints are not unique, these constraints can be parameterised into various ways for different models. Hence, the constraints can be simplified by choosing a proper model structure both in terms of specification and interpretation. We consider an alternative approach by replacing the non-stationary AR process in Equation (8.2) for the period effect with a Gegenbauer long memory structure (see section 2.3) in the mean function. The non-stationary AR process in Equation (8.3) for the cohort is also replaced by a Gegenbauer long memory structure. Essentially, we replace the period effect and cohort effect components by Gegenbauer long memory structures which are more flexible to capture different types of long memory. This replacement can also be extended to other LC models in Table 8.1. Gegenbauer Long memory structure in Equation (3.8) which is a stationary infinity MA structure is different from period effect in Equation (8.2) which is an non-stationary AR(1) process.

Definition 8.2.2 (LMLM model). *Conditional on the central mortality rate $\boldsymbol{\mu}_t = (\mu_{x_1,t}, \mu_{x_2,t}, \dots, \mu_{x_g,t})$, cohort effect $\boldsymbol{\zeta}'_t = (\zeta'_{x_1,t}, \zeta'_{x_2,t}, \dots, \zeta'_{x_g,t})$ and central exposure $\mathbf{E}_t = (E_{x_1,t}, E_{x_2,t}, \dots, E_{x_g,t})$, we assume that $\mathbf{Y}_t | \mathcal{F}_{1:t-1}, \mathbf{E}_t, \boldsymbol{\mu}_t(\boldsymbol{\zeta}'_t)$ forms an independent Markov process and the new stationary long memory mortality model with long memory cohort*

effect is defined as

$$\begin{aligned}
 Y_{x,t} | \mathcal{F}_{1:t-1}, E_{x,t}, \mu_{x,t}(\zeta'_{x,t}), \zeta'_{x,t} &\sim D(E_{x,t} \mu_{x,t}(\zeta'_{x,t}), \nu_x), \\
 \Phi_x(B) \ln \mu_{x,t} &= \alpha_x + \zeta'_{x,t} + \Theta_x(B) ((1 - 2uB + B^2)^{-d} \varepsilon_{x,t}), \quad \varepsilon_{x,t} \stackrel{i.i.d}{\sim} N(0, \sigma_{x,\varepsilon}^2), \quad (8.4) \\
 \zeta'_{x_1,t} &= \zeta' + (1 - 2u'B + B^2)^{-d'} \varepsilon'_t, \quad \varepsilon'_t \stackrel{i.i.d}{\sim} N(0, \sigma_{\zeta'}^2), \\
 \zeta'_{x_i,t} &= \zeta'_{x_{i-1},t-1}, \quad i = 2, 3, \dots, g.
 \end{aligned}$$

where the fractional difference parameters $d \in (0, 0.5)$ and $d' \in (0, 0.5)$ control the strength of long memory, the Gegenbauer parameter u with $|u| < 1$ and u' with $|u'| < 1$ control the cycle of oscillatory autocorrelation function (ACF) for the mortality rate process and cohort effect process respectively, ζ' provides level information for $\zeta'_{x_1,t}$, B is the backshift operator and $\Phi_x(B) = 1 - \Phi_{x,1}B - \dots - \Phi_{x,p}B_p$ and $\Theta_x(B) = 1 + \Theta_{x,1}B + \dots + \Theta_{x,q}B_q$ are the autoregressive and moving-average operators, respectively with no common roots. We remark that we assume a common long memory structure with consistent u and d applied to the mortality processes of different age groups and the same for the cohort effect process with consistent u' and d' . We note that the cohort effect with long memory structure can be simplified into short memory AR process when the long memory parameter $d' = 0$.

We can write the state equation in matrix notation as follows:

$$\begin{pmatrix} \zeta'_{x_1,t} \\ \zeta'_{x_2,t} \\ \vdots \\ \zeta'_{x_{g-1},t} \\ \zeta'_{x_g,t} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \zeta'_{x_1,t-1} \\ \zeta'_{x_1,t-1} \\ \vdots \\ \zeta'_{x_{g-2},t-1} \\ \zeta'_{x_{g-1},t-1} \end{pmatrix} + \begin{pmatrix} \zeta' \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} (1 - 2u'B + B^2)^{-d'} \varepsilon'_t \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Remark 8.2.1. In order to demonstrate the similarities of mean function structure between MELCC model and LMLM model, we represent Equation 8.1 into

$$\begin{aligned}
 \ln \mu_{x,t} &= \alpha_x + \beta_x^\zeta \left(\zeta + \sum_{j=0}^{\infty} \lambda^j \varepsilon_{t-j-h}^\zeta \right) + \beta_x \left(\zeta^\kappa + \sum_{j=0}^{\infty} \eta^j \varepsilon_{t-j}^\kappa \right) + \varepsilon_{x,t}, \\
 \varepsilon_t^\zeta &\stackrel{i.i.d}{\sim} N(0, \sigma_\zeta^2) \quad \varepsilon_t^\kappa \stackrel{i.i.d}{\sim} N(0, \sigma_\kappa^2) \quad \text{and} \quad \varepsilon_{x,t} \stackrel{i.i.d}{\sim} N(0, \sigma^2),
 \end{aligned}$$

$$\ln \mu_{x,t} = \alpha_x + \left(\varsigma' + \sum_{j=0}^{\infty} \psi'_j \varepsilon_{t-j-h}^{\varsigma'} \right) + \sum_{j=0}^{\infty} \psi_j \varepsilon_{x,t-j},$$

$$\varepsilon_t^{\varsigma'} \stackrel{i.i.d}{\sim} N(0, \sigma_{\varsigma'}^2) \quad \text{and} \quad \varepsilon_{x,t} \stackrel{i.i.d}{\sim} N(0, \sigma^2),$$

where $\Phi_x(B) = \Theta_x(B) = 1$ for computation convenience. The parameters are defined as $\eta = 1$, ψ'_j and ψ_j are defined in Equation (2.17) using (u', d') and (u, d) respectively. We note that $h = \dot{x} - 0.5$ is the number of years to backshift the age group x to the first age group x_1 in which a long memory process is assigned. For example, age group 10-15 has mid age 12.5 and it needs 12 year on average to be backshifted to the first age group.

8.2.3 Understanding Gegenbauer long memory and random walk period model

The following study demonstrates some mathematical evidences of the relationship between a non-stationary period effect process in Equation (8.2) and Gegenbauer long memory process. For a standard stationary period effect model where $|\eta| < 1$, the Wold representation (See Theorem 3.1.1) can be found. Under such representation, we can easily find the limit behaviour of ACF as η tends to one. This will satisfy the general characteristic of Gegenbauer long memory in Equation (1.1). Firstly, we assume the period effect is a stationary AR process with $|\eta| < 1$, that is,

$$\kappa_t = \eta \kappa_{t-1} + \varepsilon_t^{\kappa}. \quad (8.5)$$

This stationary AR process can be easily written into the Wold representation as

$$\kappa_t = \frac{1}{1 - \eta B} \varepsilon_t^{\kappa} = \sum_{j=0}^{\infty} \eta^j \varepsilon_{t-j}^{\kappa}.$$

The ACF ρ_k^{κ} at lag k for the period effect in Equation (8.5) is

$$\rho_k^{\kappa} = \eta^k$$

and we note that the limit behaviour of ACF is

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n |\rho_k^\kappa| \xrightarrow{\eta \uparrow 1} \infty, \quad (8.6)$$

which satisfies the characteristic of Gegenbauer long memory provided in Equation (1.1). Hence, this non-stationary period effect shows a behaviour of long range dependence. Furthermore, given a period effect with a Gegenbauer long memory structure κ'_t , it can be written as

$$\kappa'_t = \frac{1}{(1 - 2uB + B^2)^d} \varepsilon_t^\kappa = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} \quad (8.7)$$

and

$$\rho(j) \sim \text{constant} \cdot j^{2d-1} \sin(\pi d - j\lambda_0), \text{ as } j \rightarrow \infty,$$

where $u = \cos(\lambda_0) = \cos\left(\frac{2\pi}{\varpi}\right)$ (see Equation (2.23)). We note that a condition for a stationary Gegenbauer long memory process is given by

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n |\rho_k| \rightarrow \infty \text{ if } d \in (0, 0.5). \quad (8.8)$$

Consequently, the characteristic of the ACF in Equation (8.6) for a non-stationary period effect in Equation (8.5) is similar to the ACF in Equation (8.8) for a Gegenbauer long memory process in Equation (8.7). Furthermore, the main advantage of replacing the non-stationary period effect by a Gegenbauer long memory structure is that the Gegenbauer long memory structure can be easily written into Wold representation. Stationarity and invertibility can also be studied (see Section 2.4.1). Moreover, the model has no identification issue and so the constraints for the non-stationary period effect can be released. [Pedroza \(2006\)](#) and [Kogure and Kurachi \(2010\)](#) pointed out that the identification constraints increased the difficulties in building an efficient MCMC procedure and deriving proper posterior distributions. This agrees with our experience that parameter estimation in stationary process can be more stable and forecast can be more accurate. Figure 8.1 plots the time series and ACF for data simulated from period effect with different η values from Equation (8.5) and from long memory process with $d = 0.45$ from Equation (8.7). With η approaching to 1, there are clear trends in time series plots and the ACF plots show stronger long memory pattern.

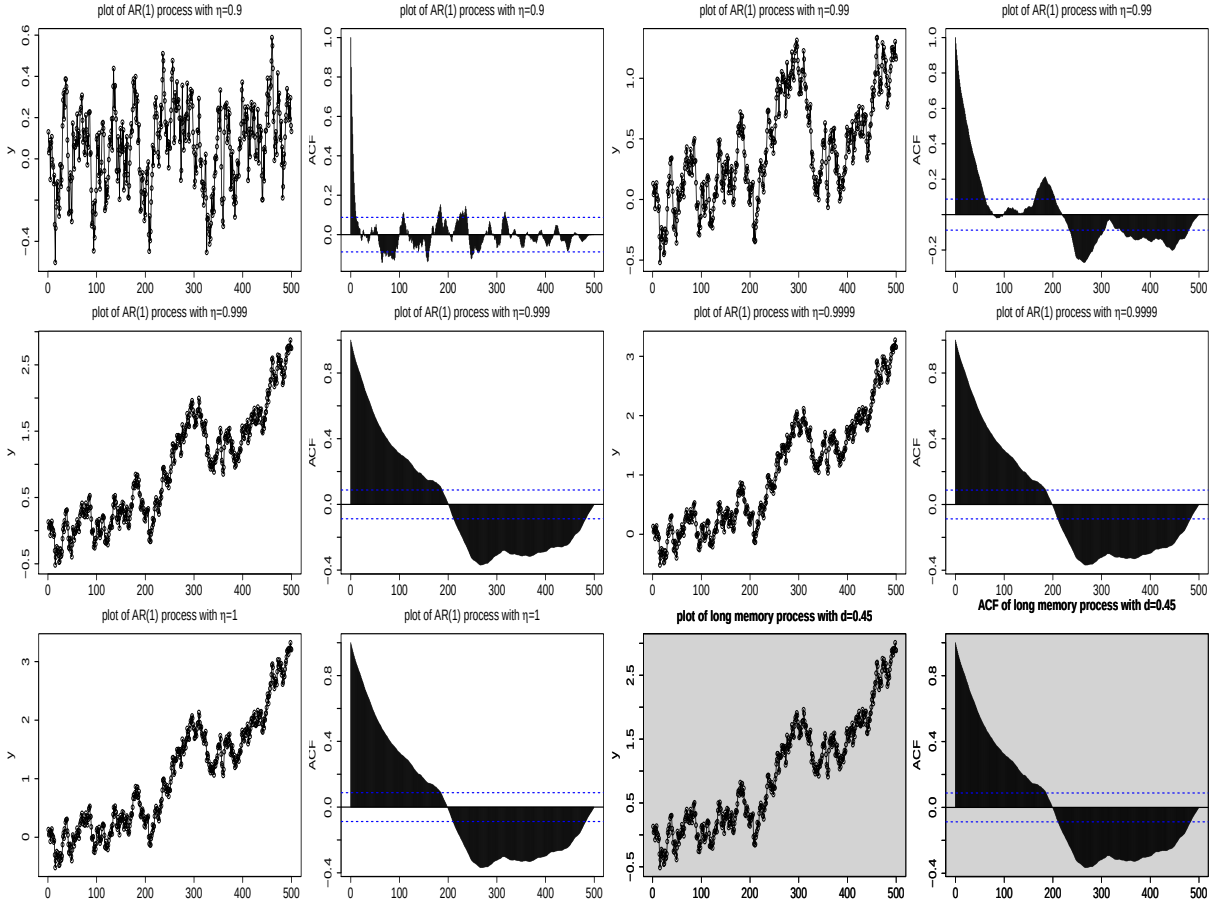


Figure 8.1. The time series plots and ACF plots for data simulated from AR(1) period effect process in Equation (8.5) with different η values. The last two plots with grey background are obtained from data simulated from a Gegenbauer long memory process in Equation (8.7) with $d = 0.45$ and $u = 1$.

8.3 Bayesian model and implementation

One clear advantage of Bayesian approach using posterior sampling again applies to the two proposed model because the models also involve latent variables in mean functions. Moreover, posterior predictive distributions can be obtained to provide distributional forecast summaries. In this section, we introduce the Bayesian approach to estimate the parameters of the proposed in model definition 8.2.2. Under the model structure

(8.2.2) with GP distribution, the complete likelihood function is

$$\begin{aligned}
f(\mathbf{y}_{x,1:T}, \mathbf{z}_{x,1:T} | \mathbf{E}_{x,1:T}, \boldsymbol{\vartheta}_x) &= \prod_{x=x_1}^{x_g} \prod_{t=1}^T f(y_{x,t} | E_{x,t}, \boldsymbol{\vartheta}_x, \mu_{x,t}(\zeta'_{x,t})) f(\mu_{x,t} | \boldsymbol{\vartheta}_x, \zeta'_{x,t}) f(\zeta'_{x,t} | \boldsymbol{\vartheta}_x) \\
&= \prod_{x=x_1}^{x_g} \prod_{t=1}^T \frac{E_{x,t} \exp(\alpha_x + \zeta'_{x,t} + (1 - 2uB + B^2)^{-d} \varepsilon_{x,t}) (1 - \nu_x)}{\Gamma(y_{x,t} + 1) \exp(y_{x,t} \nu_x)} \\
&\quad \times \frac{[E_{x,t} \exp(\alpha_x + \zeta'_{x,t} + (1 - 2uB + B^2)^{-d} \varepsilon_{x,t}) (1 - \nu_x) + y_{x,t} \nu_x]^{y_{x,t}-1}}{\exp\{E_{x,t} \exp[\alpha_x + \zeta'_{x,t} + (1 - 2uB + B^2)^{-d} \varepsilon_{x,t}] (1 - \nu_x)\}} \\
&\quad \times \frac{1}{\sqrt{2\pi} \sigma_{x,\varepsilon}^2} \exp\left(-\frac{\varepsilon_{x,t}^2}{2\sigma_{x,\varepsilon}^2}\right) \frac{1}{\sqrt{2\pi} \sigma_{\varepsilon,\zeta'}^2} \exp\left(-\frac{(\varepsilon_{x,t}^{\zeta'})^2}{2\sigma_{\varepsilon,\zeta'}^2}\right),
\end{aligned}$$

The following priors $\pi(\boldsymbol{\vartheta}_x)$ are adopted in this paper

$$\begin{aligned}
\nu_x &\sim \text{U}(-1, 1), \quad u \sim \text{U}(-1, 1), \quad d \sim \text{U}(0, 1/2), \quad u' \sim \text{U}(-1, 1), \quad d' \sim \text{U}(0, 1/2), \\
\alpha_x &\sim \text{N}(0, 1), \quad \zeta' \sim \text{N}(0, 1), \quad \sigma_{x,\varepsilon}^2 \sim \Gamma(1, 1) \quad \text{and} \quad \sigma_{\varepsilon,\zeta'}^2 \sim \Gamma(1, 1),
\end{aligned}$$

where $\text{U}(a_u, b_u)$ denotes the non-informative uniform priors on the range (a_u, b_u) for the long memory parameters u and d with particular range constraints, $\Gamma(a, b)$ with positive support denotes the gamma prior with shape and scale parameters a and b respectively. We have tried different settings of a and b and found that the estimations are not sensitive to choices of a and b . The vector of model parameters is $\boldsymbol{\vartheta}_x = (\alpha_x, \zeta', u, u', d, d', \sigma_{x,\varepsilon}^s, \sigma_{\varepsilon,\zeta'}^s, \nu_x)$.

8.3.1 Posterior predictive distribution

Bayesian forecasting is another application of Bayesian inference. The predictive distribution can be derived from the Bayesian filtering equations $f(y_{x,t} | E_{x,t}, \mathcal{F}_{x,1:t})$ which compute the marginal posterior distribution or filtering distribution of the state $y_{x,t}$ at each time t given the history of the measurements up to time t . The Bayesian filter for computing the predicted distribution at time T are given by the following Bayesian filtering equations. Suppose we want m -step forecasts $\mathbf{y}_{x,T+1:T+m}$ with the observed data filtration $\mathcal{F}_{1:T}$ defined in Equation (7.1) and the observed data filtration for the random vector of central exposure $\mathbf{E}_t = (E_{x_1,t}, E_{x_2,t}, \dots, E_{x_g,t})$ for g number of age groups is defined as

$$\mathcal{F}_{1:t-1}^E = \sigma(\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_{t-1}). \quad (8.9)$$

In order to calculate the out-of-sample forecast for the number of deaths $\mathbf{y}_{x,T+1:T+m}$, we need to know the exposures $\mathbf{E}_{x,1:T+m}$. We note that $\mathbf{E}_{x,T+1:T+m}$ may or may not be directly observed. However, there are many methods to estimate $\mathbf{E}_{T+1:T+m}$ if we cannot directly observe future exposures. In this section, we discuss two methods to generate future exposures $\mathbf{E}_{x,T+1:T+m}$ condition on whether $\mathbf{E}_{x,T+1:T+m}$ can be observed.

The first method takes the exposure smoothed posterior predictive approach based on observed exposures $\mathbf{E}_{x,1:T+m}$. This approach is usually applied when the performances of different models are assessed by dividing the data set with length $T + m$ into two parts, the training $\mathbf{Y}_{x,1:T}$ with exposure $\mathbf{E}_{x,1:T}$ and forecast $\mathbf{Y}_{x,T+1:T+m}$ with exposure $\mathbf{E}_{x,T+1:T+m}$. Hence, the exposure smoothed posterior predictive density per age group for $\mathbf{y}_{x,T+1:T+m}$ is defined as

Definition 8.3.1 (Smoothed posterior predictive approach).

$$\begin{aligned}
& f(\mathbf{y}_{x,T+1:T+m} | \mathcal{F}_{1:T}, \mathcal{F}_{1:T+m}^E) \\
&= \int \dots \int f(\mathbf{y}_{x,T+1:T+m} | \boldsymbol{\mu}_{x,1:T+m}(\boldsymbol{\zeta}'_{x,1:T+m}), \boldsymbol{\vartheta}_x, \mathcal{F}_{1:T}, \mathcal{F}_{1:T+m}^E) f(\boldsymbol{\mu}_{x,1:T+m}(\boldsymbol{\zeta}'_{x,1:T+m}) | \boldsymbol{\mu}_{x,1:T}(\boldsymbol{\zeta}'_{x,1:T}), \boldsymbol{\vartheta}_x, \mathcal{F}_{1:T}) \\
&\quad \times f(\boldsymbol{\zeta}'_{x,1:T+m} | \boldsymbol{\zeta}'_{x,1:T}, \boldsymbol{\vartheta}_x, \mathcal{F}_{1:T}) \pi(\boldsymbol{\vartheta}_x | \mathcal{F}_{1:T}) d\boldsymbol{\mu}_{x,1:T+m}(\boldsymbol{\zeta}'_{x,1:T+m}) d\boldsymbol{\zeta}'_{x,1:T+m} d\boldsymbol{\vartheta}_x, \\
&= \int \dots \int \prod_{s=1}^m [f(y_{x,T+s} | \boldsymbol{\mu}_{x,T+s}(\boldsymbol{\zeta}'_{x,T+s}), \boldsymbol{\vartheta}_x, \mathcal{F}_{1:T}, \mathcal{F}_{1:T+s}^E) f(\boldsymbol{\mu}_{x,T+s}(\boldsymbol{\zeta}'_{x,T+s}) | \boldsymbol{\mu}_{x,T+s-1}(\boldsymbol{\zeta}'_{x,T+s-1}), \boldsymbol{\vartheta}_x, \mathcal{F}_{1:T}) \\
&\quad \times f(\boldsymbol{\zeta}'_{x,T+s} | \boldsymbol{\zeta}'_{x,T+s-1}, \boldsymbol{\vartheta}_x, \mathcal{F}_{1:T})] f(\boldsymbol{\mu}_{x,1:T}(\boldsymbol{\zeta}'_{x,1:T}) | \boldsymbol{\vartheta}_x, \mathcal{F}_{1:T}) f(\boldsymbol{\zeta}'_{x,1:T} | \boldsymbol{\vartheta}_x, \mathcal{F}_{1:T}) \\
&\quad \times \pi(\boldsymbol{\vartheta}_x | \mathcal{F}_{1:T}) d\boldsymbol{\mu}_{x,1:T+m}(\boldsymbol{\zeta}'_{x,1:T+m}) d\boldsymbol{\zeta}'_{x,1:T+m} d\boldsymbol{\vartheta}_x,
\end{aligned}$$

and this integral can be approximated by the Monte Carlo estimator, constructed from posterior samples according to:

$$\hat{f}(\mathbf{y}_{x,T+s} | \mathcal{F}_{1:T}, \mathcal{F}_{1:T+s}^E) = \frac{1}{L} \sum_{l=1}^L f(y_{x,T+s} | \boldsymbol{\mu}_{x,1:T+s}^{(l)}(\boldsymbol{\zeta}'_{x,1:T+s}^{(l)}), \boldsymbol{\vartheta}_x^{(l)}, \mathcal{F}_{1:T}, \mathcal{F}_{1:T+s}^E),$$

where L is the number of iterations after burn-in in each MCMC sampler run given the current window of information and $\boldsymbol{\mu}_{x,1:T+s}^{(l)}$, $\boldsymbol{\zeta}'_{x,1:T+s}^{(l)}$ and $\boldsymbol{\vartheta}_x^{(l)}$ are the l -th draw in the posterior sample of $\boldsymbol{\mu}_{x,1:T+s}$, $\boldsymbol{\zeta}_{x,1:T+s}$ and $\boldsymbol{\vartheta}_x$, respectively.

We note that the marginal predictive distribution posterior per age group can be obtained by conditioning on d, u, d' and u' and so they have to be estimated jointly.

Remark 8.3.1. By adopting the concept of rational expectation model in economics, the future exposures $\mathbf{E}_{x,T+1:T+m}$ can be defined as a piecewise constant extension

$$\mathbf{E}_{x,T+s} = \mathbb{E}(\mathbf{E}_{x,1:T+s-1}) \text{ for } s = 1, 2, \dots, m.$$

If we do this extrapolation, the extrapolated smoothed posterior predictive approach can still be adopted with constant extended future exposures.

For second approach, we assume a stochastic model of future exposures $f(\mathbf{E}_{x,T+1:T+m} | \boldsymbol{\vartheta}_x, \mathcal{F}_{1:T}, \mathcal{F}_{1:T}^E)$. Hence, the posterior predictive distribution for $\mathbf{y}_{x,T+1:T+m}$ is defined as

Definition 8.3.2 (Posterior predictive approach with a stochastic model of future exposures).

$$\begin{aligned} & f(\mathbf{y}_{x,T+1:T+m} | \mathcal{F}_{1:T}, \mathcal{F}_{1:T}^E) \\ &= \int \dots \int f(\mathbf{y}_{x,T+1:T+m} | \mathbf{E}_{x,T+1:T+m}, \boldsymbol{\mu}_{x,1:T+m}(\boldsymbol{\zeta}'_{x,1:T+m}), \boldsymbol{\vartheta}_x, \mathcal{F}_{1:T}, \mathcal{F}_{1:T}^E) \\ & \quad \times f(\boldsymbol{\mu}_{x,1:T+m}(\boldsymbol{\zeta}'_{x,1:T+m}) | \boldsymbol{\mu}_{x,1:T}(\boldsymbol{\zeta}'_{x,1:T}), \boldsymbol{\vartheta}_x, \mathcal{F}_{1:T}) f(\boldsymbol{\zeta}'_{x,1:T+m} | \boldsymbol{\zeta}'_{x,1:T}, \boldsymbol{\vartheta}_x, \mathcal{F}_{1:T}) f(\mathbf{E}_{x,T+1:T+m} | \boldsymbol{\vartheta}_x, \mathcal{F}_{1:T}, \mathcal{F}_{1:T}^E) \\ & \quad \times \pi(\boldsymbol{\vartheta}_x | \mathcal{F}_{1:T}) d\boldsymbol{\mu}_{x,1:T+m}(\boldsymbol{\zeta}'_{x,1:T+m}) d\boldsymbol{\zeta}'_{x,1:T+m} d\mathbf{E}_{x,T+1:T+m} d\boldsymbol{\vartheta}_x, \end{aligned}$$

Remark 8.3.2. For calculating life expectancy, annuity prices, guaranteed annuity option and other financial products, mortality rates forecast is necessary rather than number of death. Consequently, we do not need to calculate the future exposure $\mathbf{E}_{x,T+1:T+m}$ values and we only apply smoothed posterior predictive approach.

8.4 Simulation Study

In this section, we perform simulation study to assess the accuracy of parameter estimates for the proposed LMLM model. Each simulation of repeated data sets for different model fittings are initialised using the same random number generator seed. The length of the data is $n = 100$ which is similar in length to the mortality data sets for Australia, U.K. and U.S. in Section 8.5. Moreover, true parameter values are chosen to generate death counts in similar magnitude to the observed rescaled death counts, often in the range of 0 to 250. We compare the effect of varying levels of u and u' (which indicate cycles of oscillatory ACFs) and the levels of d and d' (which indicate strength of long memory) on the parameter accuracy. We also consider the parameter accuracy

when u and u' are negative which give instantaneous oscillatory ACFs. As our proposed LMLM model contains two long memory processes assigned to the mean function and cohort effect component respectively, we aim to assess the accuracy of parameter estimates in two cases: case 1 considers $d = d'$ and $u = u'$ whereas case 2 sets $d \neq d'$ and $u \neq u'$. The purpose of these two case studies is to investigate if the strength and pattern of long memory between the mean and cohort effect can be distinguished from each other regardless of whether they are equal or not. We also evaluate such effects on the accuracy of other parameter estimates. We do not consider data replication as this study only aims to check the issue as stated. Instead, we consider several sets of true parameters in each case to assess the issue under different conditions. According to the results in Tables 8.2 and 8.3, the model parameters can be properly estimated by LMLM model for different sets of true values of d , d' , u and u' . Furthermore, the estimations of long memory parameters in mean function d and u do not affect the estimations of long memory parameters in cohort component d' and u' .

Table 8.2. True values, estimates and credible intervals for parameters in LMLM model with data length $n = 100$.

Case study 1: $d = d'$ and $u = u'$												
	d				u				d'			
	True value	$\hat{d}$	2.5%	97.5%	True value	$\hat{u}$	2.5%	97.5%	True value	$\hat{d}'$	2.5%	97.5%
set 1	0.45	0.4697	0.4583	0.4734	1	1.0000	0.9999	1.0000	0.45	0.4578	0.4389	0.4839
set 2	0.3	0.2681	0.2409	0.2777	1	0.9995	0.9977	0.9999	0.3	0.3064	0.2089	0.3417
set 3	0.1	0.0697	0.0402	0.0994	1	0.9820	0.8999	0.9985	0.1	0.0742	0.0029	0.1760
set 4	0.45	0.4102	0.3786	0.4432	0.9	0.8991	0.8927	0.9054	0.45	0.4308	0.3386	0.4955
set 5	0.45	0.4497	0.4032	0.4922	0.3	0.3009	0.2873	0.3140	0.45	0.4027	0.2745	0.4932
set 6	0.45	0.4674	0.4289	0.4978	-0.3	-0.2917	-0.3010	-0.2817	0.45	0.4226	0.3556	0.4937
set 7	0.45	0.4686	0.4391	0.4950	-0.9	-0.8992	-0.9038	-0.8947	0.45	0.4516	0.3715	0.4981
	u'				α_x				ζ'			
	True value	$\hat{u}'$	2.5%	97.5%	True value	$\hat{\alpha}_x$	2.5%	97.5%	True value	$\hat{\zeta}'$	2.5%	97.5%
set 1	1	0.9932	0.9907	0.9997	-5	-5.0481	-5.1096	-4.9870	1	1.0937	0.8274	1.0973
set 2	1	0.9731	0.9125	0.9904	-5	-5.0125	-5.1423	-4.8904	1	0.8775	0.7963	0.9030
set 3	1	0.9547	0.9224	0.9795	-5	-5.0024	-5.0502	-4.9534	1	0.9768	0.9321	1.0202
set 4	0.9	0.9077	0.8787	0.9324	-5	-4.9994	-5.0758	-4.9229	1	0.9434	0.8810	1.0067
set 5	0.3	0.2885	0.2494	0.3361	-5	-5.0013	-5.0424	-4.9607	1	0.996	0.9594	1.0317
set 6	-0.3	-0.3636	-0.7724	-0.2621	-5	-4.9991	-5.0283	-4.9716	1	0.9923	0.9634	1.0243
set 7	-0.9	-0.9041	-0.9304	-0.8728	-5	-5.0005	-5.0264	-4.9742	1	1.0018	0.9755	1.0283

Table 8.3. True values, estimates and credible intervals for parameters in LMLM model with data length $n = 100$.

Case study 2: $d \neq d'$ and $u \neq u'$												
	d				u				d'			
	True value	$\hat{d}$	2.5%	97.5%	True value	$\hat{u}$	2.5%	97.5%	True value	$\hat{d}'$	2.5%	97.5%
set 1	0.45	0.4285	0.4057	0.4527	1	1.0000	0.9999	1.0000	0.3	0.2928	0.1836	0.4070
set 2	0.45	0.4236	0.4008	0.4468	1	1.0000	0.9998	1.0000	0.1	0.1120	0.0062	0.2383
set 3	0.3	0.2663	0.2387	0.2950	1	0.9993	0.9973	1.0000	0.45	0.4571	0.3918	0.4980
set 4	0.1	0.0702	0.0394	0.1022	1	0.9848	0.9534	0.9983	0.45	0.4552	0.3881	0.4977
set 5	0.45	0.4170	0.3852	0.4502	0.9	0.8994	0.8934	0.9048	0.45	0.4529	0.3875	0.4970
set 6	0.45	0.4135	0.3661	0.4614	0.3	0.2986	0.2852	0.3119	0.45	0.4555	0.3905	0.4979
set 7	0.45	0.4635	0.4222	0.4963	-0.3	-0.2930	-0.3024	-0.2835	0.45	0.4542	0.3600	0.4984
set 8	0.45	0.4651	0.4336	0.4942	-0.9	-0.8985	-0.9031	-0.8936	0.45	0.4573	0.3870	0.4981
set 9	0.45	0.4290	0.4062	0.4526	1	0.9999	0.9998	1.0000	0.45	0.4475	0.3685	0.4975
set 10	0.45	0.4227	0.4005	0.4454	1	1.0000	0.9998	1.0000	0.45	0.3842	0.2621	0.4888
set 11	0.45	0.4289	0.4061	0.4518	1	0.9999	0.9998	1.0000	0.45	0.2473	0.0653	0.4340
set 12	0.45	0.4322	0.4094	0.4558	1	0.9999	0.9998	1.0000	0.45	0.4559	0.3750	0.4984
	u'				α_x				ζ'			
	True value	$\hat{u}'$	2.5%	97.5%	True value	$\hat{\alpha}_x$	2.5%	97.5%	True value	$\hat{\zeta}'$	2.5%	97.5%
set 1	1	0.9364	0.8487	0.9955	-5	-5.0391	-5.2520	-4.8295	1	0.9194	0.7967	1.0382
set 2	1	0.8601	0.7186	0.9885	-5	-5.0185	-5.5050	4.5544	1	0.9645	0.8604	1.0687
set 3	1	0.9992	0.9961	1.0000	-5	-5.0119	-5.1445	-4.8942	1	0.9170	0.7047	1.1252
set 4	1	0.9992	0.9963	1.0000	-5	-5.0000	-5.0447	-4.9730	1	0.9537	0.7354	1.1669
set 5	1	0.9993	0.9968	1.0000	-5	-4.9937	-5.1049	-4.8918	1	0.9295	0.7133	1.1394
set 6	1	0.9992	0.9964	1.0000	-5	-5.0035	-5.1053	-4.9050	1	0.9582	0.7386	1.1751
set 7	1	0.9992	0.9964	1.0000	-5	-5.0004	-5.0272	-4.9700	1	0.9686	0.7518	1.1621
set 8	1	0.9989	0.9951	1.0000	-5	-5.0009	-5.0303	-4.9723	1	0.9284	0.7090	0.9492
set 9	0.9	0.9081	0.8871	0.9290	-5	-5.0324	-5.2479	-4.8231	1	0.8889	0.7733	1.0051
set 10	0.3	0.2859	0.2491	0.3291	-5	-5.0015	-5.4811	-4.5389	1	0.9710	0.8788	1.0643
set 11	-0.3	-0.2582	-0.4878	-0.0773	-5	-4.9786	-5.1882	-4.7696	1	0.9542	0.8581	1.0486
set 12	-0.9	-0.9065	-0.9275	-0.8838	-5	-4.9767	-5.1877	-4.9046	1	0.9753	0.8878	1.0613

8.5 Estimation and forecasting of mortality rates

The mortality data sets of Australia, U.K. and U.S. are downloaded from HMD (Vladimir et al., 2017). We choose Australia because Australia is a country of migrants with small population and unique environment. So its mortality experience may be quite different from the rest of the world and is worth a study. On the other hand, U.K.'s mortality may be greatly affected by the two world wars and hence may display some interesting pattern using our long memory mortality models. Lastly, U.S. has big population and is a country that always draws world attention. So perhaps its mortality features are also of interest to people. We adopt AU, UK and US as abbreviations for these three country names and they have 94, 92 and 83 years of historical data, respectively. Each data set contains the scaled number of death $y_{x,t}^*$ and the scaled exposure $E_{x,t}^*$ divided by genders, years and ages. Age is aggregated into 21 age groups, namely, age 0 group, age 1-4 group, age 5-10 group, age 11-15 group etc. See Section 6.2 for further details.

8.5.1 In-sample fitting

In order to study the features of long memory, we perform in-sample fitting using the death counts for 21 age groups and both genders of AU, UK and US. Table 8.4 reports the parameter estimates using MELCC and LMLM models. The estimated long memory parameters d , d' , u and u' show strong long memory features in both mean function and cohort component but without oscillatory ACFs ($\hat{u}$ and $\hat{u}'$ are both close to 1). We also report in the mean function a few levels of intercept α_x (age groups 0, 16-20, 41-45, 66-70, and 86-90) in LMLM model and coefficients β_x and β_x^ζ (age groups 16-20 and 66-70) for period and cohort effects respectively in MELCC model. From the experience in the univariate models in Chapter 7, we set the dispersion parameter ν is set to be consistent across age groups. The level of mortality rates indicated by the intercepts α_x are similar between the two models for each age group but they vary across age groups. The negative shape parameter estimates ν show that the distribution is under-dispersed. This result also agrees with the univariate models in Chapter 7. The parameter estimates λ by MELCC model are all close to 1. This indicates that the cohort effect tends to be a non-stationary process. This feature motivates us to replace the AR(1) cohort effect structure in MELCC model by a long memory process.

Table 8.4. Parameter estimates and credible intervals for AU, UK and US using MELCC and LMLM models

AU	LMLM	$\hat{d}$	$\hat{u}$	$\hat{d}'$	$\hat{u}'$	$\hat{\alpha}_1$	$\hat{\alpha}_5$	$\hat{\alpha}_{10}$	$\hat{\alpha}_{15}$	$\hat{\alpha}_{19}$	$\hat{\zeta}'$	ν
female	2.5%	0.3675	0.9999	0.4813	0.9998	-2.9986	-6.2453	-5.1416	-3.2782	-1.5177	0.0444	-0.9999
	mean	0.3795	1.0000	0.4948	0.9999	-2.7296	-6.0956	-5.0028	-3.1128	-1.3731	0.1047	-0.9959
	97.5%	0.3918	1.0000	0.4999	1.0000	-2.7211	-5.9447	-4.8619	-2.9445	-1.2328	0.1655	-0.9849
male	2.5%	0.3818	0.9999	0.4882	0.9998	-2.7714	-6.0476	-4.8911	-3.0132	-1.3732	0.0377	-0.9999
	mean	0.3947	1.0000	0.4968	0.9999	-2.6404	-5.9027	-4.7577	-2.8538	-1.2355	0.1134	-0.9956
	97.5%	0.4079	1.0000	0.4999	1.0000	-2.5125	-5.7562	-4.6219	-2.6962	-1.0979	0.1892	-0.9839
AU	MELCC	$\hat{\alpha}_1$	$\hat{\alpha}_{10}$	$\hat{\alpha}_{19}$	$\hat{\beta}_5$	$\hat{\beta}_{15}$	$\hat{\beta}_{\zeta_5}$	$\hat{\beta}_{\zeta_{15}}$	ζ^κ	$\hat{\lambda}$	$\hat{\eta}$	ν
female	2.5%	-2.9529	-5.2084	-1.6411	0.0786	0.0021	0.0039	0.0639	-0.3213	0.9959	-0.3044	-0.9999
	mean	-2.8815	-5.1309	-1.5982	0.0872	0.0094	0.0128	0.0709	-0.2181	0.9988	-0.2450	-0.9944
	97.5%	-2.8069	-5.0485	-1.5513	0.0957	0.0170	0.0220	0.0778	-0.1182	1.0000	-0.1892	-0.9796
male	2.5%	-2.7168	-5.0114	-1.4821	0.0326	0.0022	0.0392	0.0690	-0.2882	0.9921	-0.2611	-0.9991
	mean	-2.6263	-4.9392	-1.4288	0.0386	0.0114	0.0441	0.0759	-0.2099	0.9978	-0.1759	-0.9725
	97.5%	-2.5318	-5.0717	-1.3740	0.0444	0.0202	0.0492	0.0823	-0.1308	0.9999	-0.0918	-0.9057
UK	LMLM	$\hat{d}$	$\hat{u}$	$\hat{d}'$	$\hat{u}'$	$\hat{\alpha}_1$	$\hat{\alpha}_5$	$\hat{\alpha}_{10}$	$\hat{\alpha}_{15}$	$\hat{\alpha}_{19}$	$\hat{\zeta}'$	ν
female	2.5%	0.4100	0.9998	0.4878	0.9998	-2.7916	-5.9339	-5.0727	-3.1675	-1.2918	0.0128	-0.9999
	mean	0.4225	0.9999	0.4966	1.0000	-2.6496	-5.7971	-4.9220	-2.9906	-1.0944	0.0883	-0.9967
	97.5%	0.4360	1.0000	0.4999	1.0000	-2.5038	-5.6624	-4.7721	-2.8129	-0.8924	0.1630	-0.9880
male	2.5%	0.4090	0.9998	0.4890	0.9998	-2.5529	-5.8807	-4.8325	-2.9047	-1.2470	0.0184	-0.9999
	mean	0.4228	0.9999	0.4970	1.0000	-2.4012	-5.7409	-4.6799	-2.7229	-1.0871	0.1035	-0.9970
	97.5%	0.4374	1.0000	0.4999	1.0000	-2.2484	-5.6004	-4.5314	-2.5468	-0.9267	0.1895	-0.9888
UK	MELCC	$\hat{\alpha}_1$	$\hat{\alpha}_{10}$	$\hat{\alpha}_{19}$	$\hat{\beta}_5$	$\hat{\beta}_{15}$	$\hat{\beta}_{\zeta_5}$	$\hat{\beta}_{\zeta_{15}}$	ζ^κ	$\hat{\lambda}$	$\hat{\eta}$	ν
female	2.5%	-2.3805	-5.1296	-1.3782	0.0485	0.0297	0.1340	0.0001	-0.5741	0.9559	-0.2900	-0.9999
	mean	-2.2238	-5.0457	-1.3144	0.0567	0.0333	0.1419	0.0027	-0.4055	0.9788	-0.1322	-0.9954
	97.5%	-2.0085	-4.9383	-1.2396	0.0646	0.0373	0.1498	0.0081	-0.2474	0.9981	0.0067	-0.9828
male	2.5%	-2.1062	-4.9311	-1.3091	0.0568	0.0212	0.0676	0.0000	-0.7090	0.9915	0.0170	-0.9997
	mean	-1.8923	-4.8438	-1.2661	0.0592	0.0250	0.0709	0.0004	-0.5072	0.9977	0.1773	-0.9907
	97.5%	-1.6307	-4.7365	-1.2194	0.0615	0.0290	0.0745	0.0015	-0.3058	0.9999	0.3442	-0.9663
US	LMLM	$\hat{d}$	$\hat{u}$	$\hat{d}'$	$\hat{u}'$	$\hat{\alpha}_1$	$\hat{\alpha}_5$	$\hat{\alpha}_{10}$	$\hat{\alpha}_{15}$	$\hat{\alpha}_{19}$	$\hat{\zeta}'$	ν
female	2.5%	0.3934	0.9994	0.4835	0.9993	-2.9837	-6.1581	-4.9575	-3.1517	-1.3772	0.0421	-0.9999
	mean	0.4063	0.9996	0.4956	0.9998	-2.8590	-6.0477	-4.8253	-3.0314	-1.2424	0.0977	-0.9974
	97.5%	0.4195	0.9998	0.4999	1.0000	-2.7363	-5.9350	-4.6940	-2.9143	-1.1110	0.1546	-0.9905
male	2.5%	0.3861	0.9996	0.4877	0.9998	-2.8207	-5.9321	-4.7143	-2.8276	-1.2885	0.0524	-0.9999
	mean	0.4002	0.9998	0.4966	0.9999	-2.6954	-5.8174	-4.5823	-2.9486	-1.1436	0.1200	-0.9974
	97.5%	0.4147	0.9999	0.4999	1.0000	-2.5674	-5.7039	-4.4504	-2.7042	-1.0024	0.1871	-0.9902
US	MELCC	$\hat{\alpha}_1$	$\hat{\alpha}_{10}$	$\hat{\alpha}_{19}$	$\hat{\beta}_5$	$\hat{\beta}_{15}$	$\hat{\beta}_{\zeta_5}$	$\hat{\beta}_{\zeta_{15}}$	ζ^κ	$\hat{\lambda}$	$\hat{\eta}$	ν
female	2.5%	-2.8524	-5.1127	-1.6041	0.0660	0.0269	0.0820	0.0142	-0.6658	0.9943	0.1025	-0.9999
	mean	-2.7740	-5.0345	-1.5501	0.0696	0.0293	0.0911	0.0207	-0.5518	0.9984	0.1947	-0.9968
	97.5%	-2.6887	-4.9515	-1.4965	0.0738	0.0315	0.1002	0.0267	-0.4405	1.0000	0.2952	-0.9876
male	2.5%	-2.6642	-4.8371	-1.5846	0.0597	0.0015	0.0196	0.0544	-0.2529	0.9882	-0.2689	-0.9999
	mean	-2.5731	-4.7510	-1.5104	0.0660	0.0078	0.0262	0.0627	-0.1440	0.9962	-0.1928	-0.9956
	97.5%	-2.4756	-4.6476	-1.4328	0.0721	0.0143	0.0327	0.0702	-0.0298	0.9999	-0.1254	-0.9836

In order to visualise the level information for 21 age groups, Figure 8.2 plots the intercept estimates for $\hat{\alpha}_x$ estimated by MELCC and LMLM models. The shape of the curves agree with the of $a_{x,i}^{(0)}$ in the graduation models in Chapter 7. The estimates $\hat{\alpha}_x$ decline from age 0 to the lowest level around age 15-20. Thereafter, there is a sharper increase and then a slower increase between age 20-34.

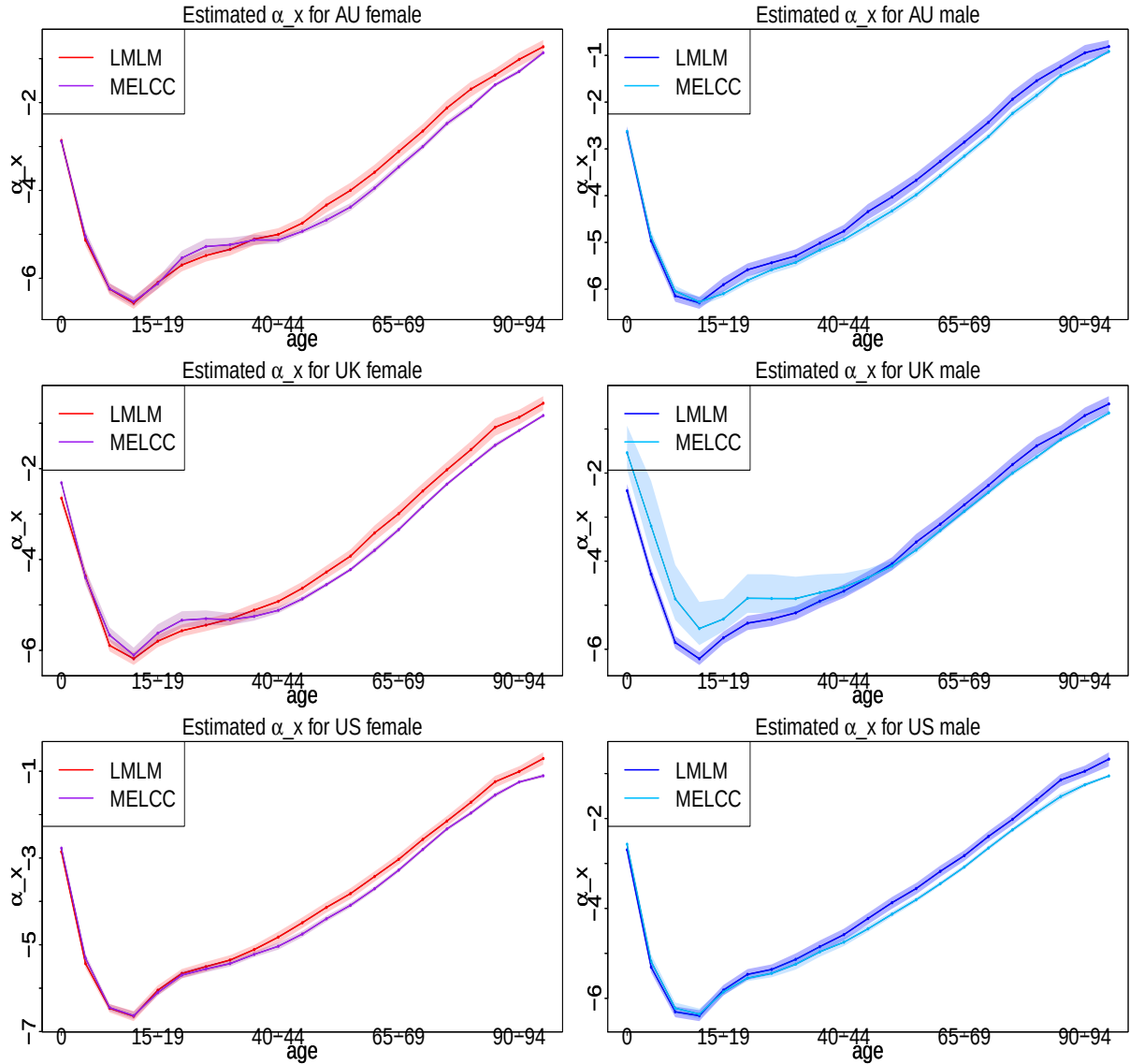


Figure 8.2. The plots for $\hat{\alpha}_x$ estimated by LMLM (red for female and blue for male) and MELCC (purple for female and skyblue for male) models for AU, UK and US.

Table 8.5 shows the performance of MELCC and LMLM models using four selection criteria, namely, MAEs, RMSEs, MAPEs and MASEs defined in Section 5.3.3 by comparing the estimated mortality rates $\hat{\mu}_{x,t}$ with the observed ones for AU, UK and US from HMD. Generally speaking, LMLM model performs better than MELCC model as the selection criteria values for LMLM model are smaller than those of MELCC model.

Table 8.5. Model selection criteria for comparing mortality rate estimates using MELCC and LMLM models for AU, UK and US

	MAEs	RMSEs	MAPEs	MASEs	MAEs	RMSEs	MAPEs	MASEs
	Australia female				Australia male			
MELCC	0.00097	0.00366	0.0300	0.0299	0.00024	0.00053	0.0296	0.0210
LMLM	0.00087	0.00359	0.0228	0.0267	0.00023	0.00060	0.0180	0.0196
	U.K. female				U.K. male			
MELCC	0.00112	0.00327	0.0251	0.0298	0.00188	0.00554	0.0356	0.0456
LMLM	0.00112	0.00402	0.0181	0.0298	0.00152	0.00552	0.0187	0.0368
	U.S. female				U.S. male			
MELCC	0.00115	0.00462	0.0329	0.0472	0.00162	0.00518	0.0333	0.0639
LMLM	0.00074	0.00435	0.0148	0.0303	0.00092	0.00516	0.0155	0.0364

In order to visualise the improved performance of LMLM model as compared with the MELCC model, Figures 8.3, 8.4 and 8.5 present the time series of observed and fitted mortality rates each superimposed with their 95% credible intervals for six age groups (0, 15-19, 35-39, 50-54, 60-64 and 70-74) representing youth, adult, middle, senior and older senior age groups for female and male respectively in Australia, U.K. and U.S.. The performance of LMLM model with long memory structure is much better than MELCC model as LMLM model can fit the trend and capture the peaks of the time series better. Moreover the credible intervals using LMLM model are also sharper in general. Comparing across age groups and gender, we find consistently faster drop of mortality rates in earlier years and more gentle drops in recent years. The only exception is for the senior ages when the rate of drop is a reverse. We also observe the trend of mortality rates to drop from 0 age to age groups 15-19 and increase thereafter. All these general trends agree with the findings in literature. Interestingly, we note a small peak of mortality rise for U.K. young male during the second world war in Figure 8.4 and both models are able to capture such peak. We also highlight the general trend for the width of creditable interval which increases with age indicating more versatile residuals behaviour. This is not surprising due to the less information from survivors of higher age groups.

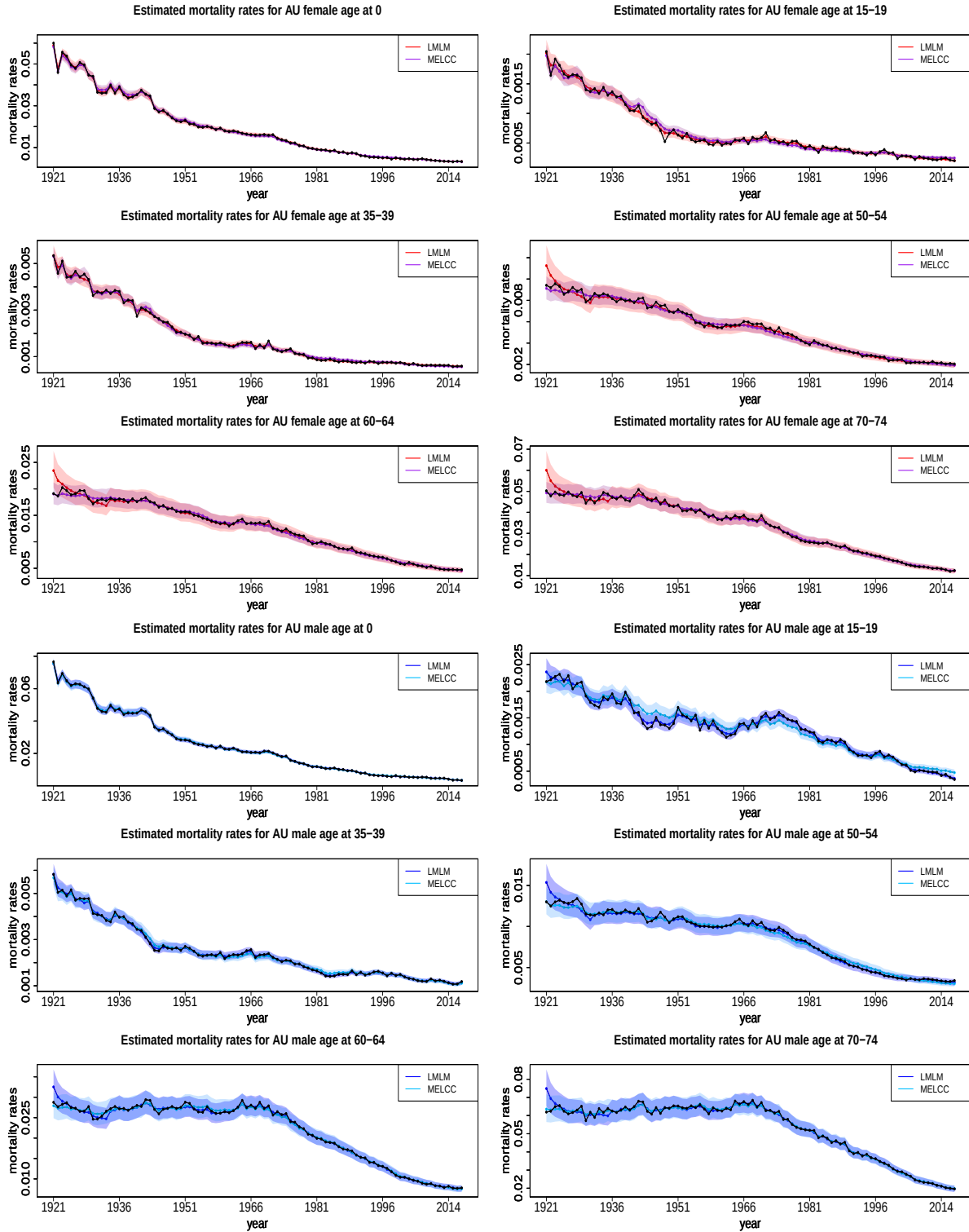


Figure 8.3. Observed mortality rate $\mu_{x,t}$ (black line) from HMD and estimated mortality rate $\hat{\mu}_{x,t}$ LMLM (red for female and blue for male) and MELCC (purple for female and skyblue for male) for the six age groups (0, 15-19, 35-39, 50-54, 60-64 and 70-74) in Australia.

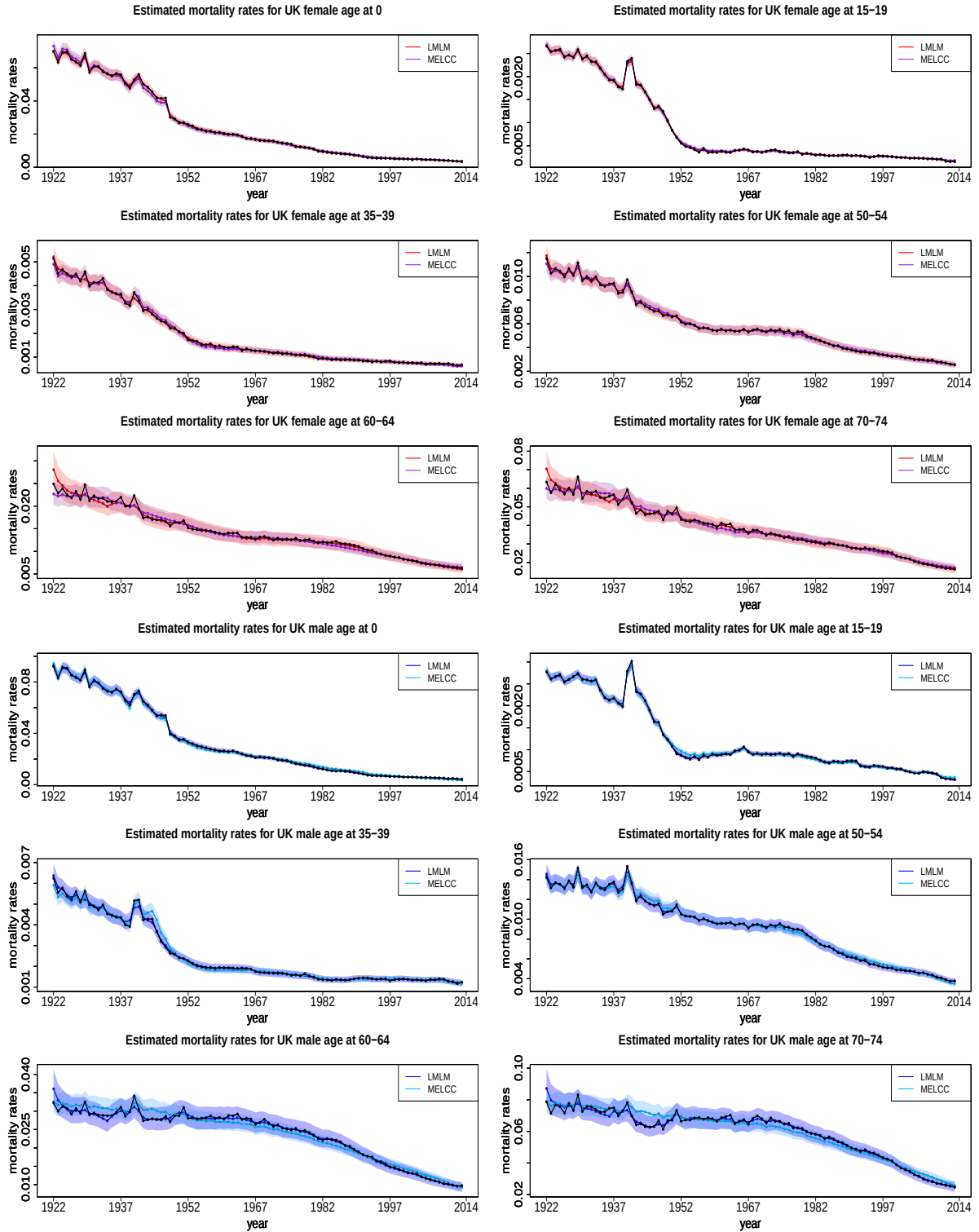


Figure 8.4. Observed mortality rate $\mu_{x,t}$ (black line) from HMD and estimated mortality rate $\hat{\mu}_{x,t}$ LMLM (red for female and blue for male) and MELCC (purple for female and skyblue for male) for the six age groups (0, 15-19, 35-39, 50-54, 60-64 and 70-74) in U.K..

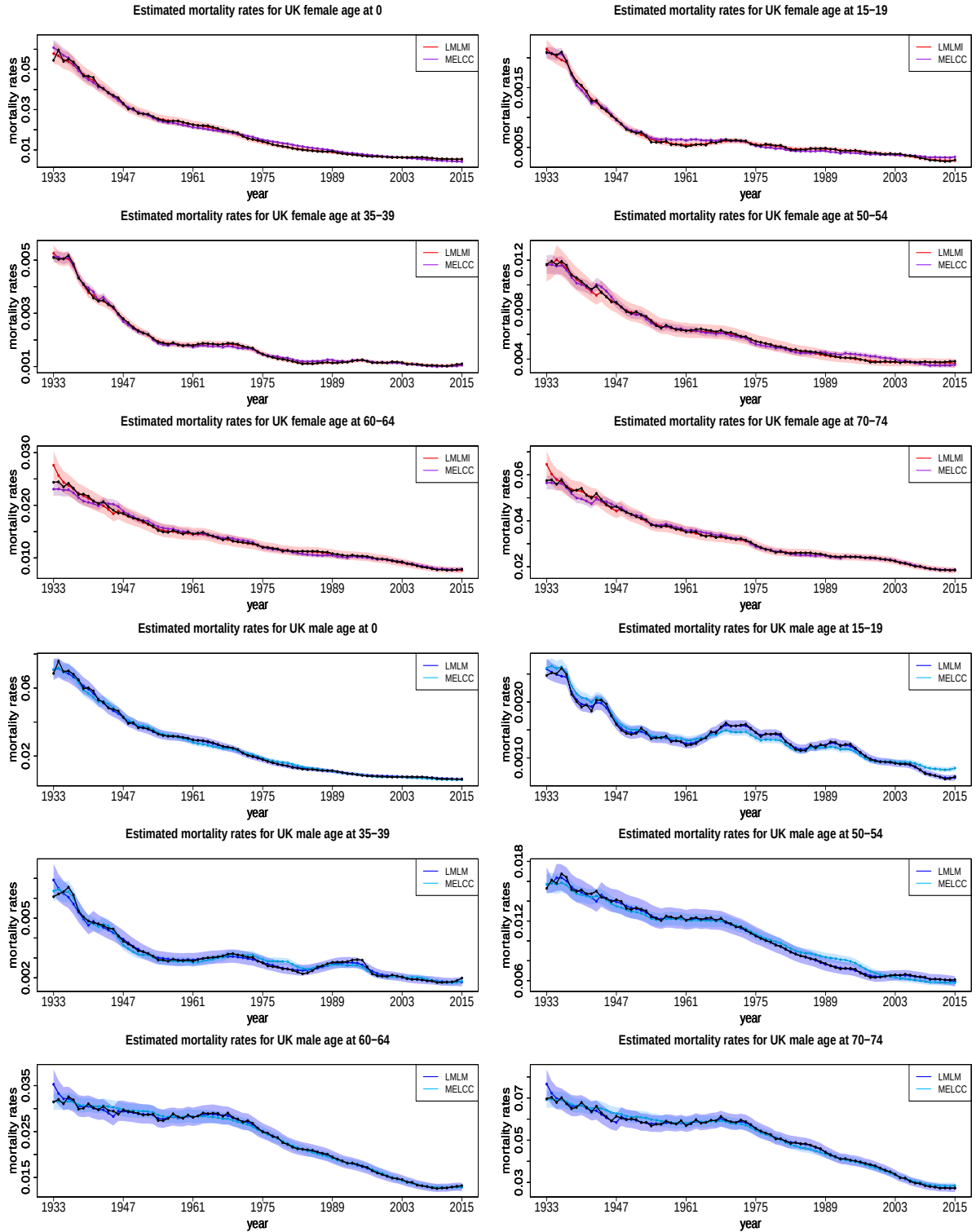


Figure 8.5. Observed mortality rate $\mu_{x,t}$ (black line) from HMD and estimated mortality rate $\hat{\mu}_{x,t}$ LMLM (red for female and blue for male) and MELCC (purple for female and skyblue for male) for the six age groups (0, 15-19, 35-39, 50-54, 60-64 and 70-74) in U.S..

Table 8.6 presents the performances of MELCC and LMLM models by using four selection criteria, namely, MAEs, RMSEs, MAPEs and MASEs which compare the life expectancy calculated by estimated mortality rates $\hat{\mu}_{x,t}$ with the observed life expectancy $e_{\tilde{x},t}$ from the HMD. These selection criteria are given in Section 5. Generally speaking, LMLM model performs better than MELCC model as the selection criteria values for LMLM model are smaller than MELCC model in most cases.

Table 8.6. Comparison of estimated life expectancy by in-sample fitting using MELCC and LMLM models for Australia, U.K. and U.S.

	MAEs	RMSEs	MAPEs	MASEs	MAEs	RMSEs	MAPEs	MASEs
	Australia female				Australia male			
MELCC	0.094	0.135	0.0042	0.0157	0.097	0.140	0.0044	0.0171
LMLM	0.084	0.124	0.0036	0.0142	0.087	0.133	0.0046	0.0153
	U.K. female				U.K. male			
MELCC	0.1402	0.2037	0.0070	0.0248	0.2140	0.2795	0.0100	0.0394
LMLM	0.1302	0.2076	0.0066	0.0230	0.1219	0.1988	0.0066	0.0225
	U.S. female				U.S. male			
MELCC	0.0995	0.1488	0.0044	0.0185	0.1093	0.1421	0.0059	0.0214
LMLM	0.0831	0.1711	0.0036	0.0155	0.0834	0.1676	0.0039	0.0163

In order to visualise the improved performance of LMLM model as compared with the LCC model, Figures 8.6, 8.7 and 8.8 plot the time series of observed and calculated life expectancy each superimposed with their 95% credible intervals for six age groups (0, 15-19, 35-39, 50-54, 60-64 and 70-74) representing youth, adult, middle, senior and older senior age groups for female and male respectively in Australia, U.K. and U.S.. The performance of LMLM model with long memory structure is much better than MELCC model as LMLM model can fit the trend and capture the peaks of the time series better.

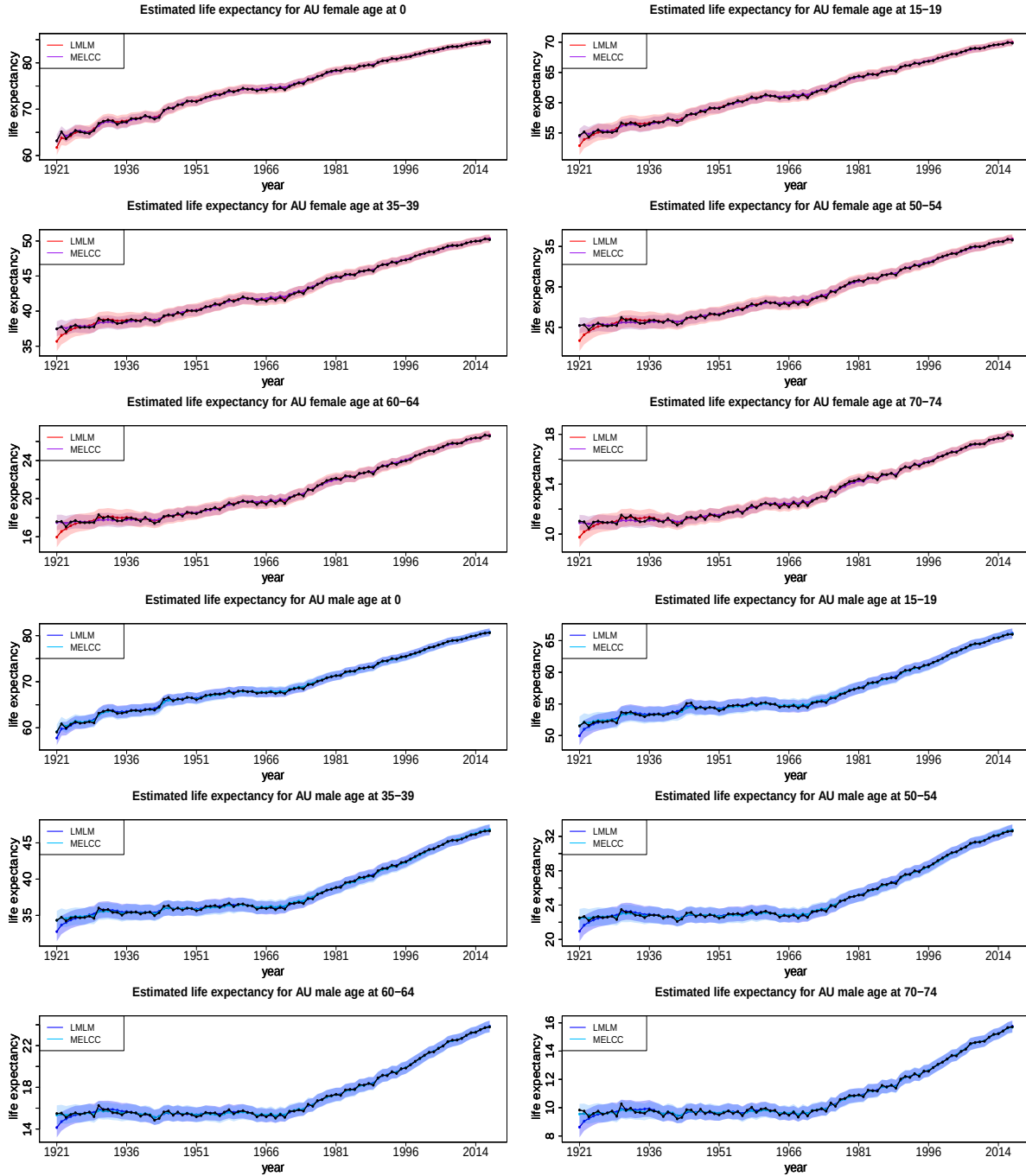


Figure 8.6. Observed life expectancy $e_{\bar{x},t}$ (black line) from HMD and life expectancy calculated by the estimated mortality rates by LMLM (red for female and blue for male) and MELCC (purple for female and skyblue for male) for the six age groups (0, 15-19, 35-39, 50-54, 60-64 and 70-74) in Australia.

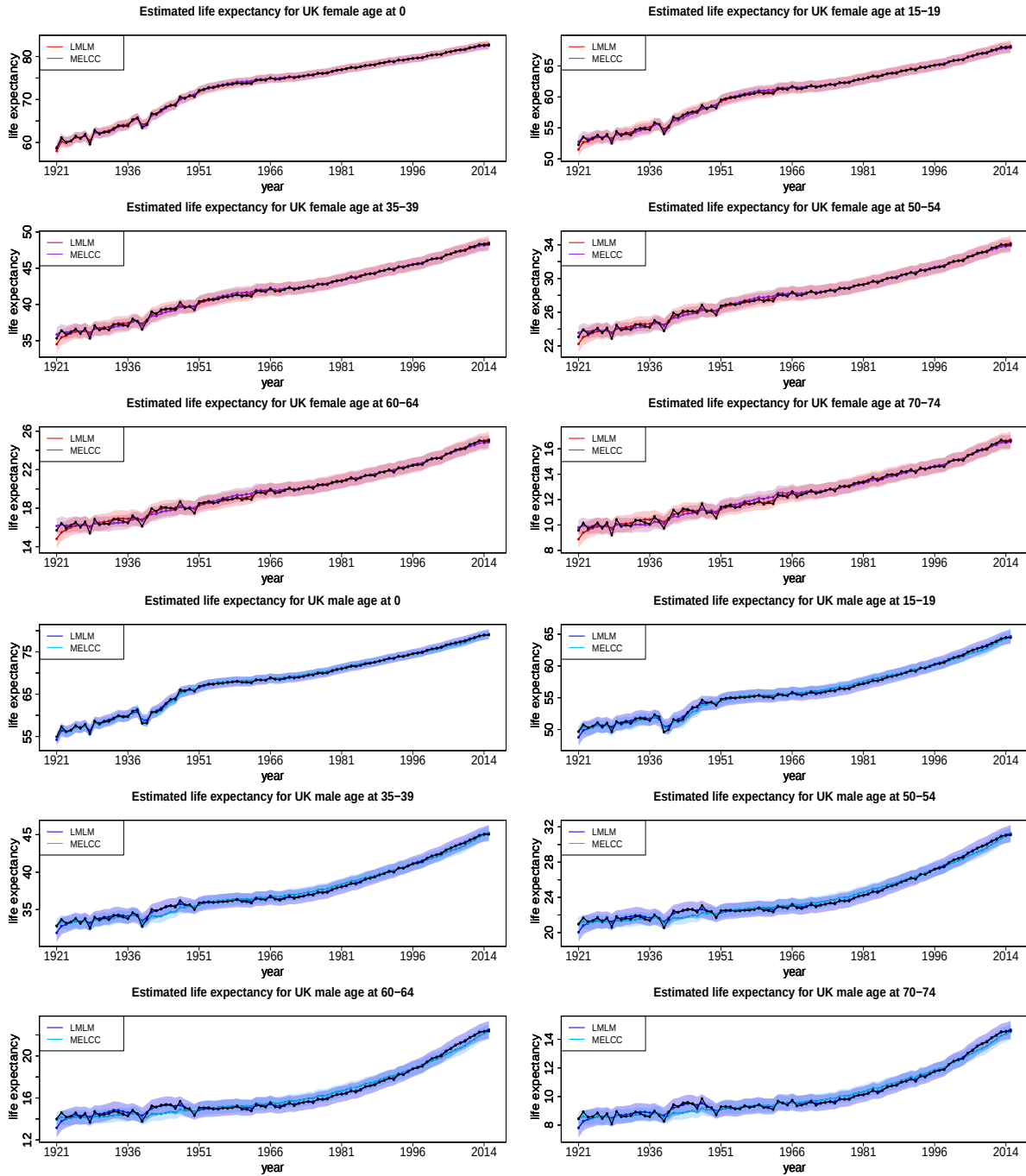


Figure 8.7. Observed life expectancy $e_{\bar{x},t}$ (black line) from HMD and life expectancy calculated by the estimated mortality rates by LMLM (red for female and blue for male) and LCC (purple for female and skyblue for male) for the four six groups (0, 15-19, 35-39, 50-54, 60-64 and 70-74) in U.K..

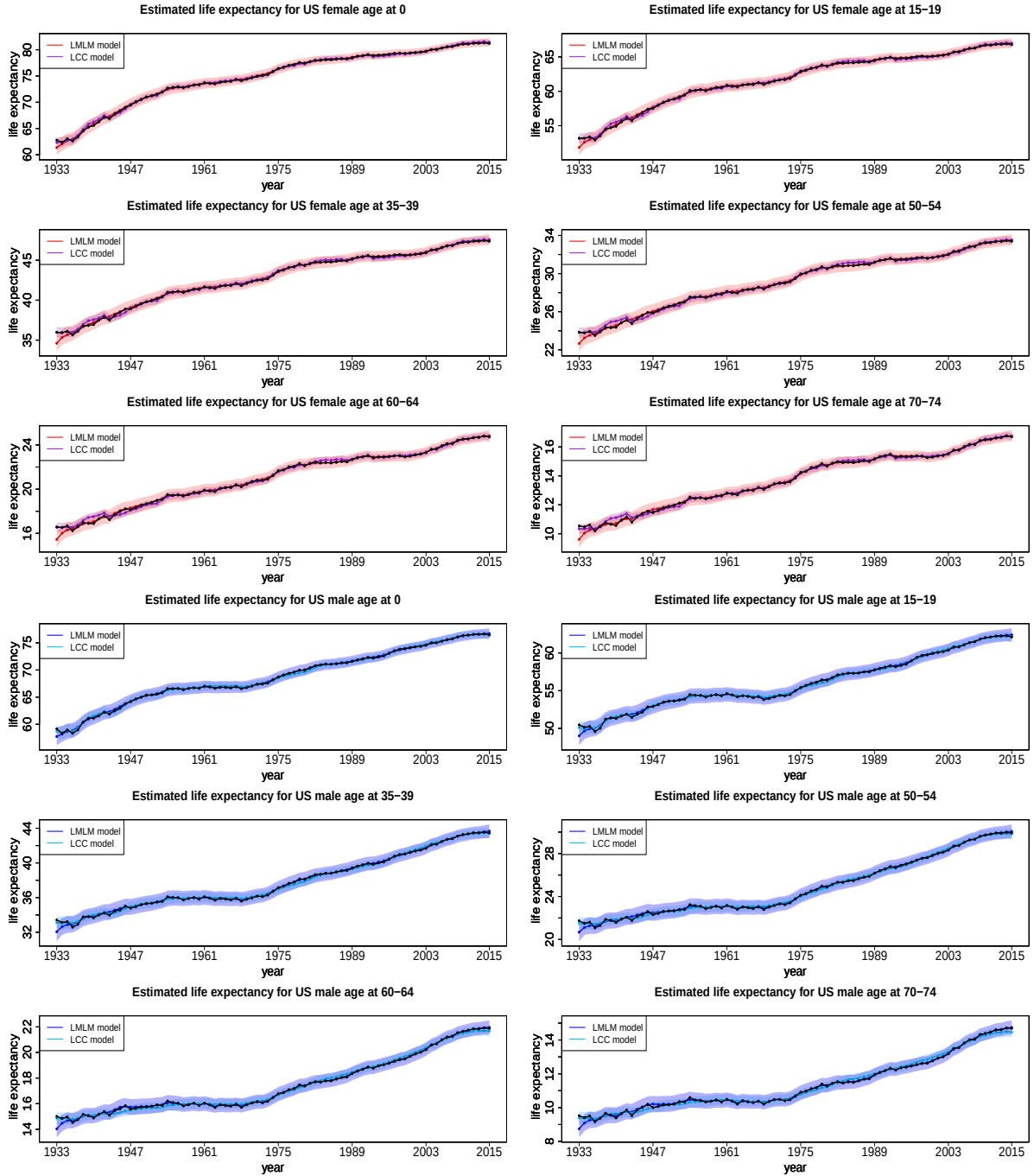


Figure 8.8. Observed life expectancy $e_{\bar{x},t}$ (black line) from HMD and life expectancy calculated by the estimated mortality rates by LMLM (red for female and blue for male) and LCC (purple for female and skyblue for male) for the four six groups (0, 15-19, 35-39, 50-54, 60-64 and 70-74) in U.S..

8.5.2 Out-of-sample forecast

In order to measure the forecast performance of both models, we perform of out-of-sample forecast by dividing the data sets for the number of death counts $\mathbf{Y}_{x,1:T}$ into two parts, the training $\mathbf{Y}_{x,1:(T-20)}$ and forecast $\mathbf{Y}_{x,(T-19):T}$. Table 8.7 reports the four model selection criteria based on mortality rate forecasts 20 years ahead $\hat{\boldsymbol{\mu}}_{x,(T-19):T}$ to compare the forecast performance between LMLM and MELCC models. Again, LMLM model provides more accurate mortality rate forecasts as the four criteria are lower for LMLM model.

Table 8.7. Model selection criteria for comparing mortality rate forecasts using MELCC and LMLM models for AU, UK and US

country		AU				UK				US			
gender	model	MAE	RMSE	MAPE	MASE	MAE	RMSE	MAPE	MASE	MAE	RMSE	MAPE	MASE
female	MELCC	0.00227	0.00465	0.146	0.084	0.00393	0.00826	0.150	0.140	0.00481	0.01050	0.170	0.205
	LMLM	0.00106	0.00290	0.068	0.039	0.00198	0.00615	0.098	0.070	0.00217	0.00595	0.060	0.092
male	MELCC	0.00355	0.00841	0.178	0.112	0.01100	0.02220	0.299	0.318	0.00308	0.00673	0.102	0.107
	LMLM	0.00248	0.00823	0.089	0.078	0.00330	0.00691	0.162	0.096	0.00200	0.00510	0.078	0.069

To view the performance of LMLM model compared with MELCC model, Figures 8.9, 8.10 and 8.11 plot the time series of the observed $\boldsymbol{\mu}_{x,(T-19):T}$, forecasted $\hat{\boldsymbol{\mu}}_{x,(T-19):T}$ by MELCC model and forecasted $\hat{\boldsymbol{\mu}}_{x,(T-19):T}$ by LMLM model with 95% credible interval for age groups (0-99). The forecast performance of LMLM model with long memory component is better than MELCC model for all age groups as the amounts of over or under estimation of mortality rates are relatively less than those for MELCC model in nearly all cases. We also highlight the general trend for the width of creditable interval which increases with age indicating more versatile residuals behaviour. This is not surprising due to the less information from survivors of higher age groups as the amounts of over or under estimation of mortality rates are relatively less than those for MELCC model in nearly all cases.

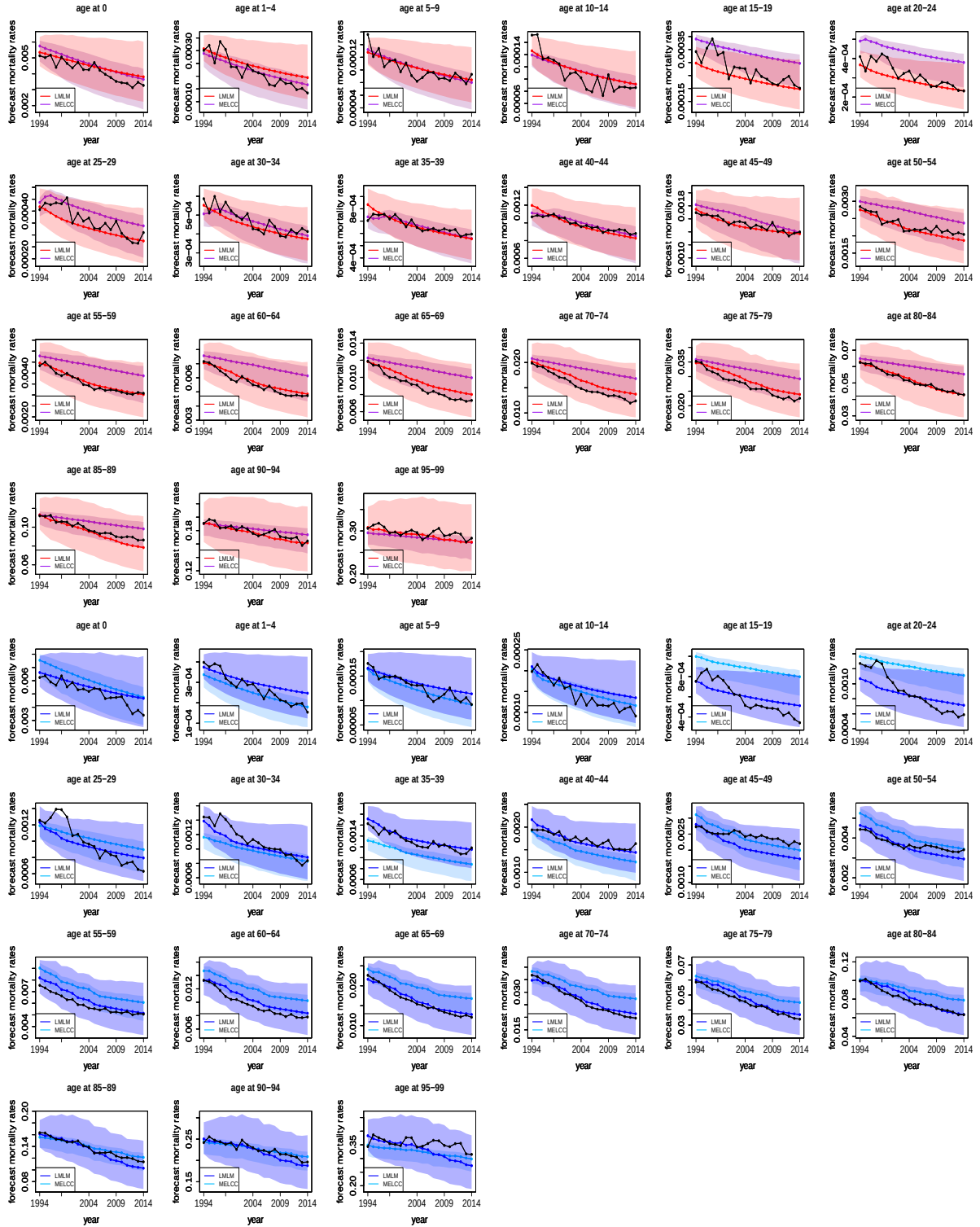


Figure 8.9. Observed mortality rates $\mu_{x,(T-19):T}$ (black line) from HMD and forecasted $\hat{\mu}_{x,(T-19):T}$ by LMLM (red for female and blue for male) and MELCC (purple for female and skyblue for male) models for age groups (0-99) in Australia.

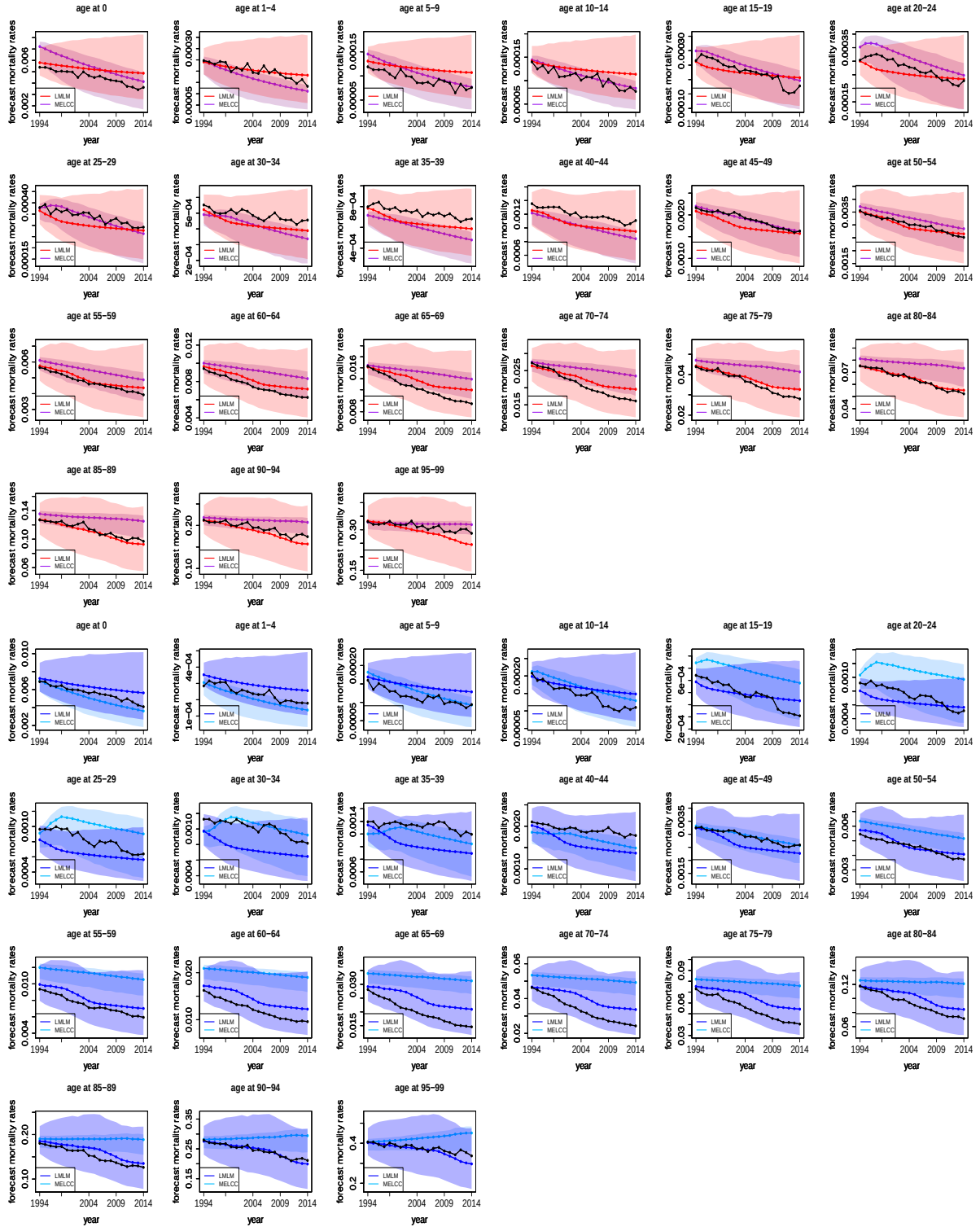


Figure 8.10. Observed mortality rates $\mu_{x,T-20:T}$ (black line) from HMD and forecasted $\hat{\mu}_{x,T-20:T}$ by LMLM (red for female and blue for male) and MELCC (purple for female and skyblue for male) models for age groups (0-99) in U.K..

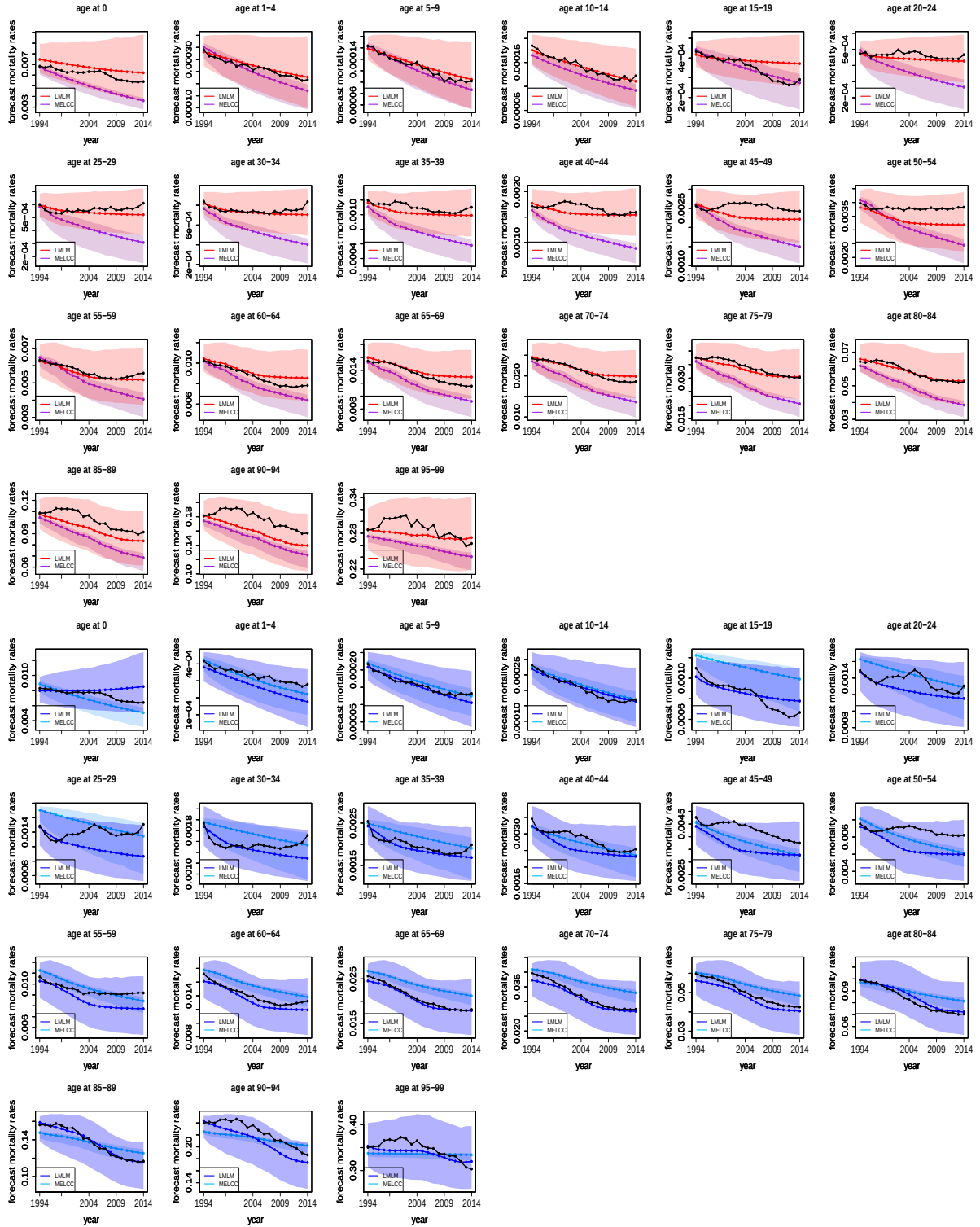


Figure 8.11. Observed mortality rates $\mu_{x,T-20:T}$ (black line) from HMD and forecasted $\hat{\mu}_{x,T-20:T}$ by LMLM (red for female and blue for male) and MELCC (purple for female and skyblue for male) models for age groups (0-99) in U.S..

Apart from in-sample fitting, we also forecast life expectancy 20 years ahead based on forecasted mortality rates using models fitted to the whole except the last 20 years (1994 to 2014) which are reserved for forecast comparison. Table 8.8 shows the performance of MELCC and LMLM models based on the four selection criteria by comparing the life expectancy calculated using forecasted mortality rates $\hat{\mu}_{x,(T-19):T}$ with the observed life expectancy $e_{\tilde{x},t}$ which is calculated using the mortality rates from HMD. Generally speaking, LMLM model performs better than MELCC model as the selection criteria values for LMLM model are consistently much smaller than those of MELCC model.

Table 8.8. Model selection criteria for comparing life expectancy forecasts using MELCC and LMLM models for Australia, U.K. and U.S.

	MAEs	RMSEs	MAPEs	MASEs	MAEs	RMSEs	MAPEs	MASEs
	Australia female				Australia male			
MELCC	0.984	1.157	0.028	0.244	0.998	1.169	0.030	0.265
LMLM	0.134	0.172	0.005	0.033	0.172	0.221	0.008	0.046
	U.K. female				U.K. male			
MELCC	1.206	1.425	0.041	0.305	2.727	3.161	0.095	0.735
LMLM	0.225	0.305	0.008	0.057	0.889	1.049	0.032	0.240
	U.S. female				U.S. male			
MELCC	1.375	1.595	0.046	0.350	0.546	0.640	0.021	0.150
LMLM	0.211	0.262	0.011	0.054	0.353	0.417	0.015	0.097

Figures 8.12, 8.13 and 8.14 depict the time series plots of observed life expectancy from HMD and forecasted life expectancy calculated using the mortality rate forecast from MELCC and LMLM models. In general, the forecast performance of LMLM model with long memory component is much better than that of MELCC model for all age groups as it often underestimates life expectancy whereas the LMLM model forecasts are fairly accurate particularly for female of all age groups because of the gradual improvement in life expectancy throughout the years. The only exception for MELCC model is the forecasts for Australian female which are overestimates. For males, both models underestimate life expectancy in U.K. due to the recent rise in life expectancy particular for senior age groups.

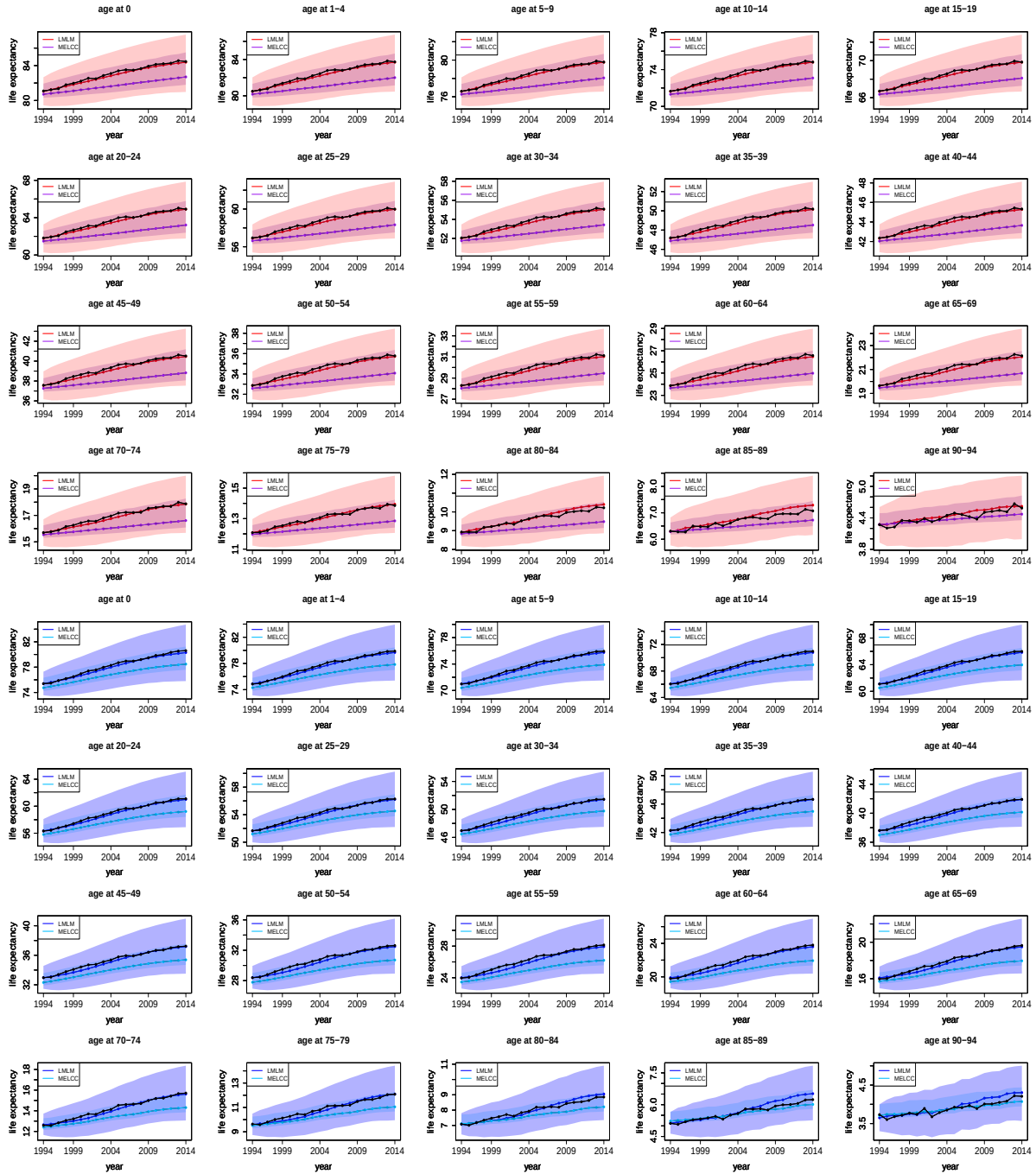


Figure 8.12. Observed life expectancy $e_{\bar{x},t}$ (black line) from HMD and forecasted life expectancy calculated using estimated mortality rates by LMLM (red for female and blue for male) and MELCC (purple for female and skyblue for male) models for age groups (0-94) in Australia.

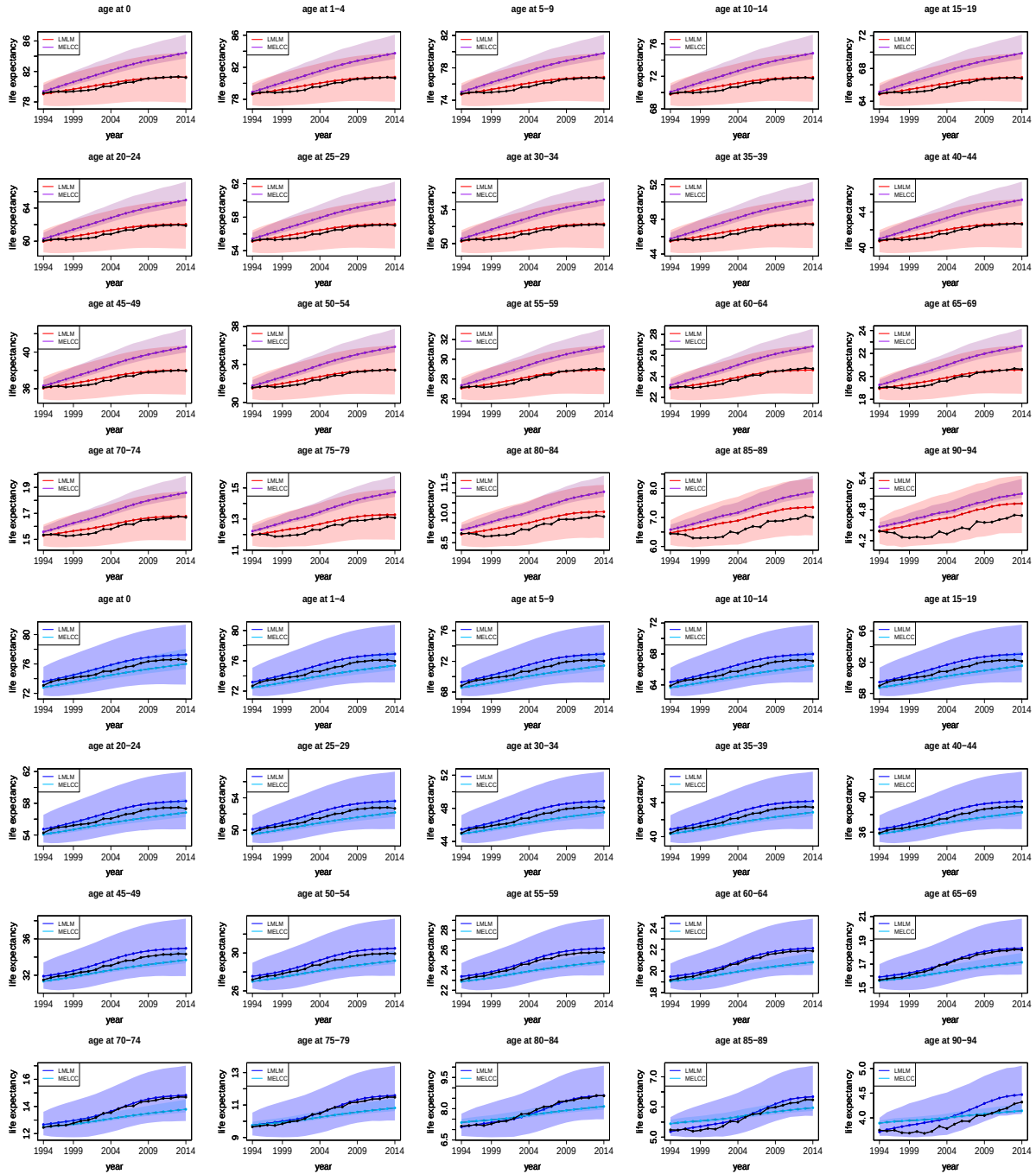


Figure 8.13. Observed life expectancy $e_{\bar{x},t}$ (black line) from HMD and forecasted life expectancy calculated using estimated mortality rates by LMLM (red for female and blue for male) and MELCC (purple for female and skyblue for male) models for age groups (0-94) in U.S..

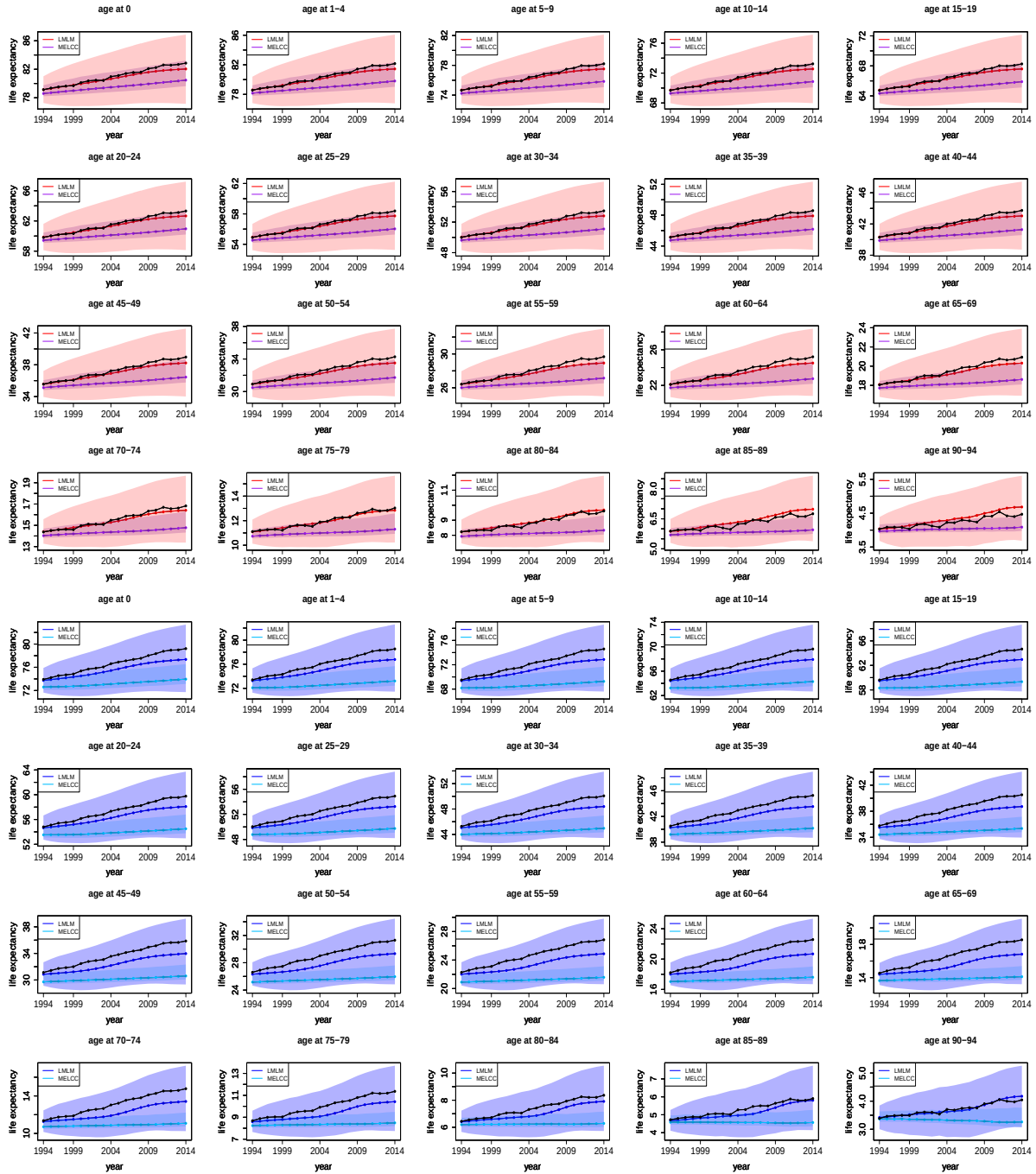


Figure 8.14. Observed life expectancy $e_{\bar{x},t}$ (black line) from HMD and forecasted life expectancy calculated using estimated mortality rates by LMLM (red for female and blue for male) and MELCC (purple for female and skyblue for male) models for age groups (0-94) in U.K..

8.6 Conclusion

This chapter introduces MELCC and LMLM models extended from LC model to death counts with GP distribution. The connection between these two models is explored. Compared to MELCC model, LMLM model drops the non-stationary period effect component and incorporates long memory to both mean function and cohort effect. Furthermore, we explain how the identification issue of MELCC model can be avoided in LMLM model. Bayesian inference is applied to perform the model parameter estimation and mortality rate and life expectancy forecasting. Results show that LMLM outperforms MELCC model in both in-sample fitting and out-sample forecasting using the four selection criteria given in Section 5.3.3 which measure discrepancies of mortality rates with those from HMD and discrepancies of life expectancies calculated using mortality rates from HMD. In summary, the life expectancy forecasts using LMLM model outperform those using MELCC model and so can be used to construct life tables as in Chapter 7. Moreover life expectancy forecasts from both models will be applied to price life annuities and guaranteed annuity options (GAO) in the next chapter.

CHAPTER 9

The insurance applications of multivariate long memory cohort mortality model

9.1 Background

This chapter explores the role of model risk when it comes to the quantification of longevity risk in insurance and the resulting mispricing that may arise. Longevity risk, defined as the risk that a person lives longer than expected is a core attribute to be addressed and modelled when determining the stochastic discounting of life contingent cash flows into the future for common life insurance products. The measurement and management of longevity risk includes low risk-free interest rates, changes in accounting regulations and changes in the regulatory environment ([Olivieri and Pitacco, 2009](#)).

As such, life insurance product design and pricing depends crucially on the ability of actuarial modelling to develop frameworks to accurately capture complex risk factors from stochastic relationships between three core stochastic drivers: firstly, interest rate dynamics over multi-decade time horizons; secondly, mortality rate dynamics by age group and gender for a given population and thirdly the ability to model the causal relationship between mortality events and interest rate fluctuations. We tackle all three aspects of these challenging problems faced by actuaries seeking to robustly price life products and share risk through mortality-linked securities such as annuities and generalised annuity options (GAO). We demonstrate with real data for three major populations, U.K., U.S.A. and Australia with different mortality profiles.

We begin our discussion on interest rate based on an important example of life annuity which is an important, growing part of the pension system nowadays. A life annuity

provides an individual with the opportunity to convert a stock of wealth into an income until the end of life. This annuity insures individual against out-living his/her wealth in the event of living longer than expected. The price of annuities will be determined by several factors, such as the prevailing interest rates at the time the annuity is purchased, the life expectancy of the annuitant and the size of the premium paid for the annuity (Cannon and Tonks, 2005). There are various approaches to model the term structure of interest rates in continuous time. For example, the short-term interest rates can be assumed to follow a diffusion process. Under this assumption, Vasicek (1977) proposed a one factor interest rates model called Vasicek model to describe interest rate movements as driven by only one source of market risk. This model is the first one to capture mean reversion feature which is an essential characteristic of interest rates. Because of this important property and its simplicity, we adopt this popular Vasicek model in modelling the interest rate dynamics in this chapter, see Svoboda (2003) for more details.

We then explain how to price mortality linked securities within both independent and dependent frameworks, where we price under a risk neutral framework. However, the calibration can be performed under two different assumptions that we explain. In the first case we can assume that we calibrate the model and risk premium based on observed yield data at tenors appropriate for the single factor short rate model, via either making an assumption that we observe directly a true risk free rate. In this case we do not need to explicitly estimate the risk premium as we are assuming we directly observe the risk neutral process, this is commonly assumed, though not often explicitly stated in for instance the results in Mamon (2004). Alternatively, we may also calibrate the Bayesian models we develop for the short rate model, in which we assume that there is a risk premium present as the observe government interest rate curve or LIBOR is no longer assumed to be truly risk free. If we assume that it suffices to estimate a constant in time unknown deterministic risk premium, then in such cases we need to calibrate the real world process on a single tenor time series, at some reasonable tenor daily or weekly observed in maturity interval one month to one year. Then we could simply extend this single factor short rate model to a term structure model to explain the term structure of interest rates between say one month to five years maturity, conditional on the real world model parameter calibration and we use this extension to estimate the risk premium, see discussion in Andreasson and Shevchenko (2017).

Next, we address the stochasticity issue of mortality rates. Until more recently, it was more prevalent that mortality models were constructed from deterministic modelling frameworks by traditional pension scheme actuaries. However, the uncertainty of mortality rates has become increasingly important to quantify as the effect of longevity risk has increased, as such it cannot be captured by deterministic mortality models. In order to quantify the uncertainty related to longevity risk and provide a more accurate mortality projection, stochastic mortality models are required. However classical Lee-Carter (LC) model was found to consistently underestimate mortality rates. To reduce the model risk and associated forecast errors of LC models, we develop in Chapter 8 effect (LMLM) models as extensions of classical LC models. We then utilise these model frameworks to study the influence of model risk on the pricing of life products including annuity portfolios and GAO in this chapter. We demonstrate that the MELCC model with no long memory provides can produce less accurate forecasts of life annuity and GAO prices than our proposed LMLM model and we quantify the mispricing cost of this model risk. This finding is substantial, since we demonstrate with a simple example that a small underpricing of annuity and GAO relative to a bench mark model due to longevity risk has a significant impact on insurance, re-insurance and government pension. A literature review of mortality and interest rates modelling can be found in Sections 1.2.4 and 1.2.5.

Lastly, we incorporate dependence dynamics which links with our stochastic interest rate model and mortality rate model. This formulation is developed in several contexts and can be viewed as extensions to the works of [Gao et al. \(2015\)](#) in a Bayesian approach. Specifically, we explore extensions of the dependence quantification between mortality rate and interest rate dynamics using independence, linear and robust rank correlation estimators in the marginal distribution approach as well as the jointly dependent estimator in the joint distribution approach. Throughout, we adopt Bayesian modelling frameworks to undertake both the calibration and forecasting framework, which is implemented via the `Rstan` package using the Hamiltonian Markov chain Monte Carlo (MCMC) algorithm.

In the next section we outline why it is important to consider longevity risk and to understand the aspect of model risk and subsequent mispricing when studying life products. Furthermore, we discuss the different strategies that practitioners often adopt

to address such problems, linking this to commentary on the role of model risk in such contexts.

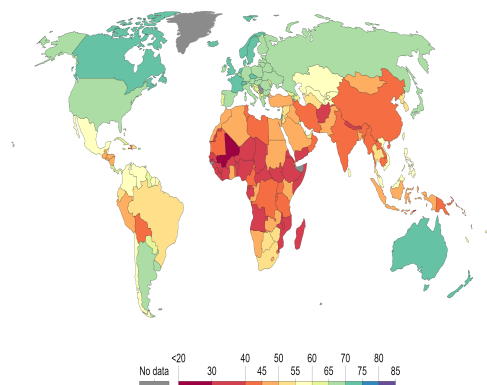
9.2 Longevity Risk and Practical Mechanisms of Risk Transfer

Longevity risk affects primarily the corporate sponsors of pension plans and insurers providing life assurance facilities. These two groups principally face the risk of higher than expected pay-outs as a result of the increasing longevity of pension plan or insurance policyholders. It is an observed fact that pensioners today are living significantly longer than 20 years ago and we demonstrate the increase in life expectancy throughout the world in each twenty years lapse during 1960 to 2010 in Figure 9.1. Furthermore, one can observe in reports that the life expectancy has risen by three to five years over the past two decades for many countries. This demonstrates the growing impact of longevity risk and the importance of reducing model risk and subsequent mispricing of life insurance products, when longevity risk is on the rise. Figure 9.1 shows further that the longevity risk becomes increasingly pronounced in recent years. As pensioners live longer, they also receive a pension for a longer time. This longevity risk for pension funds and other annuity providers is explicitly linked to medical advances in prolonging life.

Adding the effects of historically low risk-free interest rates, changes in accounting regulations and changes in the regulatory environment (Olivieri and Pitacco, 2009), one begins to see a picture in which longevity risk and associated financial risks combine together to pose a serious concern for extended time period life contingent liabilities. This is critical for pension planning, for instance the defined benefit pension plans, which guarantees the payment of benefits over the lifetime of the pension plan holder, are particularly exposed to longevity risk. Such pension plans often also involve financial instruments such as annuities. A life annuity provides an individual the ability to convert a stock of wealth into an income until the end of life. This annuity insures the individual against out-living their wealth in the event of living longer than expected. Furthermore, the annuity market is an increasingly important aspect of the pension

Life expectancy, 1960

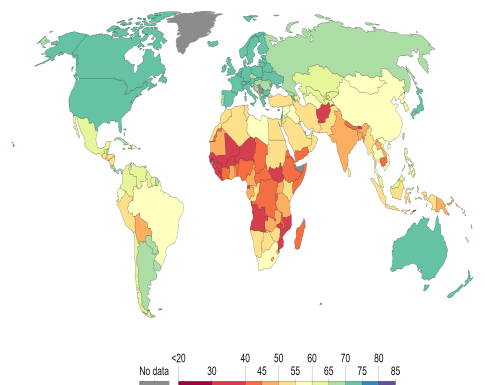
Shown is period life expectancy at birth. This corresponds to an estimate of the average number of years a newborn infant would live if prevailing patterns of mortality at the time of its birth were to stay the same throughout its life



Source: Clio-Infra estimates until 1949; UN Population Division from 1950 to 2015
OurWorldinData.org/life-expectancy-how-is-it-calculated-and-how-should-it-be-interpreted/ • CC BY-SA

Life expectancy, 1970

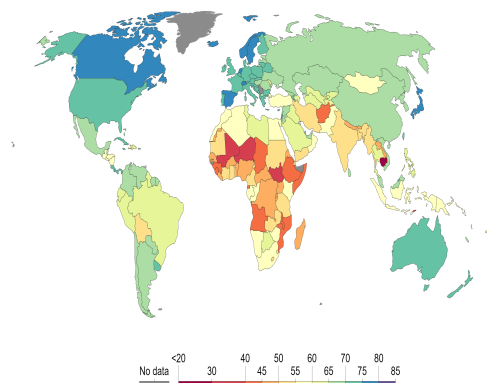
Shown is period life expectancy at birth. This corresponds to an estimate of the average number of years a newborn infant would live if prevailing patterns of mortality at the time of its birth were to stay the same throughout its life



Source: Clio-Infra estimates until 1949; UN Population Division from 1950 to 2015
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Life expectancy, 1980

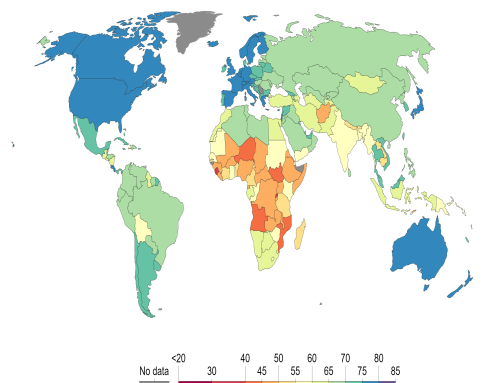
Shown is period life expectancy at birth. This corresponds to an estimate of the average number of years a newborn infant would live if prevailing patterns of mortality at the time of its birth were to stay the same throughout its life



Source: Clio-Infra estimates until 1949; UN Population Division from 1950 to 2015
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Life expectancy, 1990

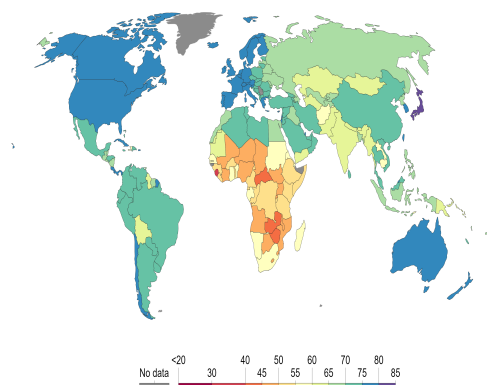
Shown is period life expectancy at birth. This corresponds to an estimate of the average number of years a newborn infant would live if prevailing patterns of mortality at the time of its birth were to stay the same throughout its life



Source: Clio-Infra estimates until 1949; UN Population Division from 1950 to 2015
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Life expectancy, 2000

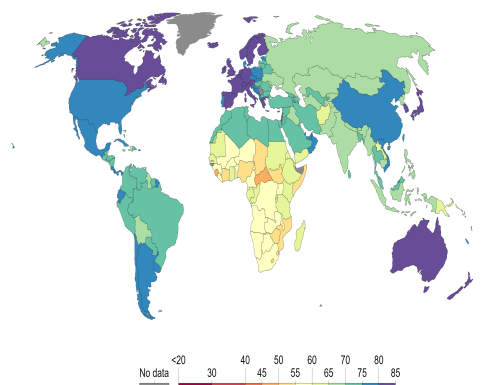
Shown is period life expectancy at birth. This corresponds to an estimate of the average number of years a newborn infant would live if prevailing patterns of mortality at the time of its birth were to stay the same throughout its life



Source: Clio-Infra estimates until 1949; UN Population Division from 1950 to 2015
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Life expectancy, 2010

Shown is period life expectancy at birth. This corresponds to an estimate of the average number of years a newborn infant would live if prevailing patterns of mortality at the time of its birth were to stay the same throughout its life



Source: Clio-Infra estimates until 1949; UN Population Division from 1950 to 2015
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Figure 9.1. Observed life expectancy. Figures created by Our World in Data, Roser (2018)

system. The price of annuities will be determined by several factors, such as the prevailing interest rates at the time the annuity is purchased, the life expectancy of the annuitant and the size of the premium paid for the annuity (Cannon and Tonks, 2005).

There are a number of papers tackling different aspects of the challenge of trying to mitigate, transfer or hedge longevity risk. For instance in Cairns (2013), they adopted an index-based longevity hedging instrument to reduce the longevity risk for a pension plan or annuity provider. Cairns et al. (2014) further showed that the hedge effectiveness can be broken down into contributions from a number of distinct types of risk factors.

Other life contingent financial instruments include mortality-linked securities and mortality-related guarantees in the capital market, such as GAO. Under a GAO, the insurance company guarantees to convert a policyholders accumulated funds to a life annuity at a fixed rate when the policy matures (Boyle and Hardy, 2003). London (1972) reported that the origin of GAOs dates back to 1839. However, these instruments became prevalence in UK in the era of 1970-1990. With the increase of life expectancy, the model risk and associated pricing uncertainties of GAOs have drawn a lot of attention. The underpricing of such guarantees has already caused serious solvency problems to insurers in practice. For example in the UK, as an after effect of encashment of too many GAOs, the world's oldest life insurer - Equitable Life had to close to new business in 2000 (Bahl and Sabanis, 2017).

In practice, insurance companies consider a range of different approaches to tackle impacts of longevity risk, some of these observed strategies adopted in practice to transfer longevity risk from the insurer to other parties are discussed in Gunter (2012). Examples of the approaches they can adopt in practice can include: diversification of the risk, constructing natural hedges to longevity risk, buy out of the plan, distributing the risk to the annuity beneficiaries, securitising the risk and transferring the risk with capital market participants.

In the case of natural hedges, one could set up a portfolio that can be balanced by trying to maximise natural hedges. This would involve setting up an annuity business exposed to longevity risk and concurrently setting up a term assurance business exposed to mortality risk. If a financial institution decides to adopt a sharing the risk framework with annuity beneficiaries, then in such a case the annuity providers may decide to

pass on the exposure to longevity risk in part to policyholders. Such frameworks were discussed in [Gunter \(2012\)](#).

In this chapter, we are interested in aspects of model risk directly associated with the last strategy mentioned. That is we focus on the strategy of sharing the risk of longevity with capital market participants. In such cases the company can decide to manage the risk through mortality-linked securities, such as mortality and longevity bonds, or other financial instruments.

9.3 Assessing Mispricing in Annuity Pricing Under Constant Interest Rate

In this section we progressively develop the complexity of the pricing framework, bringing together the stochastic mortality models with the stochastic interest rate models in stages. We first present the case of constant interest rate, for a classical analysis of the influence of model risk, comparing the MELCC and the LMLM life table forecasts and subsequent distribution of annuity prices that are obtained from each. This will demonstrate how the effect of a demonstrated less accurate life expectancy prediction, under the MELCC model will result in a mispricing relative to the improved forecasting model of LMLM.

9.3.1 Model Risk from Mortality Component

In this analysis, the impact of longevity risk on the pricing of annuities will be studied using the forecasted mortality rates by MELCC and LMLM models. The τ year survival probability of a person aged x currently at time t is determined by

$${}_t p_{x,t} = \prod_{j=1}^{\tau} {}_t p_{x+j-1,t+j-1} = \prod_{j=1}^{\tau} e^{-\mu_{x+j-1,t+j-1}},$$

which is a random variable since $\mu_{x+j-1,t+j-1}$ for $j = 1, \dots, \tau$ are random quantities forecasted by the MELCC and LMLM models. We note that we assume that the current time is year t for all subsequent annuity and GAO formula for convenience.

Definition 9.3.1 (Price of an annuity). *Assuming a large enough annuity portfolio and conditioning on the mortality rates path $\boldsymbol{\mu}_{x^+,t,\mathcal{T}} = (\mu_{x,t}, \mu_{x+1,t+1}, \dots, \mu_{x+\mathcal{T},t+\mathcal{T}})$, the price of an annuity $a(t, \mathcal{T}; x)$ where $\mathcal{T}$ is the maturity period, written for a people aged x with benefit \$1 per year is given by*

$$a(t, \mathcal{T}; x) = \sum_{\tau=1}^{\mathcal{T}} B(0, \tau) \mathbb{E}(\mathbb{1}_{[e_{x,t} \geq \tau]} | \boldsymbol{\mu}_{x^+,t,\tau}) = \sum_{\tau=1}^{\mathcal{T}} B(0, \tau) {}_{\tau}p_{x,t}(\boldsymbol{\mu}_{x^+,t,\tau}), \quad (9.1)$$

where $B(0, \tau)$ is the τ year zero coupon bond price, $\boldsymbol{\mu}_{x^+,t,\tau}$ is the first τ elements of $\boldsymbol{\mu}_{x^+,t,\mathcal{T}}$, ${}_{\tau}p_{x,t}(\boldsymbol{\mu}_{x^+,t,\tau})$ denotes the survival probability given $\boldsymbol{\mu}_{x^+,t,\tau}$ which is random and $e_{x,t}$ (for $e_{\tilde{x},t}$) is the random future survival time or life expectancy for age group x starting at age $\tilde{x}$.

According to the definition of annuity $a(t, \mathcal{T}; x)$ in Equation (9.1), we can now obtain a relationship $a(t, \mathcal{T}; x) = \varrho_a(\boldsymbol{\mu}_{x^+,t,\mathcal{T}})$ between $a(t, \mathcal{T}; x) \in \mathbb{R}^+$ and mortality rates $\boldsymbol{\mu}_{x^+,t,\mathcal{T}} \in \mathbb{R}^{+\mathcal{T}}$ directly with $\varrho_a : \mathbb{R}^{+\mathcal{T}} \rightarrow \mathbb{R}^+$. We note the posterior distribution of annuity $f_a(a(t, \mathcal{T}; x) | \mathcal{F}_{1:T})$ is then obtained by the posterior predictive distribution of mortality derived in Equation (7.8). The distribution summaries for the distribution of annuity price for age group x , such as the marginal mean and variance, can be calculated as below:

$$\begin{aligned} \mathbb{E}(a(t, \mathcal{T}; x)) &= \int a(t, \mathcal{T}; x) f_a(a(t, \mathcal{T}; x)) da(t, \mathcal{T}; x), \\ \text{Var}(a(t, \mathcal{T}; x)) &= \int (a(t, \mathcal{T}; x) - \mathbb{E}(a(t, \mathcal{T}; x)))^2 f_a(a(t, \mathcal{T}; x)) da(t, \mathcal{T}; x). \end{aligned}$$

In practice, this procedure can be approximated by the Monte Carlo estimator, constructed from posterior samples according to Algorithm (2). Denuit and Dhaene (2007) showed that some bounds of ${}_{\tau}p_{x,t}(\boldsymbol{\mu}_{x^+,t,\mathcal{T}})$ can be computed analytically. Biffis (2005) evaluated annuity prices allowing for longevity risk using financial theory. From an annuity provider perspective, what is important, however, is that the annuity price is a random quantity depending on the random paths of $\boldsymbol{\mu}_{x^+,t,\mathcal{T}}$. Moreover, it is important to determine a survival curve ${}_{\tau}p_{x,t}$ (as a function of τ at year t) in Equation (9.1) that best captures the mortality experience of the portfolio for risk management purposes (Fung et al., 2015). Figure 9.2 plots the survival curves for Australian, U.K. and U.S. policyholders aged 65 with maturity $\mathcal{T} = 20$ years. The performance of LMLM model is much better than MELCC model because the survival curve calculated by LMLM

Algorithm 2 Compute posterior distribution for annuity price $f_a(a(t, \mathcal{T}; x) | \mathcal{F}_{1:T})$

- 1: Draw L posterior samples $\boldsymbol{\mu}_{x^+, t, \mathcal{T}}$ under the filtration $\mathcal{F}_{1:T}$ by setting $T = t$ and $m = \mathcal{T}$.
- 2: Calculate L samples of $a(t, \mathcal{T}; x)$ based on $\boldsymbol{\mu}_{x^+, t, \mathcal{T}}^{(l)}$ according to Equation (9.1) as the approximated predictive distribution.
- 3: Construct the kernel density estimation based on posterior distribution of the L samples $a^{(l)}(t, \mathcal{T}; x)$.
- 4: Calculate posterior summaries based on the approximated distribution of $a^{(l)}(t, \mathcal{T}; x)$

$$\hat{\mathbb{E}}(a(t, \mathcal{T}; x)) = \frac{1}{L} \sum_{l=1}^L a^{(l)}(t, \mathcal{T}; x),$$

$$\hat{\mathbb{V}}\text{ar}(a(t, \mathcal{T}; x)) = \frac{1}{L-1} \left[\sum_{l=1}^L (a^{(l)}(t, \mathcal{T}; x))^2 - L[\hat{\mathbb{E}}(a(t, \mathcal{T}; x))]^2 \right],$$

where L is the number of iterations after burn-in in each MCMC sampler run given the window of information $\mathcal{F}_{1:T}$ and $\boldsymbol{\mu}_{x^+, t, \mathcal{T}}^{(l)}$ is the l -th draw in the posterior sample of $\boldsymbol{\mu}_{x^+, t, \mathcal{T}}$.

model is closer to the survival curve calculated based on the observed mortality rates from HMD.

9.3.2 Annuity Pricing Under Constant Interest Rate

We evaluate different quantiles of the annuity price $a(t, \mathcal{T}; x)$ setting the forecast period m to be the maturity period $\mathcal{T}$ based on the corresponding survival curves. For computation convenience, we assume the mortality rates within each age groups are the same. For example, for age group 65-69, we assume $\mu_{65,t} = \mu_{66,t} = \mu_{67,t} = \mu_{68,t} = \mu_{69,t}$. To facilitate comparison, Table 9.1 reports the mean, 2.5% quantile and 97.5% quantile of annuity prices by countries and genders calculated using MELCC and LMLM models assuming a constant interest rate $r = 4.7\%$ which is the average interest rate of London Interbank Offered Rate (LIBOR) based on British Pound from 1994 to 2014 and a τ year zero coupon bond price $B(0, \tau) = e^{-r\tau}$. The annuity prices $a(t, 20; 65)$ condition on $\mathcal{F}_{1:t}$ where t corresponds to the starting year of 1994 are calculated by the forecasted mortality rates $\boldsymbol{\mu}_{65^+, t, 20}$ at age 65 for different maturities $\mathcal{T} = 5, 10, 15$ and 20. Although the price difference might appear to be overall small, mispricing can be a significant risk when considering a large annuity portfolio. For an annuity

portfolio consisting of N policies where the benefit per year is $\mathcal{B}$, an under-pricing of $\mathcal{D}\% = (\text{observed} - \text{estimated})/\text{observed}$ of the correct annuity price will result in a shortfall of $N\mathcal{B}a(t, \mathcal{T}; x)\mathcal{D}/100$ where $a(t, \mathcal{T}; x)$ is the observed annuity price being charged with benefit \$1 per year calculated based on the mortality rates from HMD. As a demonstration, $N = 10,000$ policies written to U.K. male policyholders aged 65 with maturity $\mathcal{T} = 15$ years and $\mathcal{B} = \$20,000$ benefit per year will result in a shortfall of \$118,400,000 (observed and estimated $a(t, 15; 65)$ being 8.687 and 8.095 respectively in Table 9.1) if they apply MELCC model and a shortfall of \$51,600,000 (estimated $a(t, 15; 65)$ being 8.429) if they apply LMLM model. Consequently, \$66,800,000 amount of shortfall can be avoided if we apply LMLM model.

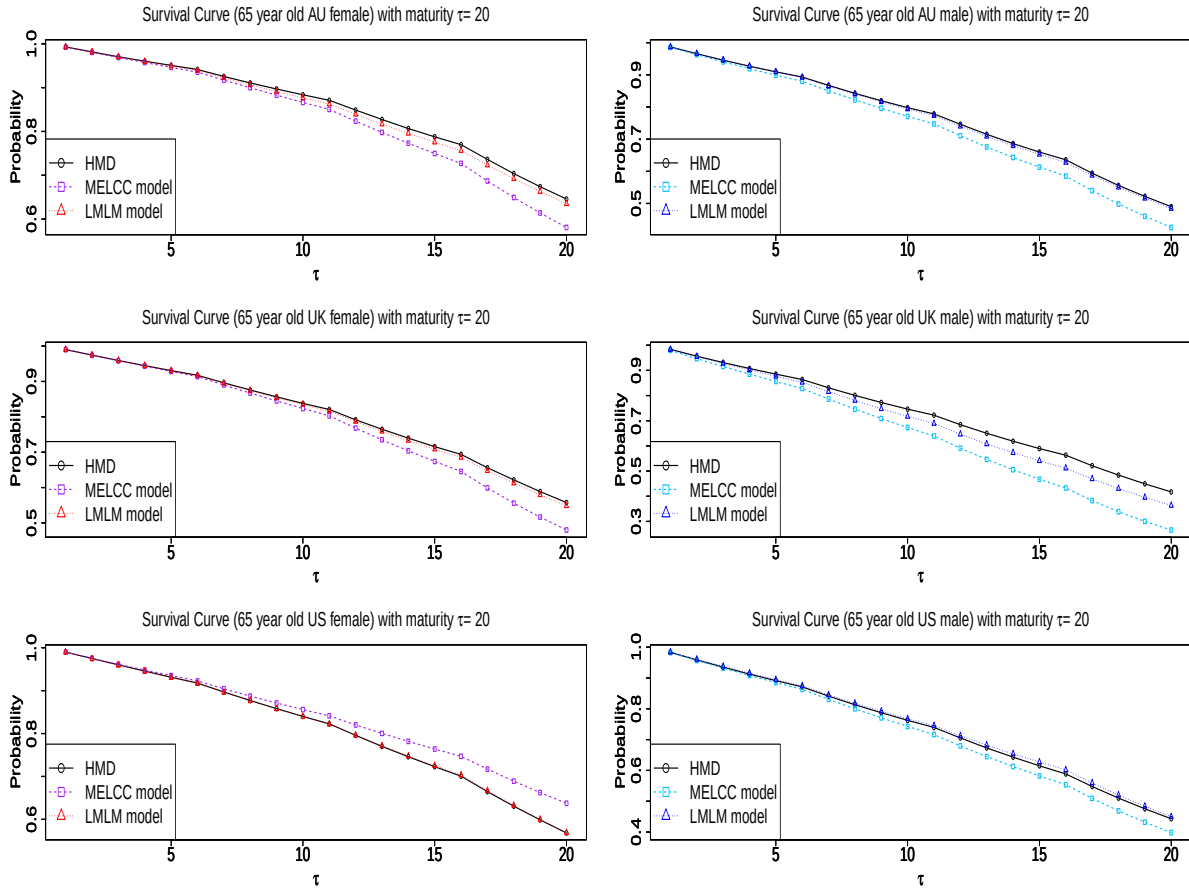


Figure 9.2. Forecasted survival curves by LMLM (red for female and blue for male) and MELCC (purple for female and skyblue for male) models for Australian, U.K. and U.S. policyholders aged 65 with maturity $\mathcal{T} = 20$ years from 1994 to 2014.

Table 9.1. Forecasted pricing of annuities by MELCC and LMLM models for Australia, U.K. and U.S. policyholders aged 65 with different maturity time $\mathcal{T}$ from 1994 to 2014. HMD stands for the annuities and the survival curve calculated by the mortality rates from HMD data set.

		mean	2.50%	97.50%	mean	2.50%	97.50%
model	maturity	Australia female			Australia male		
HMD	$\mathcal{T} = 5$	4.231	NA	NA	4.128	NA	NA
MELCC		4.222	4.216	4.240	4.105	4.094	4.137
LMLM		4.229	4.202	4.253	4.131	4.081	4.174
HMD	$\mathcal{T} = 10$	7.373	NA	NA	7.040	NA	NA
MELCC		7.325	7.300	7.398	6.949	6.911	7.058
LMLM		7.356	7.251	7.447	7.033	6.845	7.199
HMD	$\mathcal{T} = 15$	9.631	NA	NA	8.999	NA	NA
MELCC		9.505	9.446	9.675	8.801	8.726	9.022
LMLM		9.586	9.350	9.790	8.975	8.580	9.328
HMD	$\mathcal{T} = 20$	11.155	NA	NA	10.209	NA	NA
MELCC		10.914	10.811	11.217	9.888	9.772	10.237
LMLM		11.085	10.675	11.443	10.171	9.545	10.750
model	maturity	U.K. female			U.K. male		
HMD	$\mathcal{T} = 5$	4.189	NA	NA	4.078	NA	NA
MELCC		4.179	4.172	4.199	4.001	3.988	4.038
LMLM		4.185	4.141	4.221	4.053	3.978	4.118
HMD	$\mathcal{T} = 10$	7.224	NA	NA	6.873	NA	NA
MELCC		7.172	7.145	7.253	6.588	6.546	6.710
LMLM		7.202	7.032	7.344	6.756	6.473	7.001
HMD	$\mathcal{T} = 15$	9.334	NA	NA	8.687	NA	NA
MELCC		9.185	9.124	9.367	8.095	8.016	8.329
LMLM		9.278	8.903	9.596	8.429	7.853	8.942
HMD	$\mathcal{T} = 20$	10.701	NA	NA	9.768	NA	NA
MELCC		10.396	10.293	10.708	8.843	8.728	9.193
LMLM		10.607	9.973	11.159	9.370	8.499	10.191
model	maturity	U.S. female			U.S. male		
HMD	$\mathcal{T} = 5$	4.185	NA	NA	4.083	NA	NA
MELCC		4.192	4.187	4.207	4.069	4.066	4.080
LMLM		4.182	4.155	4.207	4.093	4.044	4.136
HMD	$\mathcal{T} = 10$	7.212	NA	NA	6.896	NA	NA
MELCC		7.253	7.233	7.313	6.836	6.823	6.876
LMLM		7.207	7.102	7.300	6.922	6.743	7.078
HMD	$\mathcal{T} = 15$	9.317	NA	NA	8.740	NA	NA
MELCC		9.438	9.395	9.570	8.605	8.576	8.689
LMLM		9.314	9.088	9.517	8.790	8.429	9.111
HMD	$\mathcal{T} = 20$	10.684	NA	NA	9.851	NA	NA
MELCC		10.929	10.856	11.152	9.629	9.582	9.767
LMLM		10.685	10.306	11.032	9.922	9.363	10.432

Table 9.2 reports the 20 year ahead forecast of annuity prices for Australia, U.K. and U.S. policyholders at age 65 from 2014 to 2034. In this study, we adopt a constant interest rate $r = 3\%$ because, currently, the interest rates show a decreasing trend.

Table 9.2. Comparison of forecasted pricing of annuities using MELCC and LMLM models for Australia ,U.K. and U.S. policyholders aged 65 with different maturity time $\mathcal{T}$ from 2014 to 2034.

		mean	2.50%	97.50%	mean	2.50%	97.50%
model	maturity	Australia female			Australia male		
MELCC	$\mathcal{T} = 5$	4.494	4.502	4.491	4.442	4.460	4.435
LMLM		4.495	4.509	4.478	4.442	4.465	4.415
MELCC	$\mathcal{T} = 10$	8.212	8.244	8.200	8.014	8.083	7.987
LMLM		8.209	8.271	8.137	8.023	8.119	7.911
MELCC	$\mathcal{T} = 15$	11.225	11.298	11.198	10.772	10.930	10.712
LMLM		11.205	11.358	11.025	10.827	11.053	10.562
MELCC	$\mathcal{T} = 20$	13.564	13.700	13.515	12.754	13.039	12.649
LMLM		13.514	13.814	13.159	12.918	13.340	12.431
model	maturity	U.K. female			U.K. male		
MELCC	$\mathcal{T} = 5$	4.456	4.471	4.450	4.405	4.426	4.397
LMLM		4.466	4.489	4.438	4.409	4.444	4.368
MELCC	$\mathcal{T} = 10$	8.060	8.117	8.039	7.868	7.953	7.837
LMLM		8.101	8.200	7.982	7.901	8.045	7.729
MELCC	$\mathcal{T} = 15$	10.872	10.999	10.825	10.451	10.642	10.381
LMLM		10.972	11.211	10.682	10.564	10.903	10.164
MELCC	$\mathcal{T} = 20$	12.929	13.154	12.846	12.205	12.545	12.082
LMLM		13.113	13.575	12.560	12.464	13.089	11.742
model	maturity	U.S. female			U.S. male		
MELCC	$\mathcal{T} = 5$	4.444	4.457	4.439	4.359	4.372	4.354
LMLM		4.445	4.464	4.423	4.370	4.399	4.336
MELCC	$\mathcal{T} = 10$	8.049	8.108	8.029	7.727	7.776	7.709
LMLM		8.031	8.115	7.939	7.777	7.897	7.639
MELCC	$\mathcal{T} = 15$	10.927	11.076	10.876	10.194	10.305	10.155
LMLM		10.832	11.034	10.608	10.314	10.593	9.997
MELCC	$\mathcal{T} = 20$	13.154	13.443	13.055	11.849	12.043	11.781
LMLM		12.895	13.281	12.471	12.065	12.569	11.508

9.4 Assessing Mispricing in Annuity Pricing Under Stochastic Interest Rate

9.4.1 Annuities, Endowments and GAO formulation

In this section we consider two sources of stochasticity, the mortality and interest rates used for discounting cash flows and assess a different class of product, known as GAO which represents one of several possible mortality-related guarantees that are increasingly introduced to insurance markets. A GAO is a contract that gives policyholders the right to convert their survival benefit into an annuity at a pre-specified conversion rate. In this section, we give the formulation of these insurance products.

Theorem 9.4.1 (Zero-coupon bond). *The risk-neutral price of a zero-coupon bond at any instant t with maturity $\mathcal{T}^*$ is given by*

$$\begin{aligned} B(t, \mathcal{T}^*) &= \mathbb{E}^Q \left[e^{-\int_t^{t+\mathcal{T}^*} r_u du} | \mathcal{F}_t^J \right] = e^{-A(\mathcal{T}^*)r_t + D(\mathcal{T}^*)}, \\ A(\mathcal{T}^*) &= \frac{1 - e^{-c_1 \mathcal{T}^*}}{c_1}, \end{aligned} \quad (9.2)$$

$$D(\mathcal{T}^*) = \left(c_2 - \frac{c_3^2}{2c_1^2} \right) [A(\mathcal{T}^*) - \mathcal{T}^*] - \frac{c_3^2 A(\mathcal{T}^*)^2}{4c_1}. \quad (9.3)$$

The proofs are given in [Mamon \(2004\)](#).

Let $M(t, \mathcal{T}^*; x)$ be the fair value at time t of a unit pure endowment payment at time $t + \mathcal{T}^*$, provided that a person ages x is alive at time t . The pure endowment is an insurance contract promising to pay the insured a stated sum if he survives a specified period with nothing payable in case of prior death.

Definition 9.4.1 (Pure endowment). *Under the risk-neutral case, the fair price of pure endowment with random survival time $e_{x,t}$ for age group x at current time t is given by*

$$\begin{aligned} \mathcal{M}(t, \mathcal{T}^*; x) &= \mathbb{E}^Q \left[e^{-\int_t^{t+\mathcal{T}^*} r_u du} \cdot \mathbb{1}_{[e_{x,t} \geq \mathcal{T}^*]} | \mathcal{F}_t^J \right] \\ &= \mathbb{1}_{[e_{x,t} \geq t]} \mathbb{E}^Q \left[e^{-\int_t^{t+\mathcal{T}^*} r_u du - \int_t^{t+\mathcal{T}^*} \mu_{x,v} dv} | \mathcal{F}_t^J \right] \\ &= \mathbb{1}_{[e_{x,t} \geq t]} M(t, \mathcal{T}^*; x). \end{aligned}$$

A life annuity $\mathcal{A}(t, \infty; x)$ is simply a linear combination of pure endowments, giving its fair value (Liu et al., 2013).

Definition 9.4.2 (Life annuity). *Let $\mathcal{A}(t, \infty; x)$ be the risk-neutral value, evaluated at time t , of a life annuity contract paying \$1 to the insured aged x at time t annually conditional on his/her survival at the moment of payments, provided that the people aged x is alive at the time of valuation. Hence, we have*

$$\begin{aligned}\mathcal{A}(t, \infty; x) &= \sum_{\tau=0}^{\infty} \mathbb{E}^Q \left[e^{-\int_t^{t+\tau} r_u du - \int_t^{t+\tau} \mu_{x,v} dv} | \mathcal{F}_{1:t-1}^J \right] \\ &= \sum_{\tau=0}^{\infty} M(t, \tau; x).\end{aligned}\tag{9.4}$$

Remark 9.4.1. *In practice, $\mathcal{A}(t, \infty; x)$ is a finite summation because $\mathbb{1}_{[e_{x,t} > \tau]}$ will eventually always be 0 as τ tends to infinity.*

We note that Equation (9.1) is the special case of Equation (9.4) when the period of payment is $\mathcal{T}$ instead of ∞ , the short rate process r_t is a fixed value independent of the force of mortality rate process $\mu_{x,t}$ and the mortality rate process $\mu_{x,t}$ is discretised into a mortality rates path $\boldsymbol{\mu}_{x^+,t,\mathcal{T}} = (\mu_{x,t}, \mu_{x+1,t+1}, \dots, \mu_{x+\mathcal{T},t+\mathcal{T}})$. Furthermore, the mortality rate process $\mu_{x,t}$ in Equation (9.1) is calculated by our proposed LMLM and MELCC model whereas the mortality rate process $\mu_{x,t}$ in Equation (9.4) is defined using Equation (9.8) which is a relatively simple model. If we assume independence between the fixed short rate r and the mortality rates path $\boldsymbol{\mu}_{x^+,t,\mathcal{T}}$, Equation (9.4) with maturity $\mathcal{T}$ year can be represented as

$$\begin{aligned}\mathcal{A}(t, \mathcal{T}; x) &= \sum_{\tau=1}^{\mathcal{T}} \mathbb{E}^Q \left[e^{-r\tau} \cdot \mathbb{1}_{[e_{x,t} \geq \tau]} | \mathcal{F}_{1:t-1}^r, \mathcal{F}_{1:t-1}^{\mu} \right] \\ &= \sum_{\tau=1}^{\mathcal{T}} \mathbb{E}^Q \left[e^{-r\tau} | \mathcal{F}_{1:t-1}^r \right] \mathbb{E}^Q \left[\mathbb{1}_{[e_{x,t} \geq \tau]} | \mathcal{F}_{1:t-1}^{\mu} \right] \\ &= \sum_{\tau=1}^{\mathcal{T}} B(0, \tau) \mathbb{E}(\mathbb{1}_{[e_{x,t} \geq \tau]} | \boldsymbol{\mu}_{x^+,t,\tau}) = a(t, \mathcal{T}; x).\end{aligned}$$

Furthermore, a call option $C(t, \mathcal{T}^*)$ can be used for the valuation of a particular type of GAO.

Definition 9.4.3 (Call option). *The time t risk-neutral value of a call option to acquire a life annuity at time $t + \mathcal{T}^*$ with strike price $\mathcal{K}$ can be calculated from*

$$\begin{aligned}\mathcal{C}(t, \mathcal{T}^*; x) &= \mathbb{E}^Q \left[e^{-\int_t^{t+\mathcal{T}^*} r_u du} \cdot \mathbb{1}_{[e_x, t \geq \mathcal{T}^*]} (\mathcal{A}(t, \infty; x) - \mathcal{K})^+ | \mathcal{F}_{1:t-1}^J \right] \\ &= \mathbb{1}_{[e_x, t \geq 0]} \mathbb{E}^Q \left[e^{-\int_t^{t+\mathcal{T}^*} r_u du - \int_t^{t+\mathcal{T}^*} \mu_{x,v} dv} (\mathcal{A}(t, \infty; x) - \mathcal{K})^+ | \mathcal{F}_{1:t-1}^J \right] \\ &= \mathbb{1}_{[e_x, t \geq 0]} C(t, \mathcal{T}^*; x).\end{aligned}$$

We note that this starting time t can be different from the starting time in annuity for different types of GAO as long as it is within the annuity payment period. For illustrative purpose, we assume both starting times to be consistent but this assumption can be easily generalised as long as suitable periods for mortality rates are considered in the calculation of annuity and call option. In order to calculate the price of GAO, the guaranteed conversion rate $\mathcal{G}$ needs to be defined. If an annuity is sold at a rate higher than $1/\mathcal{G}$ in the primary market, the GAO is of positive value; otherwise, the GAO is valueless.

Definition 9.4.4 (GAO). *Letting $\mathcal{K} = 1/\mathcal{G}$, the price for GAO at time t can be expressed as*

$$\begin{aligned}P_{GAO}(t, \infty, \mathcal{T}; x) &= \mathbb{E}^Q \left[e^{-\int_t^{t+\mathcal{T}^*} r_u du} e^{-\int_t^{t+\mathcal{T}^*} \mu_{x,v} dv} (\mathcal{G}\mathcal{A}(t, \infty; x) - 1)^+ | \mathcal{F}_{1:t-1}^J \right] \\ &= \mathcal{G} \mathbb{E}^Q \left[e^{-\int_t^{t+\mathcal{T}^*} r_u du} e^{-\int_t^{t+\mathcal{T}^*} \mu_{x,v} dv} \left(\mathcal{A}(t, \infty; x) - \frac{1}{\mathcal{G}} \right)^+ | \mathcal{F}_{1:t-1}^J \right] = \mathcal{G} C(t, \mathcal{T}^*; x).\end{aligned}$$

In order to obtain a valuation formulae, [Liu et al. \(2013\)](#) applied a well-developed change of numeraire technique to derive a more tractable valuation formulae for mortality-dependent GAO. The pure endowments $M(t, \mathcal{T}^*; x)$ can be represented as

$$\begin{aligned}M(t, \mathcal{T}^*; x) &= \exp \left\{ - \left[A(\mathcal{T}^*) r_t + \tilde{G}(\mathcal{T}^*) \mu_{x,t} \right] + D(\mathcal{T}^*) + \left[\frac{\rho c_3 c_5}{c_1 c_4} - \frac{(c_5)^2}{2(c_4)^2} \right] [\tilde{G}(\mathcal{T}^*) - \mathcal{T}^*] \right. \\ &\quad \left. + \frac{\rho c_3 c_5}{c_1 c_4} \left[A(\mathcal{T}^*) - \frac{1 - \exp(-(c_1 - c_4)\mathcal{T}^*)}{c_1 - c_4} \right] + \frac{(c_5)^2}{4c_4} (\tilde{G}(\mathcal{T}^*))^2 \right\},\end{aligned}\quad (9.5)$$

where $\tilde{G}(\mathcal{T}^*) = \frac{\exp(c_4 \mathcal{T}^*) - 1}{c_4}$, $A(\mathcal{T}^*)$ is defined in Equation (9.2) and $D(\mathcal{T}^*)$ is defined in Equation (9.3).

The upper limit of the sum for the annuity in Equation (9.4) is set to $\mathcal{T} = 20$ instead of infinity for computational simplicity, which means the annuity payments start from

age 65 and cease at age 85 or the time of death whichever occurs earlier. With a similar change of numeraire argument, a valuation formulae (Liu et al., 2013) can be given by

$$C(t, \mathcal{T}^*; x) = M(t, \mathcal{T}^*; x) \mathbb{E}^{\hat{Q}} \left[(\mathcal{A}(t, \mathcal{T}; x) - \mathcal{K})^+ | \mathcal{F}_{1:t-1}^J \right],$$

where the new measure $\hat{Q}$ is associated with the pure endowment $\mathcal{M}(t, \mathcal{T}^*; x)$ as the numeraire and defined via the Radon-Nikodym derivative

$$\frac{d\hat{Q}}{dQ} := \frac{e^{-\int_t^{t+\mathcal{T}^*} r_u du} \mathcal{M}(t + \mathcal{T}^*, 0; x)}{M(t, \mathcal{T}^*; x)}.$$

Under measure $\hat{Q}$, $(r_t, \mu_{x,t})$ is a bivariate normal random variable. Hence, a valuation formulae can be represented as

$$P_{GAO}(t, \mathcal{T}, \mathcal{T}^*; x) = \mathcal{G}M(t, \mathcal{T}^*; x) \mathbb{E}^{\hat{Q}} \left[\left(\mathcal{A}(t, \mathcal{T}; x) - \frac{1}{\mathcal{G}} \right)^+ | \mathcal{F}_{1:t-1}^J \right]. \quad (9.6)$$

9.4.2 Interest Rates Model and Discretisation

The framework of GAO studied by Milevsky and Promislow (2001), Dahl (2004), Biffis (2005) and Cairns et al. (2006a) is under the assumption of independence of mortality evolution and financial market. In this study, we adopt the assumption proposed by Liu et al. (2013). There exists a GAO under the continuous time representation with dependence structure between the dynamics of mortality $\mu_{x,t}$ and interest rates r_t . Liu et al. (2013) stated the reason that demographic factor interacts with economy and has financial impact, whether it is caused by longevity risk and its associated financial burden or catastrophic events such as those massive earthquakes in Kobe (1995), China (2008) and Japan (2011). We will further extend Liu et al. (2013) study to discrete case in order to incorporate LMLM model. The generalised modelling framework of GAO bases on the short rate process r_t and the force of mortality rate process $\mu_{x,t}$ with the filtered probability space $(\Omega, \mathcal{F}_{1:t-1}^J, Q)$ where Ω is the sample space, Q is a risk-neutral measure and $\mathcal{F}_{1:t-1}^J$ is the joint filtration generated by r_t and $\mu_{x,t}$. Under a risk-neutral measure Q , we assume that r_t follows the well-known Vasicek model (Vasicek, 1977) and the force of mortality $\mu_{x,t}$ follows a diffusion model. The Vasicek model is commonly used to describe the evolution of interest rates and it regards the short rate as a Ornstein-Uhlenbeck process exhibiting mean reversion. The Vasicek

model for interest rates and the mortality model are given by

$$dr_t = c_1(c_2 - r_t)dt + c_3dW_t^r, \quad (9.7)$$

$$d\mu_{x,t} = c_4\mu_{x,t}dt + c_5dW_t^\mu, \quad (9.8)$$

where c_1, c_2, c_3, c_4 and c_5 are positive constants. The direction of movement of short rate is controlled by its long-term mean c_2 . When the short rate moves above c_2 ($r_t > c_2$), the short rate drift becomes negative and the short rate r_t will be pulled downward. Similarly, when the short rate moves below c_2 , r_t will be pulled up. The parameter c_1 determines the speed at which the drift is pulled. W_t^r and W_t^μ are all standard Brownian motion with mean zero and variance t . In this section, we assume they hold a dependent relation

$$dW_t^r dW_t^\mu = \rho dt \text{ implies } W_t^\mu = \rho W_t^r + \sqrt{1 - \rho^2} W_t,$$

where W_t is a standard Brownian motion independent of W_t^r . Hence, the joint filtration $\mathcal{F}_{1:t-1}^J = \mathcal{F}_{1:t-1}^r \vee \mathcal{F}_{1:t-1}^\mu$ is also generated by $\mathcal{F}_{1:t-1}^{W^r} \vee \mathcal{F}_{1:t-1}^{W^\mu}$.

Remark 9.4.2. *We assume there is an association between short rate process r_t and the force of mortality rate process $\mu_{x,t}$. Furthermore, there may also be a statistical causality between them. This is a directional causality that manifests as a conditional dependence structure. The interest rates r_t can be influenced by extreme mortality events but the mortality rates $\mu_{x,t}$ will not be affected by changing of interest rates r_t . For example, earthquakes, terrorist attacks, epidemics etc. may produce massive mortality events and consequently, such events will also potentially influence financial markets and interest rates. However, the converse is unlikely to hold, that is an increase of interest rates will not lead to a rise in mortality rates. The quantification of this causality can be captured through a correlation ρ . Instead of modelling the explicit causation, we will model correlation ρ in our model.*

To evaluate annuity, pure endowment and GAO in Equations (9.4), (9.5) and (9.6) under the Vasicek model, we calibrate the short rate process r_t and the force of mortality rate process $\mu_{x,t}$. To do this, we need to discretise Equation (9.7) and Equation (9.8). To approximate the stochastic differential equation for the short rate by its first order Taylor approximation, we apply a Euler scheme to simulate the first-order discrete approximation to the continuous time short rate process r_t . We divide the time interval $[t, t + \mathcal{T}]$ into K equal intervals of length l such that $\mathcal{T} = Kl$. For example, if we set

$\mathcal{T} = 20$ with monthly short rate, we will have $l = 0.0833$ (1/12th of one year) and $K = 240$. As we will calibrate our models based on yearly rates, we have $l = 1$ and $K = 20$. In this case, the discretised time subscript k corresponds to the usual time subscript t for the yearly data we consider. Under the discrete time setting, the short rate process r_t can be represented as

$$\begin{aligned}\Delta r_k &= c_1(c_2 - r_k)l + c_3(W_{k+1}^r - W_k^r) \\ &= c_1(c_2 - r_k)l + c_3\sqrt{l}\varepsilon_{k+1}^r, \quad \varepsilon_k^r \stackrel{\text{i.i.d}}{\sim} N(0, 1).\end{aligned}$$

Hence, we have a recursive equation for r_t

$$\begin{aligned}r_{k+1} &= r_k + \Delta r_k \\ &= r_k(1 - c_1l) + c_1c_2l + c_3\sqrt{l}\varepsilon_{k+1}^r,\end{aligned}\tag{9.9}$$

then we obtain the equation for the short rate in any period k

$$r_k = r_0(1 - c_1l)^k + c_1c_2l \sum_{i=0}^{k-1} (1 - c_1l)^i + c_3\sqrt{l} \sum_{i=0}^{k-1} (1 - c_1l)^i \varepsilon_{k-i}^r \tag{9.10}$$

as a function of the short rate r_0 in period zero. Similarly, for mortality rate process $\mu_{x,t}$, we have

$$\begin{aligned}\Delta \mu_{x,k} &= c_4\mu_{x,k}l + c_5(W_{k+1}^\mu - W_k^\mu) \\ &= c_4\mu_{x,k}l + c_5\sqrt{l}\varepsilon_{k+1}^\mu, \quad \varepsilon_k^\mu \stackrel{\text{i.i.d}}{\sim} N(0, 1).\end{aligned}$$

Hence, we have a recursive equation for $\mu_{x,t}$

$$\begin{aligned}\mu_{x,k+1} &= \mu_{x,k} + \Delta \mu_{x,k} \\ &= \mu_{x,k}(1 + c_4l) + c_5\sqrt{l}\varepsilon_{k+1}^\mu.\end{aligned}\tag{9.11}$$

Then we obtain the equation for the short rate in any period k

$$\mu_{x,k} = \mu_{x,0}(1 + c_4l)^k + c_5\sqrt{l} \sum_{i=0}^{k-1} (1 + c_4l)^i \varepsilon_{k-i}^\mu \tag{9.12}$$

as a function of the mortality rate $\mu_{x,0}$ in period zero.

9.4.3 Annuities, Endowments and GAO: Model Risk from Mortality Component and Stochastic Interest Rate

To evaluate r_t and $\mu_{x,k}$ in Equation (9.10) and (9.12), we need to estimate parameters c_1 to c_5 . We consider two sources of stochasticity, the mortality and the interest rate used for discounting cash flows. We also set up independence models where we treat mortality process $\mu_{x,t}$ as independent of interest rate process r_t and dependence models where we treat the mortality process as dependent on the interest rate r_t . The dependence models can be defined in two ways: the first way is to calibrate the model marginally for r_t and $\mu_{x,t}$ and estimate the dependence parameter ρ separately using linear or rank correlation of the residuals for r_t and $\mu_{x,t}$. This method is akin to the approach known in statistics as Inference for Margins, which is often adopted when fitting copula models in practice, see discussions in Joe and Xu (1996). The second way is to adopt a joint model for r_t and $\mu_{x,t}$.

We begin by explaining briefly the Bayesian model adopted for the interest rate calibration. To estimate the unknown parameters $\boldsymbol{\vartheta}_c = (c_1, c_2, c_3, c_4, c_5, \rho)$ in the joint dependence structure between the mortality rate $\mu_{x,t}$ and interest rate r_t , we assume that $(\varepsilon_t^r, \varepsilon_t^\mu)$ in Equations (9.9) and (9.11) follow a bivariate normal and so the complete data likelihood function for the joint model of r_k and $\mu_{x,k}$ can be represented as

$$\begin{aligned}
 f(r_{1:K}, \mu_{x,1:K} | \boldsymbol{\vartheta}_c) &= \prod_{k=1}^K \frac{1}{2\pi\sqrt{1-\rho^2}c_5c_3l} \exp \left[-\frac{1}{2(1-\rho^2)} \left(\frac{[\mu_{x,k} - \mu_{x,k-1}(1+c_4l)]^2}{c_5^2l} \right. \right. \\
 &\quad \left. \left. - 2\rho \frac{[\mu_{x,k} - \mu_{x,k-1}(1+c_4l)][r_k - r_{k-1}(1-c_1l) - c_1c_2l]}{c_5c_3l} + \frac{[r_k - r_{k-1}(1-c_1l) - c_1c_2l]^2}{c_3^2l} \right) \right] \\
 &= \prod_{k=1}^K \frac{1}{2\pi\sqrt{1-\rho^2}c_5c_3l \sum_{i=0}^{k-1} (1-c_1l)^i \sum_{i=0}^{k-1} (1+c_4l)^i} \exp \left[-\frac{1}{2(1-\rho^2)} \left(\frac{[\mu_{x,k} - \mu_{x,0}(1+c_4l)^k]^2}{c_5^2l \left(\sum_{i=0}^{k-1} (1+c_4l)^i \right)^2} \right. \right. \\
 &\quad \left. \left. - 2\rho \frac{[\mu_{x,k} - \mu_{x,0}(1+c_4l)^k][r_k - r_0(1-c_1l)^k - c_1c_2l \sum_{i=0}^{k-1} (1-c_1l)^i]}{c_5c_3l \sum_{i=0}^{k-1} (1-c_1l)^i \sum_{i=0}^{k-1} (1+c_4l)^i} + \right. \right. \\
 &\quad \left. \left. \frac{\left(r_k - r_0(1-c_1l)^k - c_1c_2l \sum_{i=0}^{k-1} (1-c_1l)^i \right)^2}{c_3^2l \left(\sum_{i=0}^{k-1} (1-c_1l)^i \right)^2} \right) \right]. \tag{9.13}
 \end{aligned}$$

where r_k are obtained from LIBOR and $\mu_{x,k}$ are downloaded from HMD, estimated from model MELCC and from model LMLM respectively. In our application with $x = 65$ and $k = 0$ to represent year 1994, $\mu_{65,0}$ are 0.0162, 0.0203 and 0.0167 for the three cases respectively.

Remark 9.4.3. *For independence case with $\rho = 0$ and the marginal case with linear or rank correlation of the residuals for r_t and $\mu_{x,t}$ and the model structure (9.9) with Gaussian distribution, the complete data likelihood function for interest rates r_k can be simplified to following form*

$$f(r_{0:K}|c_1, c_2, c_3) = \prod_{k=1}^K \frac{1}{\sqrt{2\pi c_3^2 l}} \exp \left[-\frac{1}{2} \frac{[r_k - r_{k-1}(1 - c_1 l) - c_1 c_2 l]^2}{c_3^2 l} \right]. \quad (9.14)$$

Similarly, for mortality rates μ_k , the model structure (9.11) can be simplified to following form

$$f(\mu_{x,0:K}|c_4, c_5) = \prod_{k=1}^K \frac{1}{\sqrt{2\pi c_5^2 l}} \exp \left[-\frac{1}{2} \frac{(\mu_{x,k} - \mu_{x,k-1}(1 + c_4 l))^2}{c_5^2 l} \right]. \quad (9.15)$$

The following priors $\pi(\boldsymbol{\vartheta}_c)$

$$c_1, c_2, c_3, c_4, c_5 \sim \Gamma(1, 1) \text{ and } \rho \sim U(-1, 1)$$

are adopted in this section. The choice of prior $\Gamma(1, 1)$ is arbitrary chosen and we check that other choices of prior does not change the parameter estimates substantially. We note that parameters $\boldsymbol{\vartheta}_c$ can be estimated jointly in Equation (9.13) or in two stages using marginal distributions in Equations (9.14) and (9.15) for the independence/linear or rank correlated cases (Joe and Xu, 1996). Accordingly, the evaluation of the prices for annuity, pure endowment and GAO can be summarised into Algorithms 3 and 4.

We consider an application of pricing annuity and GAO with the starting time $t = 0$ to represent year 1994, the maturity for call option $\mathcal{T}^* = 15$ and the maturity for annuity $\mathcal{T} = 20$ for UK males aged 65. For this group of insured in the 80s, per £1000 cash value can be turned into an annuity of £111 per annum, implying that $\mathcal{G} = 1/9$ (Bolton et al., 1997). We further adopt annual interest rates from LIBOR based on British Pound as the short rates and they are downloaded from (IBA) for the year 1994 to 2014. Observed mortality rates $\mu_{65+,0,20}$ (1994 to 2014) from HMD and forecasted mortality rates $\mu_{65+,0,20}$ by MELCC and LMLM models for male in U.K. are adopted. Figure 9.3 compares interest rates (left panel) and mortality rates (right

panel) estimated based on various models and assumptions. For the interest rates, we compare the starting interest rates from LIBOR (case 1) in step 1 of Algorithms 3 and 4 with the interest rates re-estimated in step 4 using independence/linear or rank correlated model (case 2; Algorithm 4) and jointly distributed model (case 3; Algorithm 3). We note that the interest rates are consistent across the three plots for LIBOR (case 1) and independence/linear or rank correlated model (case 2) but they vary under the jointly distributed model (case 3) because the interest rates are independent of mortality models unless the jointly distributed model in Equation (9.13) is used. For mortality rates, we plot again the starting mortality rates in step 2 from HMD, MELCC model and LMLM model (case 1 for the top, middle and bottom plots respectively) with the re-estimated mortality rates in step 4 using independence/linear or rank correlated model (case 2) and jointly distributed model (case 3).

Algorithm 3 Compute annuity price and GAO price based on joint distribution

- 1: Obtain interest rates r_t for the country issuing annuity from LIBOR in that country currency at designed maturity of discretion.
 - 2: Obtain mortality rates $\mu_{x,t}$ from HMD or mortality rate forecasts by MELCC and LMLM models for that country.
 - 3: Perform Monte Carlo sampling from posterior using Equation (9.13) and obtain estimated $\boldsymbol{\vartheta}_c$ with credible intervals.
 - 4: Re-estimate the discretised interest rate r_k and mortality rates $\mu_{x,k}$ using the discretised Vasicek models in Equations (9.10) and (9.12) respectively.
 - 5: Adopt estimated $\boldsymbol{\vartheta}_c$ and the set of $(r_k, \mu_{x,k})$ to evaluate annuity prices of Equation (9.4), pure endowment in Equation (9.5) and GAO in Equation (9.6).
-

Algorithm 4 Compute annuity price and GAO price based on independent distributions

- 1: Obtain interest rates r_t for the country issuing annuity from LIBOR in that country currency at designed maturity of discretion.
 - 2: Obtain mortality rates $\mu_{x,t}$ from HMD or mortality rate forecasts by MELCC and LMLM models for that country.
 - 3: Perform Monte Carlo sampling from posterior to estimate c_1, c_2 and c_3 with credible intervals using Equation (9.14) and estimate c_4 and c_5 with credible intervals using Equation (9.15).
 - 4: Re-estimate the interest rate r_k and mortality rates $\mu_{x,k}$ using the discretised Vasicek models in Equations (9.10) and (9.12) respectively.
 - 5: Set $\rho = 0$ or obtain the linear correlation or rank correlation ρ of the residuals.
 - 6: Adopt estimated $\boldsymbol{\vartheta}_c$ and the set of $(r_k, \mu_{x,k})$ to evaluate prices of annuity in Equation (9.4), pure endowment in Equation (9.5) and GAO in Equation (9.6).
-

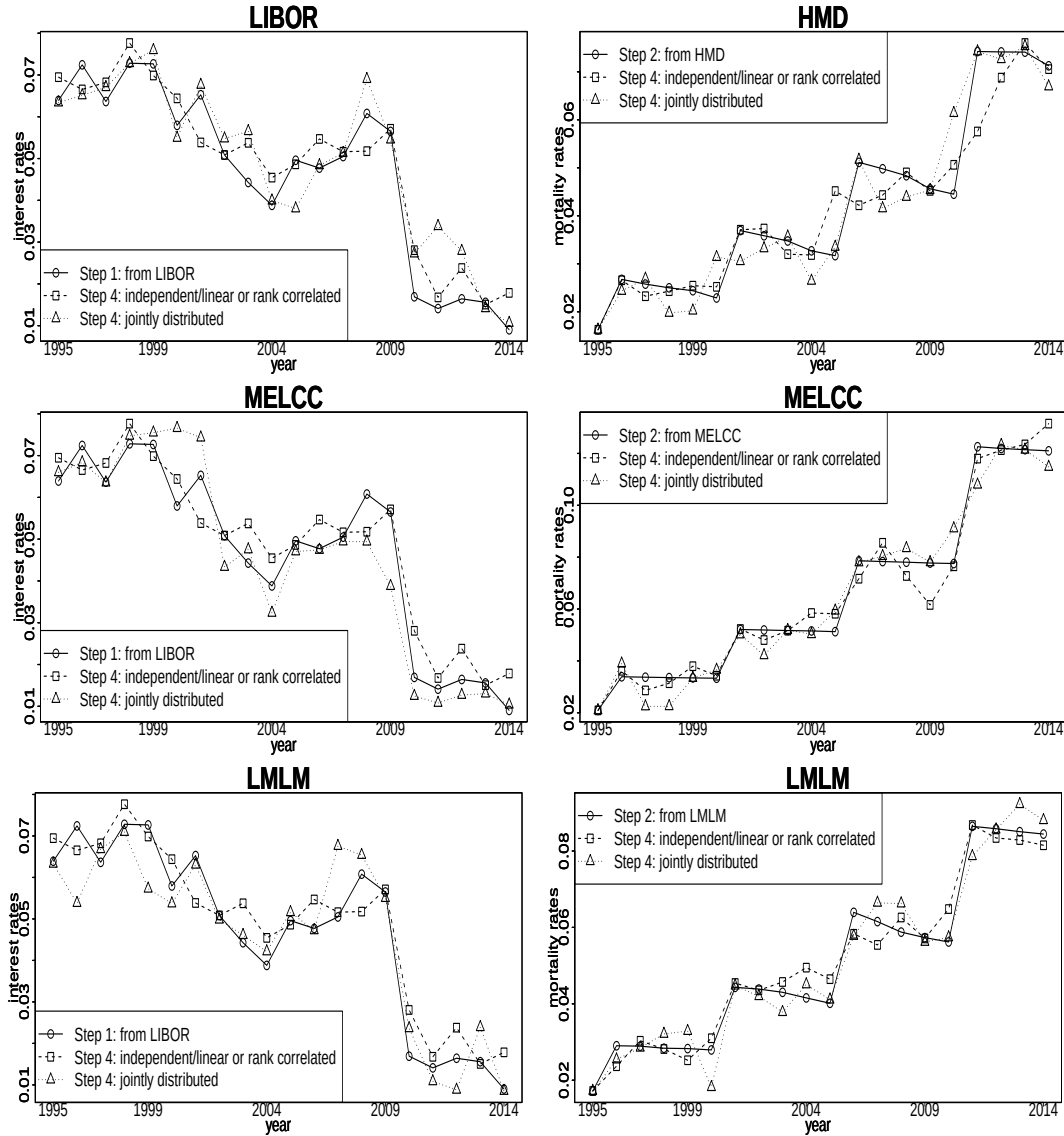


Figure 9.3. Interest rates (left hand panel) and mortality rates (right hand panel) at start (with the observed interest rates from LIBOR and mortality rates from HMD) and re-estimated based on both independent and joint distribution. The estimated mortality rates from MELCC and LMLM models are adopted as additional training sets.

Based on Algorithm (3) and Algorithm (4), we report in Table 9.3 parameter estimates ϑ_c for four different structures of correlation, namely independence between interest rates and mortality rates ($\rho_1 = 0$) and dependence between interest rates and mortality rates using marginal distribution with linear residuals correlation (ρ_2), using marginal distribution with rank residuals correlation (ρ_3) and using joint distribution (ρ_4). The

values of ρ_4 calculated from joint distribution show weak positive correlation between interest rates and mortality rates. For the dependence case with marginal distributions, the correlations based on linear residuals and rank residuals show some weak negative correlations. Furthermore, the prices of annuity, pure endowments and GAO are evaluated and their performances are compared between MELCC and LMLM models. The LMLM model performs much better than MELCC model because the LMLM model gives estimates which are closer to the values calculated using observed mortality rates. The price of annuity, pure endowments and GAO calculated based on joint distribution are higher than those from marginal distribution with separate correlation estimate.

Table 9.3. Parameter estimates and prices of annuity, pure endowment and GAO using Vasicek model for U.K. males aged 65 with maturities $\mathcal{T} = 20$ and $\mathcal{T}^* = 15$ from 1994 to 2014.

Dependency $\mu_{x,t}$ in step 2		Independence			Linearly correlated			Rank correlated			Jointly distributed		
		HMD	MELCC	LMLM	HMD	MELCC	LMLM	HMD	MELCC	LMLM	‡ HMD	MELCC	LMLM
$\dagger c_1$	2.5%	0.0080	0.0080	0.0080	0.0080	0.0080	0.0080	0.0080	0.0080	0.0080	0.0073	0.0074	0.0080
	mean	0.0908	0.0908	0.0908	0.0908	0.0908	0.0908	0.0908	0.0908	0.0908	0.1213	0.1251	0.1237
	97.5%	0.3471	0.3471	0.3471	0.3471	0.3471	0.3471	0.3471	0.3471	0.3471	0.3378	0.3440	0.3369
$\dagger c_2$	2.5%	0.0077	0.0077	0.0077	0.0077	0.0077	0.0077	0.0077	0.0077	0.0077	0.0070	0.0071	0.0072
	mean	0.0712	0.0712	0.0712	0.0712	0.0712	0.0712	0.0712	0.0712	0.0712	0.0490	0.0485	0.0487
	97.5%	0.1167	0.1167	0.1167	0.1167	0.1167	0.1167	0.1167	0.1167	0.1167	0.1261	0.1214	0.1211
$\dagger c_3$	2.5%	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0093	0.0095	0.0095	0.0095
	mean	0.0130	0.0130	0.0130	0.0130	0.0130	0.0130	0.0130	0.0130	0.0130	0.0132	0.0132	0.0133
	97.5%	0.0178	0.0178	0.0178	0.0178	0.0178	0.0178	0.0178	0.0178	0.0178	0.0190	0.0190	0.0191
c_4	2.5%	0.0038	0.0059	0.0041	0.0038	0.0059	0.0041	0.0038	0.0059	0.0041	0.0049	0.0074	0.0037
	mean	0.0594	0.0649	0.0578	0.0594	0.0649	0.0578	0.0594	0.0649	0.0578	0.0709	0.0761	0.0702
	97.5%	0.1473	0.1452	0.1383	0.1473	0.1452	0.1383	0.1473	0.1452	0.1383	0.1724	0.1704	0.1523
c_5	2.5%	0.0071	0.0100	0.0078	0.0071	0.0100	0.0078	0.0071	0.0100	0.0078	0.0074	0.0104	0.0061
	mean	0.0098	0.0137	0.0106	0.0098	0.0137	0.0106	0.0098	0.0137	0.0106	0.0103	0.0144	0.0096
	97.5%	0.0139	0.0192	0.0150	0.0139	0.0192	0.0150	0.0139	0.0192	0.0150	0.0150	0.0210	0.0126
ρ	2.5%	NA	NA	NA	NA	NA	NA	NA	NA	NA	-0.1990	-0.2441	-0.1870
	mean	0.0000	0.0000	0.0000	-0.0578	-0.0337	0.1118	-0.1895	-0.2120	0.0767	0.1629	0.1325	0.1735
	97.5%	NA	NA	NA	NA	NA	NA	NA	NA	NA	0.5256	0.5059	0.5347
$\# \mathcal{A}$	2.5%	6.927	6.632	6.840	6.921	6.646	6.836	6.907	6.641	6.817	6.515	6.411	6.869
	mean	10.84	10.31	10.78	10.82	10.35	10.76	10.78	10.34	10.70	11.69	10.98	11.52
	97.5%	15.21	14.86	15.18	15.19	14.92	15.16	15.13	14.90	15.08	15.00	14.61	15.00
$\# M$	2.5%	0.1628	0.2656	0.1731	0.1611	0.2732	0.1721	0.1572	0.2708	0.1664	0.2341	0.7346	0.1443
	mean	0.2695	0.2698	0.2725	0.2675	0.2755	0.2713	0.2630	0.2737	0.2647	0.3775	0.3905	0.3682
	97.5%	0.7466	0.7196	0.7458	0.7435	0.7279	0.7438	0.7364	0.7253	0.7333	0.7452	0.7092	0.7342
$\# P_{GAO}$	2.5%	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	mean	0.0549	0.0390	0.0534	0.0539	0.0411	0.0529	0.0518	0.0405	0.0498	0.1125	0.0856	0.1028
	97.5%	0.5140	0.4674	0.5108	0.5098	0.4779	0.5081	0.5005	0.4746	0.4944	0.4958	0.4408	0.4886

$\# \mathcal{A}$ represents $\mathcal{A}(0, 20; 65)$, M for $M(0, 15; 65)$ and P_{GAO} for $P_{GAO}(0, 20, 15; 65)$.

‡ bench mark model for performance comparison.

With the bench mark model using joint distribution and mortality rates from HMD, the first row in Table 9.4 shows the absolute differences $\mathcal{A}_\Delta$ between annuity prices calculated by the bench mark model and other models. We adopt $N = 10,000$ policies written to U.K. male policyholders aged 65 with $\mathcal{T} = 20$ years and $\mathcal{B} = \$20,000$ to calculate the annuity price short falls. The second row in Table 9.4 shows the shortfall (SF) caused by mispricing in million pounds.

In this study, under the assumption that the mortality rates and interest rates are correlated, we adopt the value calculated used HMD mortality rates with joint distribution with LIBOR interest rates. Comparing the SF calculated from fixed and stochastic interest rates, the improvement is significant. Figure 9.3 also shows that there is a clear downwards trend in interest rates. It is inappropriate to ignore this trend in calculation. By applying LMLM model rather than MELCC model, the SF can be decreased under both dependence case with marginal distributions and joint distribution case. For the dependence case with marginal distributions and different the correlations, the SF are similar across independent, linear correlated and rank correlated cases. However, the SF can be dramatically reduced if we adopt joint distribution in our model assumption. According to the SF values in Table 9.4, adopting stochastic interest rates considerably lowers the amount of SF. The SF can also be lowered by applying LMLM model instead of MELCC model. Lastly, the SF will further decrease if the joint dependency structure is adopted.

Table 9.4. Comparison of annuity prices calculated using the twelve models with stochastic interest rates and the three models with fixed interest rates for U.K. male aged 65 with maturity $\mathcal{T} = 20$ starting from 1994.

	Independent			Linear correlated			Rank correlated			Joint distributed			Fixed r_t		
	HMD	MELCC	LMLM	HMD	MELCC	LMLM	HMD	MELCC	LMLM	HMD	MELCC	LMLM	HMD	MELCC	LMLM
$\mathcal{A}_\Delta$	0.850	1.380	0.910	0.870	1.340	0.930	0.910	1.350	0.990	0.000	0.710	0.170	3.003	3.595	3.261
SF	170	276	182	174	268	186	182	270	198	0	142	34	601	719	652

9.5 Conclusion

This chapter introduces the background of longevity risk and practical mechanisms of risk transfer. Furthermore, the mortality and interest rate model developments are presented to show the importance of adopting stochastic interest rates and stochastic mortality rates models. Moreover, the reason why we need to model the causal relationship between mortality events and interest rate fluctuations is explained. Mortality estimates from LMLM model consistently give higher survival curves than those using MELCC model. By adopting fixed interest rates, we evaluate different annuity prices with credible intervals based on posterior predictive distributions of mortality rates. We demonstrate a substantial amount of shortfall in annuity price can be avoided using the mortality rates from LMLM model. The annuity pricing procedures are further extended to incorporate stochastic interest rates. Four types of dependency structure between interest rates and mortality rates under the Vasicek model are adopted. To calibrate the discretised version of Vasicek model, parameters $\boldsymbol{\vartheta}_c$ are estimated by joint or marginal models between interest rates and mortality rates. The starting interest rates in these two types of models are obtained from LIBOR whereas mortality rates are obtained from HMD or estimated from MELCC and LMLM models. As a result, we are able to provide twelve different formulations for the pricing of annuity, pure endowment and GAO. Results show that annuity prices adopting HMD for the starting mortality rates and assuming jointly distributed dependency structure are the highest (thereby set as bench mark model for comparison) whereas prices adopting estimates from LMLM model still with joint dependency structure are marginally lower. Hence the latter model provides the minimal amount of shortfall with respect to the bench mark model. These confirms the advantage of LMLM model in pricing of annuity as well as GAO. Currently, we only adopt Vasicek model for the interest rates and mortality rates. Other models should be considered to provide more accurate pricing.

CHAPTER 10

Conclusions and future work

10.1 Summary

In this thesis, an essential and novel model called Generalised Linear Gegenbauer Autoregressive Moving Average (GLGARMA) model with discrete distribution is proposed to capture the oscillatory pattern in long memory time series. Model properties in both time domain and frequency domain are explored to obtain a deeper understanding of the model structure. Bayesian approach and various model selection criteria are adopted to assess the in-sample fitting and out-sample forecast performance. Furthermore, there are two key model applications: open interest on a range of important U.S. treasury futures contracts and mortality rates from 16 selected countries cross-classified by gender and age groups.

From a modelling perspective, GLGARMA model with four choices of count data distribution with a wide range of dispersion is proposed in Chapter 2 to capture the long memory and oscillatory long memory properties in a count series. Model structures and properties including moment properties, stationarity, invertibility and long memory conditions are precisely explained. The distinct features between Gegenbauer long memory component and seasonality component are clearly demonstrated. We show that the ACF and spectral density plots for oscillatory long memory time series are different from those for seasonal time series and deseasonisation cannot eliminate such difference. The features of Gegenbauer polynomial as a special case of an orthogonal polynomial are presented in Chapter 3 to study the characterisation of the long memory features of our models in the frequency domain. The spectral density function

in Gaussian and non-Gaussian models, the GLGARMA-PD model in particular, are derived to facilitate the Whittle likelihood estimation.

From a financial perspective, Chapter 4 demonstrates the property of consistent long memory features in trading behaviour and liquidity provision that is present in many financial markets cross-classified by different market participant behaviours. This aim motivates four analyses on CFTC data which represent some of the most active futures contracts traded in markets. Our first analysis in Chapter 4 confirms the presence of long memory through the nonparametric Hurst exponent measure applying to 136 CFTC data and further identifies four types of long memory, namely basic, oscillatory, combined and irregular, through studies on their ACFs and spectral density plots. Results show that basic and irregular are more prevalence and trade type spread displays oscillatory long memory.

Then we perform in Chapter 5 the second analysis which is an extensive preliminary study fitting each CFTC data to 40 different models. In this analysis, we identify the best model structures in terms of count distribution, order of model, type of model and type of long memory and find that PD model with Gegenbauer long memory and GP distribution consistently provides better model fits. Then we apply this model structure to the next two analyses. Our third analysis investigates the difference between Gegenbauer long memory and seasonality and confirms that seasonal model cannot replace Gegenbauer long memory model even for spread data which may display seasonality due to future roll periodicity. With these information and our aim to forecast open interest, our last analysis performs a large scale forecasting experiment comparing our proposed models with a wide choices of popular time series count models including seasonal generalised linear autoregressive moving average (SGLARMA) model, autoregressive conditional Poisson (ACP) and generalised autoregressive score (GAS) models for four open interest data representing the four types of long memory. Results confirm again the superiority of our GLGARMA model in forecasting open interest. In summary, for different market participants with different types of long-memory, our models improve CFTC data forecasting accuracy and aid the ability to predict future monetary flow dynamics in futures markets. For model estimation and prediction, advantages of adopting a Bayesian approach via the efficient and user-friendly `Rstan` package for the model estimation is also explained and measures for convergence monitoring are also

introduced. Moreover, various model selection criteria are adopted to assess model performances. In summary, from such comprehensive studies on CFTC data in fitting and forecasting totally 9,224 models by running super computers for 1,844,000 hours, we conclude with confidence that the GLGARMA model provides the best model structure for all types of open interest data.

From an insurance perspective, Chapter 6 aims to study the long memory features in mortality data and explore the impacts of this stylised factor on the construction of life table. This aim motivates an extensive study of death counts from 16 countries cross classified by age groups and genders. We demonstrate for the first time that the long memory is prevalence in these mortality data using Hurst exponent and we further identify different patterns of long memory across different countries, age groups, genders and periods.

In order to foster a deeper understanding of how the long memory features improve the mortality model fitting and forecasting performance, we propose in Chapter 7 a class of generalised linear models extended from Lee Carter (LC) model for a single age group with different features including a spline component, a period effect and a long memory. The most comprehensive model (model 8) in the class contains all three components and there are seven submodels. We perform simulation studies to assess the accuracy of parameter estimation for these models and conduct in-sample fitting and out-of-sample forecasting for five selected countries, namely, Australia, Iceland, Japan, U.K. and U.S. with different mortality profiles. Results show model 8 provides the best performance consistently among all other submodels. The enhancement in life expectancy forecasts is also demonstrated by comparing results from model 8 with the univariate LC model (model 3). In addition, the life expectancy and life table forecasts 20 years ahead for 16 countries comparing model 8 and model 3 are reported with credible intervals in the Appendix..

As cohort effect after WW2 is detected by H for some countries, the univariate model in Chapter 7 is further extended to multivariate model in Chapter 8 to include cohort effect and hence allow modelling of death counts from all age groups simultaneously. This extended model is the long memory with long memory cohort effect (LMLM) model which is an extension of the LC cohort model by combining the generalised linear model with Gegenbauer long memory model to describe some stylised long memory features in mortality data. The extension of the univariate LC model (model 3) to multivariate

LC model for death counts with cohort effect is called multivariate extended LC cohort (MELCC) model. We explain the advantage of LMLM model over MELCC model by investigating and comparing the long memory structure in the non-stationary AR(1) period effect in MELCC model with the long memory structure in the mean function of LMLM model. We claim that by assigning Gegenbauer long memory structure in the mean function, we avoid the identification constraints in common LC models as these constraints can create numerical problems in parameter estimation. We perform simulation studies to assess the accuracy of parameter estimates particularly the two pairs of long memory parameters and results confirm good accuracy. Consequently, parameter estimation problems related to identification issues can be released. Compared with MELCC model, LMLM model is the best model for in-sample fitting and out-of-sample forecasting. Then the forecasted mortality rates are applied to calculate life expectancies which can then be used to construct life table for three selected countries, namely, Australia, U.K. and U.S..

Lastly, in Chapter 9, we apply the MELCC model and LMLM model to forecast mortality rates from three selected countries and demonstrate that life expectancy estimates and forecasts from LMLM model outperforms MELCC model in general based on four selection criteria which measure discrepancies with those life expectancy estimates calculated using mortality rates from HMD. LMLM model estimates consistently higher life expectancies and hence higher survival curves than those using MELCC model. We then evaluate various annuity prices with credible intervals based on posterior predictive distributions of mortality rates with fixed interest rates. Results show that the annuity prices are consistently higher using LMLM model because it estimates higher life expectancies. We demonstrate a substantial amount of shortfall in annuity price can be avoided using the mortality rate estimates from LMLM model.

We further extend the annuity pricing using stochastic interest rates apart from mortality rates and propose *four* types of dependency structure between interest rates and mortality rates under the Vasicek model. To calibrate the discretised version of the Vasicek model, parameters of Vasicek model are estimated by joint or marginal distributions between interest rates and mortality rates. The “observed” interest rates in these two types of models are obtained from LIBOR whereas the “observed” mortality rates are obtained from HMD or estimated from MELCC and LMLM models. As a result, we can provide twelve different formulations for the pricing of annuity, pure

endowment and GAO. Results show that the annuity price mortality rates from HMD and assuming jointly distributed dependency structure are the highest (thereby set as a benchmark model for comparison) whereas the price adopting estimates from LMLM model and still with joint dependency structure are marginally lower. Hence the latter model provides the minimal amount of shortfall with respect to the benchmark model. These confirm the advantage of LMLM model in the pricing of an annuity as well as GAO.

10.2 Future research directions

There are some limitations to our current study. Extension can be considered in several aspects, namely, the long memory structure, the mean function, the dispersion dynamic, multivariate effect, etc, to improve model performance.

Although our proposed GLARFIMA and GLGARMA models can fit well data with basic and oscillatory (type I and II) long memory, they fail to capture more complicated long memory structures such as the combined and irregular types (type III and IV). Figure 5.1 clearly shows that the combined type contains two oscillations whereas the irregular type seems to have several oscillations interacting with each other. To capture several oscillations simultaneously, our mean function in Equation (2.16) could be extended to include more than one Gegenbauer polynomial each with different u and d parameters. The resultant model is called k -factor GLGARMA model (Ferrara and Guegan, 2001). This model has been widely used for modeling and forecasting financial time series because it captures various dependencies including short-range and long-range dependencies. Consider a stationary time series process with constant c , a k -factor GLGARMA model with order (p, d, q) is defined by

$$Y_t | \mathcal{F}_{1:t-1} \sim D(\mu_t, \nu),$$

$$\Phi(B) \prod_{i=1}^k (1 - 2u_i B + B^2)^{d_i} (\ln \mu_t - c) = \Theta(B) \varepsilon_t \equiv \Theta(B) \prod_{i=1}^k \left(\sum_{j>0}^{\infty} C_j^{d_i}(u_i) \varepsilon_{t-j} \right),$$

$$\varepsilon_t \stackrel{\text{i.i.d}}{\sim} N(0, \sigma_\varepsilon^2),$$

where $C_j^{d_i}(u_i)$ is defined in Equation (2.17). The model estimation is complicated and challenging but is worthwhile to pursue.

Apart from extending the long memory structure, mean function can also be extended to reflect some changes in financial market. This includes the non-linear threshold model (Tong and Lim, 1980) which is very common in financial time series modelling. Besides, autoregressive distributed lag models (ARDL) (Pesaran and Shin, 1998) plays an important role in modelling the long-run relation between two economic variables. Consider a stationary time series process with constant c , an ARDL model is defined by

$$Y_t | \mathcal{F}_{1:t-1} \sim D(\mu_t, \nu),$$

$$\ln \mu_t = c + \beta(B)X_t + \varepsilon_t \equiv \left(\sum_{j=0}^{\infty} \beta_j X_{t-j} + \varepsilon_t \right), \quad \varepsilon_t \stackrel{\text{i.i.d}}{\sim} N(0, \sigma_\varepsilon^2).$$

Because of the infinite moving average representation of an explanatory variable, our GLGARMA model structure can incorporate the ARDL model to capture this long-run relationship in an oscillatory pattern through the Gegenbauer polynomial.

In addition, multivariate model allows the modelling of several time series simultaneously and this facilitates portfolio setting in finance. Multivariate mortality models in the form of state space models have been proposed in Chapter 8. This state space modelling idea can be applied to model several open interest series simultaneously. The LC model uses cohort effect as the state variable to link mortality model of each age group together. Chan and Wan (2014) also applied the state space modelling idea but they adopted a multivariate t distribution to the detrended state variables so that the correlation between components can be explicitly modelled. Adopting this idea, we may consider the following model structure

$$y_{it} \sim \text{GP}(\mu_{it}, \nu_i), \quad i = 1, \dots, g; \quad t = 1, \dots, T,$$

$$\ln \mu_{it} = \alpha_i + (1 - 2u_i B + B^2)^{-d_i} \varepsilon_{it}, \quad \varepsilon_{it} \sim N(0, \sigma_{\varepsilon,i}^2),$$

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_g \end{pmatrix} \sim \text{MT}_\nu \left(\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_g \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho_{21}\sigma_1\sigma_2 & \cdots & \rho_{g1}\sigma_1\sigma_g \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \cdots & \rho_{g2}\sigma_2\sigma_g \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{g1}\sigma_g\sigma_1 & \rho_{g2}\sigma_g\sigma_2 & \cdots & \sigma_g^2 \end{pmatrix} \right).$$

which also allows dynamic modelling of the correlation between open interest through $\rho_{ij,t}$. Furthermore, the dispersion parameter ν in the count distribution in Equations (10.1) and (2.10) can be modelled as $\nu_{i,t}$ and ν_t respectively to improve flexibility.

For financial application, other financial data sets, for example, the average daily trading volume as a measure of share liquidity of a stock market can fit our GLGARMA model. The share liquidity represents the ability that a trader can rapidly sell a stock when the share price falls. Recently, cryptocurrency made the leap and attracted significant investors and media attentions. The dynamics of trade volume of cryptocurrency can also be investigated through our GLGARMA model.

For the insurance application, there are some limitations for the current models in Chapter 9. First of all, the long memory parameters u and d are assumed to be consistent across age groups. A relaxation will be to allow the parameters to change at least across certain blocks of age groups defined by different stages of life such as junior, middle and senior, etc. Secondly, we take the mortality rate estimates as observed and refit them, together with interest rates, into Vasicek model. There are other model choices, such as Cox Ingersoll Ross (CIR) model (Cox et al., 1985), Longstaff's model (Longstaff et al., 1989) and Hull and White model (Hull and White, 1990), to provide more flexible modelling. Following the similar definition of Vasicek model in Equation (9.7), CIR can be represented as

$$dr_t = (c_1 - c_2 r_t)dt + c_3 \sqrt{r_t} dW_t^r.$$

Compared with Vasicek model, CIR model introduces a square root of interest rate $\sqrt{r_t}$ which does not allow for negative interest rate. Furthermore, as r_t decreasing, the standard deviation $\sqrt{r_t}$ also becomes very small, which dampens the influence of the random shock on the interest rate. Another interest rate model called Longstaff's model is given by

$$dr_t = c_1(c_2 - c_3 \sqrt{r_t})dt + c_4 \sqrt{r_t} dW_t^r.$$

This is a two-factor model which consists of a short-term interest rate and a instantaneous variance of changes in this interest rate. Hence, the current levels and volatility of the interest rate can be reflect by price. Hull and White model as a single-factor interest model is defined as

$$dr_t = c_1(c_2 - c_3 r_t)dt + c_4 r_t^{c_5} dW_t^r, \quad 0 < c_5 < 1.$$

The interest rate are constructed on a certain diffusion process with fixed coefficients. However, while these models are popular for interest rates, they do not have long memory structure to describe the mortality features. Treating the mortality estimates as observed and fitting them into these short memory models may provide some workable solutions but they are not optimal. A better way will be to adopt as well a state space model to link the multivariate mortality model with the univariate interest rate model through some state variables. This idea is promising to explore.

Appendices

APPENDIX A

Life tables for 16 countries

Table A.1. The forecasted Australia female life expectancy by model 3

age 0	97.5%	85.07	85.29	85.49	85.64	85.77	85.85	85.89	85.97	86.06	86.11	86.23	86.27	86.34	86.38	86.47	86.46	86.54	86.58	86.61	86.68
	mean	84.38	84.35	84.33	84.31	84.29	84.26	84.23	84.21	84.18	84.16	84.13	84.11	84.08	84.05	84.02	83.99	83.97	83.94	83.92	83.89
	2.5%	83.67	83.33	83.07	82.88	82.69	82.53	82.33	82.20	82.02	81.90	81.72	81.60	81.50	81.37	81.25	81.13	81.00	80.84	80.72	80.63
1-4	97.5%	84.32	84.53	84.72	84.87	85.00	85.06	85.10	85.18	85.26	85.31	85.42	85.46	85.53	85.56	85.64	85.65	85.71	85.75	85.79	85.85
	mean	83.66	83.63	83.61	83.60	83.57	83.55	83.52	83.50	83.47	83.45	83.43	83.40	83.38	83.35	83.32	83.29	83.27	83.25	83.22	83.20
	2.5%	82.99	82.66	82.43	82.25	82.06	81.91	81.73	81.61	81.44	81.32	81.15	81.05	80.96	80.83	80.70	80.59	80.47	80.33	80.23	80.16
5-9	97.5%	80.36	80.56	80.75	80.90	81.03	81.10	81.13	81.21	81.29	81.34	81.45	81.49	81.55	81.59	81.67	81.67	81.74	81.77	81.81	81.87
	mean	79.71	79.68	79.66	79.64	79.62	79.59	79.57	79.55	79.52	79.50	79.48	79.45	79.43	79.40	79.37	79.35	79.32	79.30	79.28	79.25
	2.5%	79.04	78.73	78.49	78.32	78.13	77.99	77.81	77.69	77.52	77.41	77.24	77.15	77.04	76.92	76.80	76.70	76.57	76.44	76.35	76.26
10-14	97.5%	75.39	75.59	75.77	75.92	76.05	76.12	76.16	76.23	76.32	76.36	76.47	76.50	76.57	76.61	76.69	76.68	76.75	76.80	76.83	76.89
	mean	74.74	74.71	74.70	74.68	74.66	74.63	74.60	74.58	74.56	74.54	74.52	74.49	74.47	74.44	74.41	74.38	74.36	74.34	74.32	74.29
	2.5%	74.08	73.77	73.54	73.36	73.18	73.04	72.86	72.74	72.57	72.47	72.30	72.21	72.11	71.99	71.87	71.77	71.65	71.52	71.43	71.35
15-19	97.5%	70.41	70.62	70.80	70.95	71.07	71.14	71.18	71.25	71.33	71.38	71.49	71.52	71.59	71.63	71.70	71.70	71.77	71.81	71.85	71.91
	mean	69.77	69.75	69.73	69.71	69.69	69.66	69.64	69.62	69.59	69.57	69.55	69.52	69.50	69.47	69.45	69.42	69.40	69.38	69.35	69.33
	2.5%	69.12	68.81	68.58	68.41	68.23	68.09	67.91	67.79	67.63	67.52	67.36	67.26	67.17	67.05	66.93	66.83	66.72	66.58	66.50	66.42
20-24	97.5%	65.48	65.68	65.85	66.00	66.13	66.20	66.23	66.31	66.39	66.43	66.54	66.57	66.64	66.68	66.75	66.75	66.83	66.86	66.89	66.96
	mean	64.84	64.82	64.80	64.78	64.76	64.74	64.71	64.69	64.67	64.65	64.63	64.60	64.58	64.55	64.53	64.50	64.48	64.46	64.43	64.41
	2.5%	64.20	63.90	63.67	63.50	63.32	63.19	63.02	62.89	62.74	62.64	62.47	62.38	62.29	62.17	62.05	61.95	61.84	61.72	61.63	61.55
25-29	97.5%	60.55	60.74	60.92	61.06	61.19	61.25	61.29	61.36	61.44	61.49	61.59	61.63	61.69	61.73	61.79	61.81	61.87	61.91	61.94	62.00
	mean	59.92	59.90	59.88	59.86	59.84	59.82	59.79	59.77	59.75	59.73	59.71	59.68	59.66	59.63	59.61	59.58	59.56	59.54	59.52	59.50
	2.5%	59.29	58.99	58.76	58.60	58.43	58.30	58.13	58.02	57.85	57.75	57.60	57.51	57.42	57.29	57.18	57.09	56.97	56.86	56.77	56.68
30-34	97.5%	55.62	55.81	55.99	56.13	56.25	56.32	56.35	56.43	56.49	56.55	56.64	56.69	56.75	56.79	56.85	56.86	56.93	56.97	56.99	57.05
	mean	55.01	54.98	54.97	54.95	54.93	54.91	54.88	54.86	54.84	54.82	54.80	54.77	54.75	54.73	54.70	54.68	54.66	54.64	54.61	54.59
	2.5%	54.39	54.10	53.88	53.71	53.54	53.41	53.24	53.14	52.99	52.88	52.74	52.64	52.55	52.43	52.33	52.24	52.12	52.01	51.92	51.85
35-39	97.5%	50.72	50.91	51.08	51.22	51.34	51.40	51.43	51.51	51.58	51.63	51.72	51.76	51.82	51.86	51.92	51.94	52.00	52.04	52.07	52.12
	mean	50.12	50.09	50.08	50.06	50.04	50.02	49.99	49.97	49.95	49.93	49.91	49.89	49.87	49.84	49.82	49.79	49.77	49.75	49.73	49.71
	2.5%	49.51	49.22	49.01	48.84	48.68	48.55	48.39	48.29	48.13	48.04	47.90	47.80	47.72	47.60	47.51	47.40	47.29	47.18	47.10	47.03
40-44	97.5%	45.85	46.03	46.20	46.33	46.46	46.52	46.55	46.62	46.68	46.74	46.82	46.87	46.93	46.96	47.02	47.04	47.10	47.13	47.17	47.22
	mean	45.26	45.24	45.22	45.20	45.18	45.16	45.14	45.12	45.10	45.08	45.06	45.04	45.01	44.99	44.97	44.94	44.92	44.90	44.88	44.86
	2.5%	44.67	44.39	44.17	44.01	43.86	43.73	43.57	43.47	43.33	43.24	43.10	43.02	42.93	42.80	42.73	42.62	42.51	42.41	42.33	42.24
45-49	97.5%	41.03	41.20	41.37	41.50	41.61	41.67	41.71	41.77	41.84	41.90	41.98	42.02	42.08	42.11	42.17	42.18	42.24	42.27	42.31	42.36
	mean	40.45	40.43	40.41	40.40	40.38	40.35	40.33	40.31	40.29	40.27	40.25	40.23	40.21	40.19	40.16	40.14	40.12	40.10	40.08	40.06
	2.5%	39.87	39.60	39.39	39.24	39.09	38.96	38.82	38.71	38.57	38.49	38.35	38.27	38.18	38.06	37.97	37.87	37.77	37.68	37.58	37.52
50-54	97.5%	36.28	36.45	36.61	36.74	36.84	36.90	36.93	37.00	37.06	37.11	37.21	37.25	37.29	37.32	37.38	37.40	37.45	37.49	37.52	37.56
	mean	35.71	35.69	35.68	35.66	35.64	35.62	35.60	35.58	35.56	35.54	35.52	35.50	35.48	35.46	35.44	35.41	35.39	35.38	35.36	35.33
	2.5%	35.15	34.88	34.70	34.54	34.40	34.27	34.13	34.02	33.89	33.81	33.68	33.61	33.51	33.40	33.31	33.22	33.13	33.04	32.95	32.87
55-59	97.5%	31.60	31.76	31.91	32.03	32.13	32.18	32.21	32.28	32.34	32.38	32.48	32.51	32.55	32.59	32.64	32.67	32.72	32.75	32.77	32.82
	mean	31.06	31.04	31.03	31.01	31.00	30.97	30.95	30.94	30.92	30.90	30.88	30.86	30.84	30.82	30.80	30.78	30.76	30.75	30.73	30.71
	2.5%	30.52	30.29	30.10	29.95	29.82	29.71	29.59	29.48	29.35	29.27	29.15	29.07	28.97	28.90	28.80	28.72	28.63	28.54	28.46	28.41
60-64	97.5%	27.02	27.17	27.30	27.42	27.51	27.55	27.59	27.65	27.71	27.75	27.84	27.87	27.90	27.93	27.98	28.01	28.06	28.09	28.12	28.16
	mean	26.51	26.49	26.48	26.46	26.45	26.43	26.41	26.39	26.38	26.36	26.34	26.32	26.31	26.29	26.27	26.25	26.23	26.22	26.20	26.18
	2.5%	26.01	25.79	25.62	25.49	25.35	25.24	25.12	25.03	24.92	24.84	24.73	24.65	24.57	24.51	24.42	24.32	24.26	24.17	24.12	24.06
65-69	97.5%	22.56	22.70	22.82	22.93	23.01	23.06	23.10	23.14	23.20	23.24	23.32	23.34	23.39	23.40	23.45	23.48	23.51	23.55	23.57	23.60
	mean	22.09	22.07	22.06	22.05	22.04	22.02	22.00	21.99	21.97	21.96	21.94	21.92	21.91	21.89	21.87	21.85	21.84	21.82	21.81	21.79
	2.5%	21.62	21.43	21.26	21.15	21.03	20.93	20.82	20.73	20.63	20.56	20.47	20.40	20.32	20.25	20.18	20.10	20.03	19.97	19.92	19.85
70-74	97.5%	18.24	18.37	18.48	18.58	18.65	18.70	18.74	18.77	18.83	18.85	18.93	18.95	19.00	19.00	19.04	19.07	19.11	19.14	19.15	19.18
	mean	17.83	17.81	17.80	17.79	17.78	17.76	17.75	17.73	17.72	17.71	17.69	17.68	17.66	17.65	17.63	17.61	17.60	17.59	17.57	17.56
	2.5%	17.41	17.23	17.09	16.99	16.87	16.79	16.69	16.61	16.53	16.46	16.38	16.33	16.26	16.18	16.14	16.06	16.00	15.94	15.89	15.84
75-79	97.5%	14.19	14.30	14.39	14.46	14.53	14.57	14.60	14.64	14.67	14.71	14.77	14.78	14.81	14.82	14.85	14.89	14.92	14.93	14.96	14.

Table A.2. The forecasted Australia male life expectancy by model 3

age 0	97.5%	81.25	81.53	81.78	81.93	82.13	82.20	82.27	82.37	82.46	82.54	82.66	82.73	82.75	82.79	82.88	82.95	82.96	83.05	83.05	83.13
	mean	80.43	80.40	80.37	80.34	80.31	80.28	80.24	80.21	80.17	80.14	80.11	80.07	80.04	80.00	79.96	79.92	79.89	79.86	79.82	79.79
	2.5%	79.57	79.17	78.85	78.65	78.39	78.17	77.95	77.83	77.64	77.44	77.26	77.08	76.95	76.80	76.62	76.46	76.37	76.17	76.05	75.93
1-4	97.5%	80.51	80.77	81.02	81.17	81.35	81.42	81.49	81.58	81.66	81.74	81.86	81.93	81.95	81.99	82.08	82.13	82.15	82.24	82.25	82.31
	mean	79.72	79.68	79.66	79.63	79.60	79.57	79.53	79.50	79.47	79.44	79.40	79.37	79.33	79.30	79.26	79.22	79.19	79.16	79.13	79.09
	2.5%	78.89	78.51	78.21	78.00	77.76	77.55	77.34	77.23	77.05	76.85	76.67	76.51	76.37	76.24	76.07	75.92	75.82	75.64	75.52	75.43
5-9	97.5%	76.55	76.81	77.06	77.20	77.38	77.45	77.52	77.61	77.70	77.76	77.89	77.96	77.98	78.02	78.11	78.16	78.17	78.26	78.28	78.34
	mean	75.77	75.74	75.71	75.69	75.65	75.62	75.58	75.55	75.52	75.49	75.46	75.42	75.39	75.35	75.32	75.28	75.25	75.22	75.19	75.15
	2.5%	74.95	74.58	74.28	74.08	73.84	73.63	73.42	73.32	73.13	72.94	72.76	72.60	72.47	72.33	72.18	72.02	71.94	71.76	71.64	71.54
10-14	97.5%	71.58	71.84	72.08	72.23	72.41	72.48	72.55	72.64	72.72	72.79	72.91	72.98	73.01	73.04	73.13	73.18	73.20	73.28	73.30	73.36
	mean	70.81	70.77	70.75	70.72	70.69	70.66	70.62	70.59	70.56	70.53	70.50	70.46	70.43	70.39	70.36	70.32	70.29	70.26	70.23	70.19
	2.5%	69.99	69.62	69.32	69.12	68.89	68.68	68.48	68.37	68.18	67.99	67.82	67.67	67.53	67.39	67.25	67.08	67.01	66.82	66.71	66.62
15-19	97.5%	66.61	66.87	67.11	67.26	67.43	67.50	67.57	67.66	67.74	67.81	67.94	68.00	68.03	68.06	68.15	68.19	68.22	68.31	68.32	68.38
	mean	65.84	65.81	65.78	65.76	65.73	65.69	65.65	65.63	65.59	65.56	65.53	65.50	65.47	65.43	65.40	65.36	65.33	65.30	65.26	65.23
	2.5%	65.03	64.66	64.37	64.17	63.94	63.73	63.53	63.43	63.24	63.04	62.88	62.73	62.59	62.44	62.31	62.14	62.07	61.89	61.78	61.69
20-24	97.5%	61.72	61.98	62.21	62.36	62.53	62.60	62.67	62.75	62.83	62.90	63.02	63.08	63.11	63.14	63.24	63.28	63.30	63.39	63.41	63.46
	mean	60.96	60.93	60.90	60.88	60.85	60.82	60.78	60.75	60.72	60.69	60.66	60.63	60.60	60.56	60.53	60.49	60.46	60.43	60.40	60.37
	2.5%	60.16	59.81	59.52	59.32	59.10	58.90	58.70	58.60	58.41	58.23	58.05	57.91	57.79	57.65	57.50	57.34	57.28	57.08	56.99	56.89
25-29	97.5%	56.87	57.12	57.35	57.49	57.66	57.73	57.79	57.88	57.95	58.02	58.14	58.20	58.22	58.26	58.36	58.40	58.41	58.50	58.52	58.57
	mean	56.14	56.11	56.08	56.06	56.03	56.00	55.96	55.94	55.91	55.88	55.85	55.82	55.79	55.76	55.73	55.69	55.66	55.64	55.60	55.57
	2.5%	55.36	55.02	54.75	54.56	54.35	54.16	53.97	53.86	53.69	53.52	53.33	53.22	53.10	52.99	52.82	52.69	52.60	52.46	52.34	52.23
30-34	97.5%	52.03	52.27	52.49	52.63	52.80	52.86	52.92	53.01	53.08	53.16	53.27	53.32	53.36	53.39	53.47	53.52	53.53	53.62	53.62	53.69
	mean	51.31	51.28	51.26	51.24	51.21	51.18	51.15	51.12	51.09	51.07	51.04	51.01	50.98	50.95	50.92	50.88	50.86	50.83	50.80	50.77
	2.5%	50.57	50.23	49.97	49.79	49.58	49.39	49.22	49.12	48.96	48.78	48.61	48.50	48.38	48.27	48.11	47.98	47.88	47.75	47.65	47.55
35-39	97.5%	47.22	47.46	47.67	47.81	47.97	48.02	48.10	48.18	48.25	48.32	48.43	48.47	48.51	48.55	48.63	48.68	48.69	48.77	48.77	48.84
	mean	46.52	46.49	46.47	46.45	46.42	46.39	46.35	46.33	46.30	46.28	46.25	46.22	46.19	46.16	46.13	46.10	46.07	46.04	46.02	45.99
	2.5%	45.79	45.46	45.20	45.04	44.83	44.65	44.47	44.38	44.22	44.05	43.88	43.78	43.68	43.54	43.39	43.28	43.19	43.05	42.94	42.85
40-44	97.5%	42.46	42.68	42.89	43.03	43.17	43.24	43.31	43.39	43.45	43.52	43.63	43.67	43.71	43.73	43.82	43.86	43.89	43.96	43.96	44.02
	mean	41.77	41.74	41.72	41.70	41.67	41.64	41.61	41.59	41.56	41.53	41.51	41.48	41.45	41.42	41.39	41.36	41.33	41.31	41.28	41.25
	2.5%	41.06	40.74	40.50	40.33	40.13	39.95	39.78	39.70	39.55	39.37	39.21	39.11	39.02	38.88	38.73	38.62	38.55	38.40	38.29	38.21
45-49	97.5%	37.75	37.97	38.17	38.31	38.45	38.50	38.58	38.64	38.71	38.79	38.88	38.93	38.97	38.99	39.06	39.11	39.13	39.21	39.21	39.27
	mean	37.08	37.06	37.03	37.01	36.99	36.96	36.93	36.90	36.88	36.85	36.83	36.80	36.77	36.74	36.71	36.68	36.66	36.63	36.60	36.58
	2.5%	36.39	36.08	35.84	35.68	35.49	35.32	35.15	35.06	34.92	34.75	34.60	34.49	34.40	34.27	34.14	34.04	33.93	33.83	33.71	33.62
50-54	97.5%	33.11	33.32	33.50	33.63	33.77	33.83	33.89	33.96	34.02	34.09	34.18	34.22	34.26	34.29	34.36	34.40	34.43	34.50	34.50	34.56
	mean	32.47	32.45	32.43	32.41	32.38	32.35	32.32	32.30	32.28	32.25	32.23	32.20	32.18	32.15	32.12	32.09	32.07	32.04	32.02	31.99
	2.5%	31.82	31.53	31.30	31.14	30.97	30.81	30.64	30.54	30.42	30.26	30.14	30.05	29.96	29.83	29.70	29.60	29.50	29.39	29.28	29.22
55-59	97.5%	28.58	28.78	28.95	29.08	29.22	29.25	29.30	29.38	29.44	29.51	29.59	29.64	29.67	29.71	29.76	29.80	29.81	29.89	29.89	29.96
	mean	27.98	27.96	27.94	27.92	27.90	27.87	27.84	27.82	27.80	27.78	27.76	27.73	27.70	27.68	27.65	27.62	27.60	27.58	27.56	27.53
	2.5%	27.37	27.10	26.89	26.74	26.58	26.43	26.29	26.17	26.05	25.91	25.81	25.70	25.63	25.51	25.41	25.32	25.23	25.10	25.02	24.94
60-64	97.5%	24.22	24.39	24.54	24.67	24.79	24.83	24.89	24.94	25.00	25.05	25.15	25.18	25.20	25.22	25.29	25.32	25.34	25.42	25.43	25.47
	mean	23.65	23.63	23.61	23.59	23.57	23.55	23.52	23.50	23.48	23.46	23.44	23.41	23.39	23.37	23.34	23.32	23.30	23.28	23.26	23.23
	2.5%	23.08	22.83	22.63	22.51	22.35	22.22	22.08	21.99	21.86	21.75	21.65	21.56	21.48	21.37	21.29	21.19	21.10	20.99	20.93	20.86
65-69	97.5%	20.01	20.17	20.31	20.41	20.52	20.56	20.61	20.66	20.72	20.76	20.84	20.88	20.90	20.92	20.97	21.00	21.02	21.10	21.10	21.14
	mean	19.50	19.48	19.46	19.45	19.43	19.41	19.38	19.37	19.35	19.33	19.31	19.29	19.27	19.24	19.22	19.20	19.18	19.16	19.14	19.12
	2.5%	18.99	18.77	18.58	18.48	18.33	18.20	18.08	18.00	17.89	17.80	17.72	17.61	17.56	17.46	17.39	17.29	17.23	17.11	17.08	17.01
70-74	97.5%	16.04	16.17	16.29	16.40	16.48	16.52	16.56	16.59	16.66	16.69	16.76	16.80	16.81	16.82	16.87	16.91	16.92	16.98	16.99	17.02
	mean	15.59	15.57	15.56	15.54	15.53	15.51	15.49	15.47	15.45	15.44	15.42	15.40	15.38	15.36	15.35	15.33	15.31	15.29	15.28	15.26
	2.5%	15.14	14.96	14.77	14.69	14.58	14.47	14.37	14.27	14.18	14.12	14.04	13.96	13.91	13.81	13.75	13.69	13.59	13.53	13.47	13.40
75-79	97.5%	12.36	12.48	12.59	12.66	12.72	12.77	12.80	12.83	12.87	12.92	12.96	13.01	13.02	13.02	13.06	13.				

Table A.3. The forecasted Australia female life expectancy by model 8

age 0	97.5%	85.02	85.36	85.70	86.01	86.33	86.64	86.92	87.23	87.52	87.82	88.14	88.40	88.74	89.00	89.29	89.56	89.84	90.13	90.39	90.62
	mean	84.38	84.63	84.87	85.13	85.37	85.61	85.87	86.12	86.37	86.62	86.87	87.12	87.37	87.62	87.86	88.10	88.35	88.59	88.83	89.07
	2.5%	83.72	83.86	83.99	84.15	84.30	84.52	84.69	84.87	85.08	85.25	85.45	85.65	85.86	86.06	86.28	86.46	86.70	86.89	87.12	87.31
1-4	97.5%	84.27	84.60	84.92	85.23	85.53	85.83	86.10	86.40	86.67	86.96	87.26	87.52	87.85	88.11	88.38	88.65	88.92	89.22	89.46	89.70
	mean	83.66	83.89	84.13	84.37	84.61	84.84	85.08	85.33	85.57	85.81	86.05	86.29	86.53	86.77	87.01	87.24	87.49	87.72	87.95	88.19
	2.5%	83.02	83.16	83.30	83.46	83.59	83.80	83.95	84.15	84.34	84.50	84.70	84.89	85.09	85.28	85.49	85.69	85.92	86.11	86.32	86.49
5-9	97.5%	80.31	80.63	80.95	81.25	81.55	81.85	82.12	82.42	82.69	82.97	83.28	83.54	83.86	84.12	84.39	84.66	84.93	85.23	85.47	85.71
	mean	79.71	79.93	80.17	80.41	80.64	80.87	81.11	81.36	81.59	81.83	82.07	82.31	82.55	82.79	83.03	83.26	83.50	83.73	83.97	84.20
	2.5%	79.06	79.21	79.34	79.51	79.63	79.84	80.00	80.19	80.38	80.53	80.73	80.92	81.12	81.31	81.53	81.72	81.95	82.14	82.35	82.51
10-14	97.5%	75.34	75.67	75.98	76.28	76.58	76.88	77.14	77.44	77.71	77.99	78.30	78.55	78.88	79.13	79.41	79.67	79.94	80.24	80.48	80.72
	mean	74.74	74.97	75.20	75.44	75.67	75.91	76.15	76.39	76.62	76.86	77.10	77.34	77.58	77.82	78.05	78.29	78.53	78.76	78.99	79.22
	2.5%	74.11	74.25	74.38	74.55	74.68	74.88	75.04	75.23	75.42	75.58	75.78	75.97	76.17	76.35	76.57	76.75	76.99	77.18	77.39	77.55
15-19	97.5%	70.37	70.69	71.00	71.30	71.60	71.89	72.16	72.45	72.73	73.01	73.31	73.57	73.89	74.14	74.42	74.68	74.95	75.24	75.49	75.72
	mean	69.77	70.00	70.23	70.47	70.70	70.93	71.17	71.41	71.65	71.88	72.12	72.36	72.60	72.84	73.07	73.30	73.54	73.77	74.01	74.24
	2.5%	69.14	69.28	69.42	69.58	69.71	69.92	70.07	70.27	70.46	70.61	70.81	71.00	71.21	71.38	71.60	71.79	72.02	72.21	72.42	72.59
20-24	97.5%	65.44	65.75	66.06	66.35	66.65	66.95	67.21	67.50	67.77	68.05	68.35	68.60	68.92	69.17	69.45	69.70	69.98	70.27	70.51	70.75
	mean	64.84	65.07	65.30	65.54	65.76	65.99	66.23	66.47	66.70	66.94	67.18	67.41	67.65	67.89	68.12	68.35	68.59	68.82	69.05	69.28
	2.5%	64.22	64.35	64.49	64.66	64.79	65.00	65.15	65.34	65.52	65.68	65.88	66.08	66.28	66.45	66.68	66.86	67.09	67.27	67.49	67.66
25-29	97.5%	60.52	60.82	61.12	61.41	61.70	62.00	62.26	62.55	62.81	63.09	63.39	63.65	63.96	64.21	64.48	64.74	65.01	65.29	65.54	65.77
	mean	59.92	60.14	60.37	60.61	60.83	61.06	61.30	61.53	61.76	62.00	62.23	62.47	62.70	62.94	63.17	63.40	63.64	63.86	64.09	64.32
	2.5%	59.29	59.44	59.58	59.75	59.87	60.07	60.24	60.43	60.61	60.76	60.96	61.15	61.36	61.54	61.74	61.93	62.16	62.34	62.55	62.73
30-34	97.5%	55.58	55.88	56.18	56.47	56.75	57.04	57.31	57.59	57.85	58.13	58.43	58.68	58.99	59.24	59.50	59.76	60.04	60.32	60.57	60.80
	mean	54.99	55.21	55.44	55.67	55.89	56.12	56.35	56.59	56.82	57.05	57.29	57.52	57.75	57.99	58.22	58.44	58.68	58.91	59.13	59.36
	2.5%	54.37	54.52	54.66	54.83	54.95	55.15	55.31	55.50	55.68	55.84	56.03	56.24	56.43	56.62	56.81	57.00	57.23	57.41	57.62	57.79
35-39	97.5%	50.69	50.99	51.28	51.57	51.85	52.13	52.39	52.66	52.93	53.20	53.49	53.74	54.06	54.30	54.57	54.82	55.09	55.38	55.62	55.85
	mean	50.11	50.33	50.55	50.78	51.00	51.23	51.46	51.69	51.92	52.15	52.38	52.62	52.85	53.08	53.30	53.53	53.76	53.99	54.22	54.44
	2.5%	49.50	49.64	49.79	49.96	50.09	50.29	50.44	50.63	50.82	50.99	51.17	51.36	51.55	51.74	51.93	52.13	52.36	52.53	52.73	52.90
40-44	97.5%	45.84	46.12	46.41	46.69	46.98	47.25	47.50	47.77	48.04	48.30	48.60	48.84	49.15	49.40	49.67	49.91	50.18	50.45	50.69	50.93
	mean	45.27	45.48	45.70	45.93	46.15	46.37	46.60	46.83	47.05	47.28	47.51	47.74	47.97	48.20	48.42	48.65	48.88	49.10	49.32	49.54
	2.5%	44.65	44.81	44.95	45.13	45.26	45.45	45.61	45.79	45.97	46.16	46.34	46.51	46.71	46.91	47.09	47.28	47.52	47.68	47.88	48.04
45-49	97.5%	41.05	41.32	41.60	41.88	42.15	42.42	42.66	42.94	43.20	43.45	43.75	43.97	44.29	44.53	44.80	45.03	45.30	45.57	45.81	46.03
	mean	40.48	40.69	40.91	41.13	41.35	41.57	41.79	42.02	42.24	42.46	42.69	42.92	43.14	43.37	43.59	43.81	44.03	44.25	44.47	44.69
	2.5%	39.88	40.02	40.18	40.36	40.48	40.68	40.83	41.01	41.19	41.37	41.55	41.74	41.91	42.13	42.30	42.47	42.70	42.89	43.06	43.23
50-54	97.5%	36.30	36.57	36.84	37.11	37.37	37.64	37.88	38.14	38.40	38.65	38.92	39.16	39.45	39.71	39.95	40.19	40.45	40.71	40.95	41.17
	mean	35.75	35.96	36.17	36.39	36.60	36.81	37.03	37.25	37.47	37.69	37.91	38.13	38.35	38.57	38.78	39.00	39.22	39.43	39.65	39.86
	2.5%	35.16	35.31	35.48	35.63	35.76	35.96	36.10	36.28	36.46	36.63	36.81	36.98	37.17	37.37	37.54	37.71	37.93	38.11	38.27	38.46
55-59	97.5%	31.63	31.90	32.14	32.41	32.66	32.91	33.14	33.40	33.65	33.89	34.16	34.37	34.66	34.90	35.14	35.38	35.62	35.88	36.11	36.32
	mean	31.11	31.30	31.51	31.72	31.92	32.12	32.34	32.55	32.76	32.97	33.18	33.39	33.60	33.82	34.02	34.23	34.45	34.65	34.86	35.07
	2.5%	30.53	30.68	30.84	31.00	31.12	31.30	31.47	31.62	31.81	31.95	32.12	32.29	32.48	32.67	32.85	33.01	33.20	33.37	33.53	33.70
60-64	97.5%	27.07	27.31	27.55	27.78	28.03	28.27	28.48	28.72	28.97	29.18	29.45	29.66	29.93	30.14	30.38	30.61	30.85	31.09	31.31	31.51
	mean	26.56	26.74	26.93	27.13	27.32	27.52	27.72	27.92	28.12	28.31	28.51	28.72	28.92	29.12	29.32	29.51	29.71	29.91	30.11	30.30
	2.5%	26.01	26.14	26.30	26.45	26.58	26.73	26.88	27.05	27.21	27.36	27.52	27.67	27.83	28.02	28.18	28.34	28.53	28.69	28.84	29.01
65-69	97.5%	22.62	22.84	23.06	23.28	23.51	23.73	23.92	24.15	24.37	24.58	24.81	25.01	25.26	25.47	25.67	25.90	26.11	26.34	26.55	26.75
	mean	22.13	22.30	22.48	22.66	22.83	23.01	23.20	23.38	23.56	23.75	23.93	24.12	24.30	24.49	24.67	24.85	25.04	25.22	25.41	25.59
	2.5%	21.60	21.73	21.87	22.00	22.13	22.27	22.42	22.56	22.71	22.83	22.98	23.13	23.29	23.45	23.58	23.72	23.91	24.05	24.20	24.34
70-74	97.5%	18.31	18.50	18.69	18.89	19.08	19.28	19.46	19.66	19.87	20.04	20.26	20.44	20.65	20.85	21.05	21.25	21.46	21.66	21.85	22.03
	mean	17.84	17.99	18.14	18.30	18.46	18.61	18.77	18.94	19.10	19.26	19.43	19.59	19.76	19.92	20.09	20.25	20.42	20.58	20.75	20.91
	2.5%	17.32	17.43	17.56	17.67	17.80	17.92	18.02	18.17	18.30	18.40	18.53	18.66	18.78	18.94	19.05	19.16	19.31	19.45	19.60	19.69
75-79	97.5%	14.23	14.39	14.56	14.72	14.88	15.05	15.19	15.37	15.54	15.71	15.89	16.03	16.22	16.39	16.57	16.75	16.91	17.10	17.27	17.4

Table A.4. The forecasted Australia male life expectancy by model 8

age 0	97.5%	81.10	81.61	82.08	82.56	83.00	83.45	83.95	84.41	84.86	85.33	85.80	86.24	86.75	87.18	87.66	88.09	88.53	88.96	89.43	89.84
	mean	80.41	80.79	81.17	81.56	81.95	82.36	82.76	83.17	83.58	84.00	84.42	84.83	85.26	85.68	86.09	86.52	86.93	87.35	87.76	88.17
	2.5%	79.65	79.90	80.19	80.48	80.83	81.13	81.46	81.81	82.14	82.50	82.85	83.17	83.57	83.97	84.34	84.74	85.11	85.50	85.89	86.27
1-4	97.5%	80.37	80.85	81.30	81.76	82.19	82.62	83.10	83.56	83.99	84.46	84.91	85.36	85.85	86.28	86.75	87.17	87.61	88.03	88.49	88.90
	mean	79.69	80.05	80.42	80.80	81.18	81.58	81.97	82.37	82.77	83.18	83.58	83.99	84.41	84.82	85.22	85.64	86.05	86.46	86.86	87.27
	2.5%	78.94	79.20	79.49	79.76	80.09	80.40	80.71	81.05	81.35	81.74	82.06	82.40	82.78	83.16	83.52	83.90	84.28	84.66	85.03	85.41
5-9	97.5%	76.42	76.89	77.34	77.80	78.22	78.65	79.13	79.58	80.02	80.48	80.93	81.38	81.86	82.29	82.76	83.18	83.63	84.04	84.50	84.91
	mean	75.74	76.10	76.47	76.85	77.23	77.62	78.01	78.41	78.80	79.21	79.61	80.02	80.43	80.84	81.25	81.66	82.07	82.48	82.88	83.28
	2.5%	74.99	75.26	75.54	75.82	76.15	76.46	76.76	77.10	77.40	77.79	78.11	78.44	78.82	79.20	79.56	79.93	80.31	80.70	81.07	81.44
10-14	97.5%	71.44	71.92	72.36	72.82	73.25	73.67	74.15	74.60	75.03	75.50	75.95	76.39	76.87	77.31	77.77	78.19	78.63	79.05	79.50	79.92
	mean	70.77	71.13	71.50	71.87	72.25	72.64	73.03	73.43	73.82	74.23	74.63	75.04	75.45	75.86	76.26	76.68	77.08	77.49	77.89	78.29
	2.5%	70.03	70.29	70.57	70.85	71.18	71.48	71.79	72.13	72.43	72.82	73.14	73.47	73.85	74.23	74.58	74.96	75.33	75.72	76.09	76.46
15-19	97.5%	66.48	66.95	67.38	67.84	68.27	68.69	69.16	69.61	70.05	70.51	70.96	71.40	71.89	72.31	72.78	73.20	73.64	74.06	74.51	74.93
	mean	65.81	66.16	66.53	66.90	67.28	67.67	68.05	68.45	68.84	69.25	69.65	70.06	70.47	70.88	71.28	71.69	72.10	72.50	72.91	73.31
	2.5%	65.06	65.33	65.61	65.89	66.21	66.52	66.82	67.16	67.46	67.85	68.16	68.50	68.87	69.25	69.61	69.98	70.35	70.74	71.11	71.48
20-24	97.5%	61.59	62.05	62.48	62.94	63.35	63.78	64.25	64.69	65.11	65.57	66.01	66.46	66.93	67.36	67.82	68.24	68.68	69.09	69.54	69.95
	mean	60.93	61.28	61.64	62.01	62.38	62.76	63.15	63.54	63.93	64.33	64.73	65.12	65.53	65.94	66.34	66.75	67.15	67.55	67.95	68.35
	2.5%	60.19	60.45	60.73	61.02	61.34	61.64	61.94	62.27	62.57	62.94	63.27	63.59	63.97	64.33	64.69	65.06	65.43	65.79	66.16	66.54
25-29	97.5%	56.74	57.19	57.62	58.05	58.47	58.88	59.35	59.78	60.21	60.66	61.10	61.54	62.01	62.43	62.88	63.29	63.72	64.14	64.58	64.99
	mean	56.10	56.44	56.80	57.16	57.53	57.91	58.29	58.68	59.06	59.45	59.85	60.24	60.65	61.05	61.45	61.85	62.25	62.65	63.04	63.44
	2.5%	55.37	55.63	55.93	56.21	56.52	56.83	57.13	57.45	57.78	58.12	58.45	58.76	59.15	59.50	59.87	60.25	60.60	60.96	61.32	61.69
30-34	97.5%	51.91	52.34	52.77	53.18	53.59	54.01	54.47	54.89	55.31	55.76	56.19	56.62	57.10	57.52	57.96	58.37	58.81	59.20	59.65	60.06
	mean	51.27	51.61	51.96	52.32	52.68	53.06	53.43	53.82	54.20	54.59	54.98	55.37	55.77	56.17	56.56	56.96	57.36	57.76	58.15	58.54
	2.5%	50.56	50.80	51.10	51.39	51.69	52.01	52.32	52.64	52.94	53.29	53.63	53.94	54.32	54.67	55.03	55.40	55.77	56.12	56.46	56.83
35-39	97.5%	47.10	47.52	47.93	48.35	48.76	49.17	49.63	50.04	50.45	50.90	51.33	51.76	52.22	52.64	53.07	53.48	53.90	54.31	54.75	55.15
	mean	46.46	46.80	47.15	47.51	47.87	48.24	48.61	48.99	49.37	49.76	50.15	50.53	50.93	51.32	51.72	52.11	52.51	52.90	53.29	53.68
	2.5%	45.77	46.01	46.31	46.60	46.91	47.21	47.51	47.84	48.15	48.50	48.81	49.15	49.52	49.86	50.23	50.59	50.97	51.32	51.66	52.01
40-44	97.5%	42.34	42.76	43.18	43.58	43.98	44.41	44.85	45.26	45.66	46.10	46.53	46.94	47.40	47.82	48.24	48.65	49.07	49.48	49.89	50.32
	mean	41.73	42.07	42.41	42.77	43.12	43.49	43.86	44.24	44.61	45.00	45.38	45.77	46.17	46.55	46.94	47.34	47.73	48.12	48.51	48.89
	2.5%	41.05	41.29	41.59	41.88	42.20	42.50	42.81	43.14	43.43	43.80	44.10	44.43	44.79	45.14	45.50	45.87	46.24	46.59	46.93	47.27
45-49	97.5%	37.68	38.09	38.49	38.89	39.28	39.70	40.12	40.53	40.92	41.36	41.78	42.20	42.66	43.08	43.49	43.88	44.29	44.71	45.10	45.52
	mean	37.07	37.40	37.74	38.09	38.45	38.81	39.17	39.55	39.92	40.30	40.68	41.06	41.45	41.84	42.22	42.61	43.00	43.38	43.77	44.14
	2.5%	36.39	36.64	36.94	37.23	37.56	37.84	38.16	38.47	38.78	39.14	39.45	39.77	40.13	40.48	40.84	41.19	41.56	41.93	42.25	42.60
50-54	97.5%	33.13	33.53	33.91	34.30	34.69	35.08	35.50	35.90	36.29	36.72	37.12	37.53	37.98	38.39	38.79	39.17	39.59	39.98	40.37	40.78
	mean	32.54	32.87	33.20	33.54	33.89	34.25	34.60	34.97	35.33	35.70	36.08	36.45	36.84	37.21	37.59	37.97	38.35	38.73	39.10	39.47
	2.5%	31.89	32.15	32.44	32.70	33.04	33.32	33.63	33.96	34.26	34.60	34.89	35.24	35.58	35.92	36.27	36.61	36.97	37.32	37.65	37.99
55-59	97.5%	28.65	29.02	29.38	29.75	30.13	30.52	30.91	31.30	31.69	32.08	32.49	32.89	33.30	33.70	34.10	34.49	34.87	35.26	35.65	36.02
	mean	28.09	28.40	28.73	29.06	29.39	29.74	30.08	30.44	30.79	31.15	31.51	31.88	32.25	32.61	32.98	33.35	33.72	34.08	34.44	34.80
	2.5%	27.48	27.75	28.03	28.30	28.60	28.89	29.21	29.52	29.82	30.15	30.43	30.77	31.09	31.42	31.76	32.09	32.44	32.79	33.11	33.44
60-64	97.5%	24.32	24.68	25.01	25.37	25.72	26.08	26.44	26.80	27.18	27.56	27.92	28.32	28.70	29.08	29.47	29.84	30.20	30.56	30.93	31.30
	mean	23.82	24.12	24.42	24.73	25.05	25.37	25.70	26.03	26.36	26.70	27.04	27.39	27.74	28.08	28.43	28.78	29.13	29.47	29.82	30.16
	2.5%	23.29	23.53	23.79	24.06	24.35	24.62	24.91	25.20	25.49	25.79	26.08	26.37	26.70	27.00	27.32	27.64	27.94	28.28	28.58	28.90
65-69	97.5%	20.14	20.44	20.74	21.07	21.39	21.73	22.05	22.39	22.74	23.09	23.44	23.78	24.17	24.50	24.87	25.21	25.57	25.90	26.25	26.58
	mean	19.67	19.94	20.22	20.50	20.79	21.09	21.38	21.69	22.00	22.31	22.62	22.94	23.26	23.58	23.91	24.23	24.55	24.88	25.20	25.51
	2.5%	19.17	19.39	19.63	19.89	20.14	20.38	20.67	20.94	21.19	21.46	21.74	22.00	22.30	22.61	22.88	23.18	23.46	23.74	24.04	24.33
70-74	97.5%	16.17	16.43	16.70	16.97	17.26	17.56	17.84	18.13	18.43	18.74	19.06	19.37	19.71	20.01	20.34	20.65	20.97	21.29	21.60	21.91
	mean	15.74	15.97	16.21	16.46	16.71	16.97	17.23	17.50	17.77	18.04	18.32	18.60	18.89	19.17	19.46	19.75	20.04	20.33	20.62	20.91
	2.5%	15.29	15.49	15.71	15.91	16.13	16.35	16.58	16.83	17.04	17.30	17.53	17.76	18.01	18.27	18.50	18.77	19.03	19.28	19.54	19.80
75-79	97.5%	12.47	12.67	12.90	13.11	13.34	13.60	13.82	14.06	14.32	14.56	14.83	15.09	15.39	15.64	15.92	16.2				

Table A.5. The forecasted Belgium female life expectancy by model 3

age 0	97.5%	83.82	84.07	84.31	84.48	84.63	84.71	84.79	84.89	84.97	85.05	85.13	85.21	85.28	85.33	85.44	85.46	85.48	85.57	85.59	85.63
	mean	83.02	82.98	82.96	82.94	82.90	82.87	82.83	82.80	82.77	82.74	82.71	82.67	82.64	82.60	82.56	82.52	82.49	82.46	82.42	82.38
	2.5%	82.17	81.79	81.44	81.19	80.93	80.70	80.43	80.30	80.08	79.85	79.70	79.52	79.37	79.16	78.93	78.75	78.59	78.44	78.23	78.07
1-4	97.5%	83.03	83.25	83.48	83.65	83.78	83.87	83.94	84.03	84.11	84.18	84.27	84.35	84.41	84.46	84.55	84.59	84.60	84.67	84.71	84.75
	mean	82.26	82.23	82.21	82.18	82.15	82.12	82.09	82.06	82.03	82.00	81.98	81.94	81.91	81.87	81.84	81.80	81.77	81.74	81.71	81.68
	2.5%	81.46	81.10	80.79	80.57	80.31	80.12	79.86	79.73	79.53	79.34	79.17	79.01	78.88	78.68	78.49	78.31	78.16	78.00	77.83	77.73
5-9	97.5%	79.07	79.29	79.51	79.68	79.82	79.89	79.97	80.06	80.14	80.21	80.29	80.37	80.44	80.48	80.57	80.61	80.62	80.69	80.74	80.77
	mean	78.32	78.28	78.26	78.24	78.21	78.18	78.15	78.12	78.09	78.07	78.04	78.01	77.98	77.94	77.91	77.87	77.84	77.82	77.78	77.75
	2.5%	77.53	77.18	76.87	76.66	76.41	76.22	75.97	75.84	75.66	75.47	75.30	75.14	75.04	74.84	74.66	74.48	74.35	74.18	74.01	73.90
10-14	97.5%	74.09	74.31	74.53	74.70	74.83	74.91	74.99	75.07	75.15	75.22	75.30	75.38	75.45	75.49	75.59	75.62	75.64	75.70	75.75	75.77
	mean	73.35	73.32	73.29	73.27	73.25	73.22	73.18	73.16	73.13	73.10	73.08	73.05	73.02	72.98	72.95	72.92	72.89	72.86	72.83	72.80
	2.5%	72.57	72.22	71.92	71.72	71.48	71.29	71.05	70.92	70.75	70.56	70.41	70.25	70.13	69.95	69.76	69.59	69.46	69.33	69.17	69.06
15-19	97.5%	69.12	69.33	69.55	69.72	69.85	69.93	70.00	70.09	70.17	70.23	70.32	70.40	70.46	70.50	70.60	70.63	70.65	70.71	70.76	70.79
	mean	68.38	68.35	68.33	68.31	68.28	68.25	68.22	68.20	68.17	68.14	68.12	68.09	68.06	68.03	68.00	67.96	67.93	67.91	67.88	67.85
	2.5%	67.61	67.28	66.98	66.78	66.54	66.37	66.13	66.00	65.83	65.66	65.50	65.34	65.24	65.07	64.90	64.72	64.60	64.46	64.31	64.19
20-24	97.5%	64.16	64.37	64.59	64.75	64.88	64.96	65.03	65.12	65.20	65.27	65.35	65.42	65.49	65.53	65.62	65.66	65.67	65.74	65.78	65.81
	mean	63.44	63.41	63.39	63.37	63.34	63.31	63.28	63.26	63.23	63.21	63.18	63.15	63.12	63.09	63.06	63.03	63.00	62.98	62.95	62.92
	2.5%	62.68	62.35	62.06	61.87	61.64	61.46	61.24	61.11	60.94	60.77	60.62	60.47	60.38	60.21	60.04	59.88	59.75	59.61	59.48	59.36
25-29	97.5%	59.22	59.43	59.63	59.79	59.92	60.00	60.08	60.15	60.23	60.30	60.38	60.46	60.52	60.57	60.66	60.69	60.71	60.77	60.81	60.84
	mean	58.50	58.48	58.46	58.44	58.41	58.38	58.35	58.33	58.31	58.28	58.26	58.23	58.20	58.17	58.14	58.11	58.09	58.06	58.03	58.00
	2.5%	57.76	57.45	57.16	56.97	56.76	56.58	56.36	56.24	56.07	55.92	55.76	55.63	55.53	55.37	55.21	55.06	54.92	54.79	54.66	54.56
30-34	97.5%	54.28	54.48	54.68	54.84	54.97	55.04	55.12	55.20	55.27	55.34	55.43	55.49	55.55	55.60	55.69	55.72	55.74	55.81	55.85	55.87
	mean	53.58	53.55	53.53	53.51	53.49	53.46	53.43	53.41	53.39	53.36	53.34	53.31	53.29	53.26	53.23	53.20	53.18	53.15	53.12	53.10
	2.5%	52.85	52.55	52.27	52.09	51.87	51.71	51.49	51.38	51.24	51.07	50.92	50.79	50.70	50.54	50.40	50.24	50.11	49.99	49.86	49.77
35-39	97.5%	49.36	49.56	49.75	49.91	50.04	50.11	50.17	50.25	50.33	50.40	50.48	50.55	50.61	50.65	50.75	50.77	50.79	50.85	50.89	50.92
	mean	48.68	48.65	48.63	48.62	48.59	48.57	48.54	48.52	48.50	48.47	48.45	48.43	48.40	48.37	48.35	48.32	48.30	48.27	48.25	48.22
	2.5%	47.98	47.68	47.40	47.24	47.03	46.88	46.68	46.56	46.41	46.26	46.13	46.00	45.92	45.79	45.63	45.51	45.37	45.26	45.13	45.04
40-44	97.5%	44.47	44.67	44.85	45.00	45.13	45.20	45.26	45.34	45.42	45.48	45.56	45.64	45.68	45.73	45.82	45.84	45.87	45.93	45.95	45.99
	mean	43.82	43.80	43.78	43.76	43.74	43.72	43.69	43.67	43.65	43.63	43.61	43.58	43.56	43.54	43.51	43.48	43.46	43.44	43.42	43.40
	2.5%	43.15	42.87	42.62	42.45	42.25	42.10	41.91	41.81	41.69	41.55	41.40	41.28	41.21	41.07	40.95	40.83	40.70	40.61	40.48	40.41
45-49	97.5%	39.65	39.84	40.01	40.16	40.28	40.35	40.41	40.47	40.55	40.63	40.69	40.77	40.80	40.85	40.94	40.96	40.99	41.06	41.07	41.11
	mean	39.03	39.01	38.99	38.98	38.96	38.94	38.91	38.90	38.88	38.86	38.84	38.82	38.79	38.77	38.75	38.72	38.71	38.69	38.66	38.64
	2.5%	38.39	38.14	37.89	37.74	37.57	37.43	37.24	37.14	37.04	36.90	36.78	36.66	36.60	36.47	36.39	36.25	36.14	36.06	35.96	35.89
50-54	97.5%	34.92	35.09	35.25	35.39	35.51	35.57	35.63	35.70	35.77	35.83	35.90	35.97	35.99	36.05	36.13	36.15	36.18	36.25	36.25	36.29
	mean	34.34	34.32	34.30	34.29	34.27	34.25	34.23	34.22	34.20	34.18	34.17	34.15	34.13	34.11	34.09	34.06	34.05	34.03	34.01	33.99
	2.5%	33.74	33.50	33.29	33.15	32.98	32.86	32.70	32.60	32.51	32.40	32.28	32.16	32.10	32.00	31.91	31.80	31.70	31.62	31.55	31.48
55-59	97.5%	30.29	30.45	30.61	30.73	30.83	30.89	30.95	31.02	31.07	31.13	31.20	31.25	31.29	31.34	31.41	31.43	31.46	31.52	31.53	31.56
	mean	29.75	29.74	29.73	29.72	29.70	29.68	29.66	29.65	29.64	29.62	29.61	29.59	29.57	29.55	29.54	29.52	29.50	29.49	29.47	29.46
	2.5%	29.20	29.00	28.80	28.67	28.53	28.42	28.28	28.20	28.11	28.03	27.91	27.83	27.77	27.66	27.58	27.50	27.41	27.36	27.28	27.21
60-64	97.5%	25.86	26.01	26.14	26.25	26.35	26.41	26.46	26.52	26.58	26.64	26.69	26.75	26.78	26.82	26.89	26.91	26.93	26.99	27.01	27.03
	mean	25.35	25.34	25.32	25.30	25.29	25.27	25.26	25.24	25.23	25.21	25.20	25.18	25.17	25.15	25.13	25.12	25.11	25.09	25.08	
	2.5%	24.84	24.64	24.47	24.35	24.22	24.11	24.00	23.93	23.82	23.74	23.66	23.56	23.51	23.42	23.36	23.28	23.19	23.13	23.07	23.03
65-69	97.5%	21.57	21.71	21.82	21.93	22.02	22.08	22.12	22.17	22.22	22.28	22.33	22.37	22.40	22.44	22.50	22.52	22.54	22.58	22.62	22.63
	mean	21.10	21.09	21.08	21.07	21.06	21.04	21.02	21.01	21.00	20.99	20.97	20.96	20.95	20.93	20.92	20.90	20.89	20.87	20.86	20.85
	2.5%	20.64	20.46	20.29	20.19	20.07	19.97	19.86	19.80	19.73	19.64	19.56	19.48	19.43	19.35	19.27	19.21	19.12	19.08	19.05	18.99
70-74	97.5%	17.41	17.53	17.63	17.73	17.80	17.86	17.89	17.94	17.99	18.02	18.09	18.13	18.16	18.18	18.24	18.26	18.26	18.32	18.35	18.35
	mean	16.99	16.97	16.96	16.95	16.94	16.93	16.91	16.90	16.89	16.88	16.87	16.85	16.84	16.82	16.81	16.80	16.78	16.77	16.76	16.75
	2.5%	16.57	16.40	16.26	16.16	16.06	15.97	15.88	15.82	15.74	15.66	15.61	15.53	15.48	15.39	15.33	15.30	15.23	15.18	15.15	15.09
75-79	97.5%	13.47	13.57	13.67	13.73	13.81	13.86	13.89	13.92	13.97	14.02	14.05	14.09	14.10	14.13	14.17	14.20	14.21	14.26	14.28	14.29

Table A.6. The forecasted Belgium male life expectancy by model 3

age 0	97.5%	79.26	79.57	79.86	80.03	80.23	80.32	80.42	80.51	80.61	80.68	80.79	80.88	80.94	81.00	81.14	81.16	81.16	81.25	81.27	81.33
	mean	78.21	78.15	78.10	78.05	77.99	77.93	77.86	77.81	77.74	77.68	77.62	77.56	77.49	77.42	77.35	77.27	77.21	77.15	77.07	77.00
	2.5%	77.07	76.51	76.03	75.70	75.35	74.92	74.62	74.28	73.99	73.58	73.36	72.99	72.80	72.41	72.07	71.83	71.45	71.16	70.83	70.60
1-4	97.5%	78.49	78.80	79.08	79.24	79.43	79.51	79.61	79.69	79.78	79.85	79.96	80.05	80.10	80.16	80.29	80.31	80.31	80.39	80.42	80.47
	mean	77.51	77.45	77.40	77.35	77.29	77.23	77.17	77.12	77.06	77.00	76.94	76.88	76.81	76.74	76.68	76.61	76.54	76.48	76.41	76.35
	2.5%	76.43	75.89	75.44	75.12	74.78	74.39	74.10	73.80	73.51	73.12	72.91	72.55	72.38	72.03	71.70	71.46	71.11	70.80	70.49	70.31
5-9	97.5%	74.54	74.83	75.11	75.27	75.46	75.54	75.63	75.71	75.80	75.88	75.98	76.07	76.12	76.19	76.31	76.34	76.33	76.42	76.44	76.49
	mean	73.56	73.50	73.45	73.41	73.35	73.29	73.22	73.17	73.11	73.06	73.00	72.94	72.87	72.81	72.74	72.67	72.61	72.55	72.48	72.41
	2.5%	72.49	71.97	71.52	71.21	70.87	70.49	70.20	69.91	69.61	69.25	69.02	68.67	68.51	68.16	67.84	67.59	67.26	66.94	66.66	66.47
10-14	97.5%	69.56	69.85	70.13	70.29	70.47	70.55	70.65	70.73	70.82	70.89	71.00	71.09	71.13	71.20	71.32	71.35	71.34	71.43	71.45	71.51
	mean	68.59	68.53	68.49	68.44	68.39	68.33	68.26	68.21	68.15	68.10	68.04	67.98	67.92	67.85	67.79	67.72	67.66	67.60	67.53	67.47
	2.5%	67.53	67.02	66.58	66.27	65.94	65.56	65.28	65.00	64.70	64.34	64.13	63.80	63.62	63.28	62.95	62.71	62.39	62.10	61.80	61.62
15-19	97.5%	64.58	64.87	65.15	65.30	65.49	65.57	65.66	65.74	65.83	65.90	66.01	66.10	66.15	66.21	66.34	66.36	66.35	66.43	66.46	66.52
	mean	63.62	63.57	63.52	63.48	63.42	63.36	63.30	63.25	63.19	63.14	63.08	63.02	62.96	62.90	62.83	62.76	62.70	62.65	62.58	62.52
	2.5%	62.58	62.07	61.63	61.33	61.01	60.64	60.37	60.08	59.77	59.43	59.22	58.90	58.74	58.40	58.10	57.85	57.53	57.24	56.94	56.81
20-24	97.5%	59.66	59.94	60.21	60.36	60.54	60.62	60.72	60.78	60.88	60.94	61.06	61.15	61.19	61.26	61.38	61.39	61.39	61.48	61.51	61.55
	mean	58.72	58.66	58.62	58.57	58.52	58.47	58.41	58.36	58.30	58.25	58.20	58.14	58.08	58.02	57.96	57.89	57.83	57.78	57.71	57.65
	2.5%	57.70	57.20	56.79	56.49	56.18	55.83	55.56	55.29	55.00	54.66	54.46	54.15	53.99	53.66	53.37	53.16	52.84	52.56	52.28	52.12
25-29	97.5%	54.78	55.05	55.31	55.46	55.63	55.70	55.80	55.86	55.95	56.03	56.12	56.21	56.25	56.31	56.45	56.46	56.47	56.54	56.58	56.61
	mean	53.90	53.85	53.82	53.78	53.73	53.69	53.63	53.59	53.55	53.50	53.46	53.41	53.36	53.30	53.25	53.20	53.15	53.10	53.05	53.00
	2.5%	52.97	52.50	52.16	51.87	51.60	51.30	51.08	50.84	50.60	50.33	50.15	49.88	49.76	49.48	49.23	49.03	48.75	48.54	48.37	48.24
30-34	97.5%	49.90	50.16	50.41	50.55	50.71	50.78	50.88	50.94	51.03	51.10	51.19	51.28	51.33	51.39	51.51	51.52	51.55	51.59	51.65	51.67
	mean	49.07	49.03	48.99	48.96	48.92	48.88	48.83	48.80	48.76	48.72	48.68	48.64	48.60	48.55	48.51	48.46	48.42	48.38	48.34	48.29
	2.5%	48.18	47.77	47.45	47.20	46.97	46.70	46.50	46.28	46.10	45.86	45.70	45.50	45.37	45.14	44.94	44.72	44.57	44.37	44.25	44.10
35-39	97.5%	45.06	45.29	45.53	45.68	45.82	45.89	45.99	46.05	46.13	46.20	46.29	46.38	46.43	46.47	46.59	46.60	46.62	46.68	46.73	46.74
	mean	44.26	44.23	44.20	44.17	44.14	44.10	44.06	44.03	44.00	43.96	43.93	43.89	43.86	43.82	43.78	43.74	43.71	43.67	43.64	43.60
	2.5%	43.44	43.07	42.79	42.56	42.34	42.13	41.94	41.75	41.57	41.38	41.23	41.08	40.94	40.76	40.61	40.41	40.29	40.12	39.99	39.88
40-44	97.5%	40.24	40.47	40.69	40.83	40.97	41.03	41.13	41.18	41.26	41.33	41.43	41.49	41.55	41.59	41.71	41.73	41.75	41.78	41.84	41.87
	mean	39.49	39.46	39.43	39.41	39.38	39.35	39.31	39.29	39.26	39.23	39.20	39.16	39.13	39.10	39.07	39.03	39.00	38.97	38.94	38.91
	2.5%	38.71	38.38	38.13	37.91	37.72	37.52	37.35	37.18	37.03	36.86	36.74	36.61	36.50	36.30	36.19	36.02	35.92	35.76	35.68	35.56
45-49	97.5%	35.49	35.70	35.90	36.04	36.16	36.23	36.33	36.37	36.44	36.52	36.60	36.67	36.72	36.76	36.87	36.88	36.92	36.96	36.99	37.02
	mean	34.79	34.76	34.74	34.72	34.69	34.67	34.64	34.61	34.59	34.56	34.54	34.51	34.49	34.46	34.43	34.40	34.37	34.35	34.32	34.30
	2.5%	34.06	33.77	33.54	33.35	33.18	33.02	32.87	32.74	32.61	32.45	32.34	32.22	32.14	31.96	31.86	31.75	31.64	31.51	31.41	31.32
50-54	97.5%	30.87	31.06	31.25	31.38	31.50	31.57	31.65	31.69	31.75	31.83	31.90	31.97	32.02	32.07	32.15	32.18	32.21	32.26	32.29	32.30
	mean	30.21	30.18	30.17	30.15	30.13	30.10	30.07	30.05	30.03	30.01	29.99	29.96	29.94	29.91	29.89	29.86	29.84	29.82	29.79	29.77
	2.5%	29.54	29.27	29.05	28.89	28.72	28.56	28.45	28.34	28.19	28.07	27.98	27.87	27.77	27.63	27.54	27.45	27.33	27.23	27.14	27.08
55-59	97.5%	26.44	26.62	26.79	26.92	27.03	27.09	27.16	27.20	27.27	27.33	27.41	27.47	27.52	27.56	27.63	27.65	27.68	27.73	27.76	27.78
	mean	25.83	25.81	25.79	25.78	25.75	25.73	25.71	25.69	25.67	25.65	25.63	25.60	25.58	25.56	25.54	25.51	25.49	25.47	25.45	25.43
	2.5%	25.21	24.96	24.77	24.62	24.48	24.32	24.21	24.10	23.97	23.88	23.80	23.69	23.62	23.49	23.40	23.32	23.21	23.10	23.02	22.98
60-64	97.5%	22.28	22.44	22.59	22.71	22.82	22.87	22.94	22.99	23.04	23.09	23.17	23.21	23.26	23.30	23.36	23.38	23.41	23.46	23.49	23.51
	mean	21.72	21.70	21.68	21.67	21.65	21.63	21.61	21.59	21.57	21.55	21.53	21.51	21.49	21.47	21.45	21.43	21.41	21.39	21.37	21.35
	2.5%	21.15	20.92	20.75	20.62	20.48	20.35	20.24	20.17	20.04	19.96	19.86	19.76	19.69	19.59	19.50	19.42	19.34	19.26	19.18	19.12
65-69	97.5%	18.37	18.52	18.65	18.75	18.85	18.89	18.97	19.00	19.04	19.09	19.16	19.21	19.26	19.27	19.33	19.35	19.39	19.43	19.44	19.47
	mean	17.86	17.84	17.83	17.81	17.79	17.78	17.75	17.74	17.72	17.70	17.69	17.67	17.65	17.63	17.61	17.59	17.57	17.56	17.54	17.52
	2.5%	17.34	17.14	16.99	16.86	16.75	16.64	16.52	16.46	16.34	16.27	16.19	16.12	16.06	15.94	15.89	15.81	15.73	15.64	15.59	15.56
70-74	97.5%	14.71	14.85	14.96	15.05	15.13	15.16	15.24	15.26	15.31	15.33	15.41	15.46	15.47	15.50	15.55	15.58	15.59	15.64	15.64	15.67
	mean	14.27	14.25	14.24	14.22	14.21	14.19	14.17	14.16	14.14	14.12	14.11	14.09	14.08	14.06	14.04	14.02	14.01	13.99	13.98	13.96
	2.5%	13.81	13.63	13.50	13.39	13.29	13.19	13.10	13.02	12.95	12.86	12.80	12.74	12.68	12.59	12.54	12.48	12.42	12.34	12.31	12.26
75-79	97.5%	11.33	11.44	11.53	11.61	11.67	11.69	11.75	11.77	11.80	11.85	11.89	11.92	11.95	11.97	12.02	12.03				

Table A.7. The forecasted Belgium female life expectancy by model 8

age 0	97.5%	83.51	83.80	84.12	84.40	84.63	84.87	85.13	85.38	85.63	85.85	86.07	86.32	86.57	86.75	87.00	87.20	87.41	87.62	87.80	87.97
	mean	82.80	82.97	83.15	83.34	83.51	83.69	83.86	84.05	84.22	84.40	84.57	84.75	84.92	85.09	85.25	85.42	85.59	85.76	85.92	86.08
	2.5%	82.04	82.09	82.12	82.21	82.26	82.34	82.47	82.57	82.66	82.77	82.90	83.00	83.06	83.21	83.29	83.45	83.57	83.70	83.84	83.90
1-4	97.5%	82.73	83.00	83.30	83.55	83.78	84.01	84.25	84.50	84.74	84.96	85.16	85.40	85.65	85.83	86.07	86.26	86.47	86.69	86.85	87.02
	mean	82.05	82.21	82.38	82.56	82.72	82.89	83.06	83.24	83.40	83.57	83.74	83.91	84.07	84.24	84.40	84.56	84.72	84.88	85.04	85.19
	2.5%	81.33	81.37	81.43	81.50	81.56	81.64	81.76	81.85	81.93	82.06	82.16	82.26	82.33	82.48	82.56	82.72	82.79	82.93	83.05	83.11
5-9	97.5%	78.77	79.04	79.33	79.58	79.80	80.03	80.27	80.52	80.76	80.98	81.19	81.41	81.66	81.84	82.08	82.27	82.48	82.70	82.86	83.03
	mean	78.10	78.26	78.43	78.60	78.76	78.93	79.10	79.28	79.44	79.61	79.77	79.94	80.10	80.26	80.42	80.59	80.75	80.91	81.06	81.22
	2.5%	77.39	77.44	77.49	77.56	77.62	77.71	77.82	77.91	77.99	78.12	78.22	78.31	78.37	78.53	78.60	78.76	78.83	78.96	79.10	79.16
10-14	97.5%	73.79	74.06	74.36	74.61	74.82	75.05	75.28	75.53	75.77	76.00	76.20	76.42	76.67	76.85	77.09	77.28	77.49	77.70	77.87	78.04
	mean	73.13	73.29	73.46	73.63	73.79	73.96	74.12	74.30	74.46	74.63	74.79	74.96	75.12	75.28	75.44	75.60	75.76	75.92	76.08	76.23
	2.5%	72.43	72.47	72.52	72.60	72.66	72.75	72.86	72.94	73.02	73.15	73.25	73.35	73.42	73.56	73.63	73.79	73.87	73.99	74.13	74.18
15-19	97.5%	68.82	69.09	69.38	69.63	69.84	70.07	70.30	70.55	70.79	71.01	71.22	71.43	71.69	71.86	72.10	72.29	72.51	72.71	72.87	73.05
	mean	68.16	68.32	68.49	68.66	68.82	68.99	69.15	69.33	69.49	69.66	69.82	69.98	70.14	70.31	70.47	70.62	70.78	70.94	71.10	71.25
	2.5%	67.47	67.51	67.56	67.64	67.70	67.78	67.90	67.99	68.07	68.20	68.30	68.40	68.46	68.61	68.68	68.83	68.92	69.04	69.17	69.23
20-24	97.5%	63.87	64.13	64.42	64.66	64.87	65.11	65.33	65.57	65.82	66.04	66.24	66.46	66.71	66.88	67.12	67.31	67.52	67.72	67.88	68.06
	mean	63.22	63.38	63.54	63.71	63.87	64.04	64.20	64.38	64.53	64.70	64.87	65.03	65.19	65.35	65.51	65.67	65.82	65.98	66.14	66.29
	2.5%	62.54	62.59	62.63	62.72	62.78	62.87	62.97	63.07	63.16	63.27	63.38	63.47	63.54	63.68	63.77	63.90	64.01	64.11	64.24	64.32
25-29	97.5%	58.91	59.18	59.46	59.70	59.91	60.14	60.36	60.60	60.84	61.06	61.26	61.48	61.73	61.90	62.14	62.33	62.53	62.74	62.90	63.07
	mean	58.27	58.43	58.59	58.76	58.92	59.08	59.24	59.42	59.58	59.74	59.90	60.06	60.22	60.38	60.54	60.70	60.86	61.01	61.17	61.32
	2.5%	57.60	57.65	57.70	57.78	57.84	57.93	58.04	58.13	58.23	58.34	58.44	58.53	58.61	58.75	58.83	58.96	59.07	59.18	59.30	59.38
30-34	97.5%	53.98	54.23	54.51	54.75	54.96	55.20	55.41	55.65	55.88	56.10	56.30	56.52	56.76	56.93	57.17	57.36	57.56	57.76	57.92	58.09
	mean	53.35	53.50	53.67	53.83	53.99	54.15	54.31	54.48	54.64	54.80	54.96	55.12	55.28	55.44	55.59	55.75	55.91	56.06	56.21	56.36
	2.5%	52.69	52.74	52.80	52.87	52.93	53.02	53.13	53.22	53.32	53.44	53.53	53.62	53.71	53.83	53.91	54.05	54.15	54.27	54.38	54.46
35-39	97.5%	49.06	49.31	49.59	49.82	50.02	50.25	50.46	50.70	50.94	51.15	51.34	51.56	51.80	51.97	52.21	52.39	52.59	52.80	52.95	53.12
	mean	48.44	48.59	48.75	48.92	49.07	49.23	49.39	49.56	49.71	49.87	50.03	50.19	50.34	50.50	50.66	50.81	50.96	51.12	51.27	51.42
	2.5%	47.79	47.84	47.90	47.97	48.03	48.12	48.23	48.32	48.43	48.54	48.64	48.73	48.80	48.93	49.01	49.15	49.25	49.36	49.48	49.56
40-44	97.5%	44.18	44.42	44.69	44.92	45.12	45.35	45.55	45.78	46.01	46.23	46.42	46.64	46.87	47.03	47.27	47.45	47.65	47.86	48.00	48.16
	mean	43.57	43.72	43.88	44.04	44.19	44.34	44.50	44.67	44.82	44.98	45.13	45.29	45.44	45.60	45.75	45.90	46.05	46.20	46.35	46.50
	2.5%	42.93	42.98	43.04	43.11	43.19	43.26	43.37	43.46	43.56	43.68	43.79	43.85	43.93	44.06	44.14	44.27	44.38	44.47	44.60	44.67
45-49	97.5%	39.37	39.60	39.86	40.07	40.27	40.50	40.70	40.92	41.15	41.35	41.55	41.76	41.97	42.14	42.39	42.55	42.75	42.95	43.11	43.26
	mean	38.77	38.92	39.07	39.23	39.37	39.53	39.68	39.84	39.99	40.15	40.30	40.45	40.60	40.75	40.90	41.05	41.20	41.34	41.49	41.63
	2.5%	38.15	38.20	38.26	38.33	38.41	38.48	38.59	38.68	38.79	38.89	39.00	39.07	39.16	39.28	39.36	39.48	39.57	39.68	39.79	39.86
50-54	97.5%	34.66	34.88	35.13	35.34	35.53	35.73	35.94	36.14	36.38	36.57	36.76	36.96	37.17	37.33	37.57	37.71	37.90	38.11	38.26	38.43
	mean	34.08	34.22	34.37	34.53	34.67	34.82	34.97	35.12	35.27	35.42	35.57	35.71	35.86	36.01	36.15	36.30	36.44	36.59	36.73	36.86
	2.5%	33.49	33.55	33.61	33.68	33.75	33.84	33.94	34.04	34.13	34.23	34.34	34.41	34.51	34.62	34.69	34.79	34.89	35.02	35.12	35.18
55-59	97.5%	30.09	30.29	30.52	30.72	30.91	31.11	31.31	31.51	31.72	31.90	32.09	32.29	32.47	32.64	32.85	33.00	33.18	33.38	33.53	33.69
	mean	29.54	29.67	29.82	29.97	30.10	30.25	30.39	30.54	30.68	30.83	30.97	31.11	31.25	31.39	31.53	31.67	31.81	31.95	32.08	32.22
	2.5%	28.98	29.04	29.10	29.17	29.25	29.34	29.43	29.53	29.62	29.72	29.83	29.90	30.00	30.10	30.18	30.26	30.39	30.48	30.58	30.68
60-64	97.5%	25.69	25.87	26.10	26.29	26.47	26.66	26.84	27.03	27.22	27.40	27.58	27.77	27.94	28.10	28.30	28.45	28.61	28.81	28.94	29.09
	mean	25.18	25.30	25.44	25.58	25.72	25.86	25.99	26.14	26.27	26.41	26.55	26.68	26.82	26.95	27.09	27.22	27.35	27.48	27.61	27.74
	2.5%	24.66	24.71	24.77	24.86	24.93	25.01	25.10	25.19	25.29	25.39	25.48	25.57	25.65	25.77	25.86	25.96	26.04	26.12	26.24	26.32
65-69	97.5%	21.44	21.61	21.82	22.00	22.16	22.33	22.51	22.68	22.86	23.02	23.20	23.35	23.52	23.69	23.86	24.00	24.16	24.32	24.46	24.58
	mean	20.96	21.08	21.21	21.34	21.47	21.60	21.72	21.86	21.98	22.11	22.24	22.37	22.49	22.62	22.74	22.86	22.99	23.11	23.23	23.34
	2.5%	20.49	20.52	20.59	20.67	20.74	20.82	20.89	20.99	21.07	21.17	21.27	21.34	21.44	21.51	21.61	21.70	21.78	21.86	21.97	22.03
70-74	97.5%	17.32	17.47	17.64	17.82	17.97	18.12	18.27	18.42	18.58	18.73	18.89	19.01	19.17	19.32	19.47	19.60	19.74	19.86	20.00	20.11
	mean	16.87	16.98	17.09	17.21	17.32	17.43	17.54	17.66	17.77	17.88	17.99	18.10	18.21	18.32	18.43	18.53	18.64	18.75	18.85	18.95
	2.5%	16.42	16.47	16.52	16.60	16.66	16.72	16.78	16.87	16.95	17.01	17.11	17.17	17.24	17.31	17.38	17.47	17.54	17.61	17.69	17.76
75-79	97.5%	13.37	13.50	13.64	13.79	13.91	14.03	14.16	14.28	14.42	14.53	14.67	14.78	14.90	15.02	15.15	15.25	15.36	15.46	15.58	15.

Table A.8. The forecasted Belgium male life expectancy by model 8

age 0	97.5%	78.77	79.17	79.52	79.86	80.20	80.49	80.79	81.11	81.44	81.72	81.98	82.28	82.59	82.85	83.15	83.40	83.65	83.98	84.19	84.42
	mean	77.83	78.02	78.21	78.42	78.61	78.81	79.01	79.22	79.41	79.61	79.81	80.00	80.20	80.40	80.59	80.78	80.97	81.17	81.35	81.53
	2.5%	76.79	76.80	76.78	76.83	76.86	76.93	76.97	77.08	77.14	77.24	77.28	77.36	77.48	77.62	77.74	77.75	77.88	78.02	78.10	78.15
1-4	97.5%	78.00	78.38	78.72	79.04	79.36	79.65	79.93	80.24	80.56	80.82	81.08	81.37	81.68	81.93	82.22	82.48	82.71	83.03	83.24	83.47
	mean	77.09	77.28	77.46	77.65	77.84	78.02	78.22	78.42	78.60	78.80	78.98	79.17	79.36	79.55	79.73	79.92	80.11	80.29	80.47	80.64
	2.5%	76.10	76.11	76.09	76.13	76.15	76.23	76.27	76.36	76.44	76.52	76.55	76.62	76.73	76.89	76.96	76.98	77.14	77.24	77.28	77.39
5-9	97.5%	74.05	74.43	74.76	75.09	75.40	75.68	75.96	76.27	76.59	76.85	77.10	77.39	77.69	77.95	78.24	78.48	78.73	79.04	79.25	79.48
	mean	73.16	73.34	73.52	73.71	73.90	74.08	74.27	74.47	74.65	74.84	75.03	75.21	75.40	75.59	75.77	75.95	76.14	76.33	76.50	76.67
	2.5%	72.17	72.19	72.17	72.22	72.22	72.30	72.34	72.43	72.50	72.60	72.62	72.71	72.80	72.95	73.03	73.06	73.19	73.31	73.34	73.45
10-14	97.5%	69.08	69.45	69.78	70.11	70.42	70.70	70.97	71.28	71.60	71.87	72.11	72.40	72.71	72.96	73.25	73.49	73.73	74.04	74.26	74.49
	mean	68.19	68.37	68.55	68.74	68.92	69.10	69.29	69.49	69.67	69.86	70.04	70.23	70.42	70.60	70.78	70.97	71.15	71.34	71.51	71.69
	2.5%	67.21	67.22	67.20	67.25	67.26	67.33	67.37	67.48	67.54	67.62	67.65	67.74	67.82	67.98	68.05	68.08	68.21	68.33	68.36	68.48
15-19	97.5%	64.11	64.48	64.80	65.14	65.44	65.72	65.99	66.30	66.61	66.88	67.13	67.41	67.72	67.98	68.26	68.50	68.74	69.05	69.27	69.50
	mean	63.23	63.40	63.58	63.77	63.95	64.13	64.32	64.51	64.70	64.89	65.07	65.25	65.44	65.62	65.80	65.98	66.17	66.36	66.53	66.70
	2.5%	62.26	62.26	62.24	62.29	62.30	62.38	62.41	62.52	62.58	62.66	62.68	62.77	62.85	63.02	63.08	63.11	63.23	63.35	63.40	63.51
20-24	97.5%	59.19	59.56	59.88	60.20	60.51	60.78	61.05	61.37	61.66	61.92	62.18	62.47	62.76	63.02	63.31	63.53	63.78	64.09	64.29	64.53
	mean	58.32	58.50	58.68	58.86	59.04	59.22	59.40	59.60	59.78	59.96	60.14	60.32	60.51	60.69	60.87	61.05	61.24	61.42	61.59	61.76
	2.5%	57.37	57.39	57.36	57.40	57.41	57.49	57.53	57.63	57.69	57.78	57.81	57.89	57.98	58.14	58.19	58.23	58.36	58.43	58.52	58.60
25-29	97.5%	54.34	54.69	55.01	55.33	55.62	55.89	56.16	56.47	56.76	57.02	57.27	57.55	57.85	58.10	58.39	58.60	58.84	59.14	59.35	59.59
	mean	53.50	53.67	53.85	54.03	54.21	54.38	54.57	54.76	54.94	55.12	55.30	55.48	55.66	55.84	56.02	56.20	56.38	56.56	56.73	56.90
	2.5%	52.59	52.60	52.60	52.62	52.67	52.74	52.78	52.88	52.95	53.03	53.05	53.13	53.25	53.39	53.46	53.50	53.60	53.70	53.77	53.86
30-34	97.5%	49.49	49.84	50.15	50.46	50.73	51.02	51.28	51.58	51.87	52.12	52.37	52.64	52.93	53.19	53.46	53.69	53.93	54.23	54.43	54.66
	mean	48.68	48.85	49.02	49.20	49.37	49.55	49.73	49.92	50.09	50.28	50.45	50.63	50.81	50.99	51.17	51.34	51.52	51.70	51.87	52.04
	2.5%	47.79	47.82	47.81	47.85	47.89	47.97	48.01	48.10	48.18	48.26	48.27	48.39	48.48	48.61	48.69	48.71	48.86	48.94	49.01	49.08
35-39	97.5%	44.66	45.00	45.31	45.62	45.88	46.15	46.43	46.71	46.97	47.25	47.49	47.76	48.05	48.29	48.58	48.78	49.03	49.32	49.54	49.75
	mean	43.88	44.05	44.22	44.40	44.57	44.74	44.92	45.11	45.28	45.46	45.64	45.81	45.99	46.17	46.34	46.52	46.70	46.88	47.04	47.21
	2.5%	43.02	43.06	43.05	43.10	43.14	43.22	43.28	43.36	43.44	43.53	43.58	43.66	43.76	43.89	43.97	44.02	44.11	44.21	44.28	44.38
40-44	97.5%	39.85	40.19	40.48	40.78	41.04	41.31	41.59	41.85	42.12	42.37	42.61	42.89	43.17	43.41	43.67	43.89	44.12	44.42	44.64	44.85
	mean	39.09	39.26	39.43	39.60	39.77	39.94	40.12	40.30	40.47	40.65	40.82	40.99	41.17	41.35	41.52	41.69	41.87	42.04	42.20	42.37
	2.5%	38.27	38.30	38.30	38.36	38.40	38.48	38.54	38.61	38.71	38.79	38.84	38.93	39.01	39.11	39.21	39.26	39.36	39.47	39.53	39.63
45-49	97.5%	35.13	35.44	35.74	36.02	36.26	36.53	36.79	37.06	37.33	37.57	37.81	38.07	38.34	38.58	38.84	39.07	39.30	39.56	39.79	39.98
	mean	34.39	34.56	34.72	34.89	35.06	35.23	35.40	35.58	35.75	35.92	36.09	36.27	36.44	36.61	36.78	36.95	37.12	37.30	37.46	37.62
	2.5%	33.61	33.62	33.64	33.71	33.77	33.83	33.90	33.99	34.08	34.18	34.23	34.32	34.39	34.51	34.59	34.62	34.76	34.85	34.96	35.00
50-54	97.5%	30.54	30.83	31.11	31.40	31.62	31.88	32.14	32.40	32.65	32.89	33.13	33.39	33.65	33.89	34.14	34.36	34.57	34.84	35.05	35.27
	mean	29.85	30.01	30.18	30.35	30.52	30.68	30.86	31.04	31.21	31.38	31.55	31.72	31.90	32.07	32.24	32.41	32.58	32.76	32.92	33.08
	2.5%	29.09	29.13	29.18	29.27	29.35	29.43	29.50	29.60	29.70	29.79	29.85	29.98	30.04	30.17	30.28	30.33	30.44	30.58	30.67	30.74
55-59	97.5%	26.20	26.46	26.73	26.99	27.22	27.44	27.68	27.94	28.19	28.42	28.65	28.90	29.13	29.39	29.61	29.82	30.02	30.27	30.48	30.69
	mean	25.55	25.71	25.87	26.04	26.20	26.36	26.53	26.71	26.87	27.04	27.21	27.37	27.54	27.71	27.88	28.04	28.21	28.37	28.53	28.69
	2.5%	24.86	24.91	24.95	25.07	25.16	25.23	25.32	25.41	25.52	25.62	25.69	25.80	25.93	26.03	26.12	26.21	26.32	26.43	26.50	26.57
60-64	97.5%	22.12	22.37	22.61	22.86	23.08	23.28	23.52	23.75	23.98	24.19	24.41	24.64	24.86	25.10	25.30	25.51	25.72	25.91	26.12	26.33
	mean	21.53	21.68	21.84	22.00	22.15	22.31	22.47	22.63	22.79	22.95	23.11	23.27	23.42	23.58	23.74	23.89	24.05	24.20	24.35	24.50
	2.5%	20.91	20.97	21.02	21.10	21.19	21.27	21.34	21.45	21.57	21.67	21.74	21.86	21.95	22.05	22.17	22.21	22.32	22.44	22.54	22.62
65-69	97.5%	18.28	18.50	18.72	18.94	19.14	19.32	19.54	19.75	19.98	20.17	20.38	20.59	20.77	20.98	21.16	21.35	21.55	21.74	21.91	22.10
	mean	17.73	17.88	18.02	18.17	18.31	18.45	18.60	18.75	18.89	19.04	19.19	19.33	19.47	19.62	19.76	19.90	20.04	20.18	20.32	20.45
	2.5%	17.16	17.22	17.27	17.37	17.45	17.51	17.59	17.69	17.80	17.90	17.98	18.09	18.17	18.26	18.32	18.42	18.53	18.65	18.72	18.80
70-74	97.5%	14.70	14.88	15.06	15.24	15.42	15.58	15.75	15.94	16.11	16.29	16.47	16.62	16.79	16.96	17.11	17.25	17.42	17.58	17.72	17.90
	mean	14.20	14.31	14.43	14.55	14.67	14.79	14.91	15.03	15.15	15.27	15.39	15.51	15.62	15.74	15.85	15.96	16.07	16.19	16.29	16.39
	2.5%	13.69	13.74	13.79	13.85	13.92	13.96	14.04	14.09	14.18	14.26	14.32	14.38	14.47	14.54	14.62	14.65	14.74	14.85	14.90	14.93
75-79	97.5%	11.34	11.48	11.62	11.73	11.87	11.98	12.11	12.24	12.37	12.48	12.61	12.73	12.83	12.96	13.06	13.				

Table A.9. The forecasted Canada female life expectancy by model 3

age 0	97.5%	83.97	84.20	84.39	84.52	84.66	84.75	84.80	84.90	84.98	85.05	85.14	85.20	85.24	85.33	85.40	85.44	85.47	85.52	85.57	85.60
	mean	83.30	83.28	83.26	83.25	83.22	83.20	83.17	83.15	83.13	83.11	83.09	83.06	83.04	83.01	82.99	82.96	82.94	82.91	82.89	82.86
	2.5%	82.61	82.28	82.03	81.86	81.66	81.46	81.30	81.19	81.04	80.88	80.76	80.59	80.50	80.37	80.22	80.13	80.02	79.90	79.77	79.66
1-4	97.5%	83.31	83.51	83.69	83.82	83.95	84.04	84.08	84.17	84.24	84.31	84.39	84.45	84.50	84.58	84.64	84.68	84.71	84.75	84.80	84.82
	mean	82.67	82.65	82.63	82.62	82.60	82.57	82.55	82.53	82.51	82.49	82.47	82.44	82.42	82.40	82.37	82.35	82.33	82.31	82.28	82.26
	2.5%	82.02	81.73	81.49	81.32	81.14	80.96	80.81	80.70	80.57	80.43	80.31	80.15	80.08	79.95	79.82	79.72	79.63	79.51	79.39	79.32
5-9	97.5%	79.35	79.55	79.73	79.85	79.98	80.07	80.11	80.20	80.27	80.34	80.42	80.47	80.52	80.60	80.66	80.70	80.74	80.77	80.82	80.85
	mean	78.72	78.70	78.68	78.67	78.65	78.63	78.60	78.58	78.56	78.54	78.52	78.50	78.48	78.45	78.43	78.40	78.38	78.37	78.34	78.32
	2.5%	78.08	77.79	77.56	77.39	77.22	77.04	76.89	76.79	76.65	76.51	76.40	76.25	76.18	76.06	75.94	75.82	75.74	75.63	75.51	75.43
10-14	97.5%	74.37	74.58	74.75	74.87	75.00	75.09	75.13	75.21	75.29	75.36	75.44	75.49	75.54	75.62	75.68	75.72	75.75	75.79	75.84	75.86
	mean	73.75	73.73	73.72	73.70	73.68	73.66	73.63	73.62	73.60	73.58	73.56	73.53	73.51	73.49	73.47	73.44	73.42	73.40	73.38	73.36
	2.5%	73.12	72.83	72.60	72.44	72.27	72.09	71.95	71.84	71.71	71.57	71.46	71.32	71.25	71.12	71.00	70.90	70.81	70.71	70.59	70.51
15-19	97.5%	69.40	69.61	69.78	69.90	70.03	70.11	70.15	70.24	70.32	70.38	70.46	70.51	70.56	70.64	70.70	70.74	70.77	70.81	70.86	70.88
	mean	68.79	68.77	68.75	68.74	68.72	68.70	68.67	68.65	68.63	68.62	68.60	68.57	68.55	68.53	68.51	68.48	68.46	68.44	68.42	68.40
	2.5%	68.16	67.88	67.65	67.49	67.32	67.14	67.01	66.90	66.76	66.63	66.52	66.39	66.31	66.20	66.07	65.97	65.88	65.78	65.67	65.59
20-24	97.5%	64.48	64.67	64.85	64.96	65.09	65.17	65.22	65.30	65.37	65.44	65.51	65.57	65.61	65.69	65.76	65.79	65.82	65.86	65.92	65.93
	mean	63.87	63.85	63.83	63.82	63.80	63.78	63.76	63.74	63.72	63.70	63.68	63.66	63.64	63.62	63.60	63.57	63.55	63.53	63.51	63.49
	2.5%	63.26	62.98	62.76	62.60	62.44	62.26	62.12	62.02	61.90	61.77	61.65	61.52	61.45	61.33	61.22	61.13	61.02	60.94	60.82	60.75
25-29	97.5%	59.56	59.75	59.92	60.03	60.16	60.24	60.28	60.36	60.44	60.50	60.59	60.63	60.68	60.76	60.82	60.85	60.88	60.92	60.97	60.99
	mean	58.96	58.94	58.93	58.92	58.90	58.88	58.85	58.84	58.82	58.80	58.78	58.76	58.74	58.72	58.70	58.67	58.65	58.64	58.61	58.59
	2.5%	58.36	58.09	57.87	57.72	57.56	57.39	57.26	57.16	57.03	56.90	56.80	56.67	56.61	56.49	56.39	56.29	56.19	56.11	56.01	55.93
30-34	97.5%	54.64	54.83	54.99	55.11	55.23	55.31	55.35	55.43	55.50	55.57	55.65	55.70	55.74	55.82	55.88	55.91	55.94	55.98	56.03	56.04
	mean	54.06	54.04	54.02	54.01	53.99	53.97	53.95	53.93	53.91	53.90	53.88	53.86	53.84	53.82	53.79	53.77	53.75	53.74	53.71	53.70
	2.5%	53.47	53.20	52.99	52.83	52.68	52.52	52.38	52.29	52.16	52.04	51.94	51.81	51.75	51.65	51.53	51.45	51.35	51.28	51.16	51.08
35-39	97.5%	49.74	49.93	50.09	50.20	50.32	50.39	50.44	50.52	50.59	50.65	50.73	50.78	50.82	50.90	50.96	50.99	51.01	51.06	51.10	51.12
	mean	49.17	49.15	49.14	49.12	49.11	49.09	49.06	49.05	49.03	49.01	49.00	48.98	48.96	48.93	48.92	48.89	48.87	48.86	48.84	48.82
	2.5%	48.59	48.33	48.13	47.97	47.82	47.67	47.54	47.44	47.33	47.20	47.11	46.98	46.92	46.82	46.71	46.62	46.54	46.45	46.35	46.27
40-44	97.5%	44.87	45.05	45.21	45.32	45.44	45.51	45.56	45.62	45.70	45.76	45.84	45.89	45.92	46.00	46.06	46.09	46.12	46.16	46.21	46.22
	mean	44.31	44.29	44.28	44.27	44.25	44.23	44.21	44.19	44.17	44.16	44.14	44.12	44.10	44.08	44.06	44.04	44.02	44.01	43.99	43.97
	2.5%	43.75	43.49	43.30	43.14	43.00	42.84	42.72	42.63	42.51	42.39	42.30	42.18	42.12	42.02	41.92	41.82	41.75	41.66	41.57	41.49
45-49	97.5%	40.06	40.24	40.38	40.49	40.60	40.68	40.72	40.79	40.85	40.92	40.99	41.03	41.07	41.15	41.20	41.23	41.26	41.30	41.34	41.36
	mean	39.52	39.50	39.49	39.48	39.46	39.44	39.42	39.41	39.39	39.37	39.36	39.34	39.32	39.30	39.28	39.26	39.24	39.23	39.21	39.19
	2.5%	38.98	38.72	38.55	38.41	38.27	38.11	38.00	37.91	37.80	37.69	37.60	37.48	37.44	37.33	37.23	37.15	37.09	37.00	36.90	36.84
50-54	97.5%	35.34	35.50	35.65	35.75	35.86	35.93	35.97	36.03	36.10	36.16	36.22	36.26	36.29	36.37	36.44	36.45	36.49	36.52	36.56	36.58
	mean	34.82	34.80	34.79	34.78	34.76	34.75	34.73	34.71	34.70	34.68	34.67	34.65	34.63	34.61	34.60	34.58	34.56	34.55	34.53	34.51
	2.5%	34.29	34.06	33.89	33.76	33.63	33.49	33.38	33.29	33.18	33.08	32.99	32.91	32.84	32.74	32.65	32.58	32.51	32.43	32.34	32.28
55-59	97.5%	30.72	30.88	31.02	31.12	31.22	31.29	31.33	31.39	31.45	31.50	31.57	31.61	31.65	31.71	31.76	31.78	31.81	31.86	31.89	31.91
	mean	30.24	30.22	30.21	30.20	30.19	30.17	30.15	30.14	30.13	30.11	30.10	30.08	30.07	30.05	30.03	30.01	30.00	29.98	29.97	29.95
	2.5%	29.75	29.53	29.37	29.25	29.11	28.99	28.89	28.80	28.71	28.61	28.53	28.44	28.38	28.28	28.21	28.14	28.08	28.00	27.92	27.86
60-64	97.5%	26.24	26.39	26.52	26.61	26.71	26.77	26.79	26.85	26.92	26.96	27.03	27.06	27.10	27.16	27.20	27.22	27.25	27.29	27.31	27.35
	mean	25.79	25.77	25.76	25.75	25.74	25.72	25.71	25.69	25.68	25.67	25.66	25.64	25.63	25.61	25.59	25.58	25.56	25.55	25.54	25.52
	2.5%	25.32	25.12	24.98	24.86	24.74	24.63	24.55	24.46	24.37	24.27	24.20	24.13	24.05	23.99	23.92	23.83	23.76	23.72	23.66	23.58
65-69	97.5%	21.91	22.05	22.17	22.24	22.33	22.39	22.42	22.46	22.52	22.58	22.64	22.67	22.70	22.74	22.79	22.80	22.83	22.87	22.89	22.92
	mean	21.49	21.48	21.47	21.46	21.45	21.43	21.42	21.41	21.40	21.38	21.37	21.36	21.35	21.33	21.32	21.30	21.29	21.28	21.26	21.25
	2.5%	21.07	20.88	20.76	20.65	20.53	20.44	20.35	20.29	20.20	20.12	20.04	19.98	19.92	19.84	19.80	19.71	19.65	19.61	19.56	19.49
70-74	97.5%	17.80	17.91	18.02	18.09	18.17	18.21	18.24	18.28	18.32	18.38	18.43	18.45	18.48	18.54	18.57	18.58	18.59	18.63	18.65	18.69
	mean	17.42	17.40	17.40	17.39	17.38	17.36	17.35	17.34	17.33	17.32	17.31	17.30	17.28	17.27	17.26	17.24	17.23	17.22	17.21	17.20
	2.5%	17.04	16.87	16.77	16.67	16.57	16.49	16.41	16.34	16.27	16.21	16.14	16.07	16.04	15.96	15.90	15.84	15.80	15.75	15.71	15.63
75-79	97.5%	13.94	14.03	14.13	14.19	14.25	14.28	14.32	14.35	14.38	14.42	14.48	14.50	14.52	14.56	14.59	14.60	14.61	14.64	14.66	14.6

Table A.10. The forecasted Canada male life expectancy by model 3

age 0	97.5%	79.84	80.09	80.29	80.45	80.62	80.68	80.75	80.82	80.92	81.00	81.08	81.11	81.17	81.23	81.32	81.37	81.37	81.47	81.50	81.52
	mean	79.10	79.07	79.04	79.02	78.99	78.96	78.92	78.90	78.87	78.84	78.81	78.78	78.75	78.71	78.68	78.65	78.62	78.59	78.56	78.53
	2.5%	78.34	77.97	77.71	77.52	77.29	77.09	76.92	76.77	76.60	76.44	76.32	76.16	76.03	75.88	75.71	75.63	75.48	75.34	75.23	75.11
1-4	97.5%	79.21	79.44	79.64	79.78	79.93	79.99	80.07	80.13	80.22	80.29	80.38	80.39	80.47	80.52	80.60	80.66	80.65	80.72	80.77	80.78
	mean	78.52	78.49	78.47	78.44	78.42	78.39	78.35	78.33	78.30	78.27	78.25	78.22	78.19	78.15	78.13	78.09	78.07	78.04	78.01	77.98
	2.5%	77.80	77.48	77.23	77.05	76.84	76.67	76.50	76.36	76.20	76.07	75.92	75.79	75.68	75.55	75.37	75.32	75.16	75.06	74.95	74.84
5-9	97.5%	75.26	75.49	75.68	75.83	75.98	76.03	76.11	76.16	76.26	76.33	76.41	76.43	76.50	76.56	76.63	76.69	76.69	76.76	76.81	76.81
	mean	74.58	74.55	74.53	74.51	74.48	74.45	74.42	74.40	74.37	74.34	74.32	74.29	74.26	74.23	74.20	74.16	74.14	74.11	74.08	74.06
	2.5%	73.88	73.56	73.32	73.13	72.93	72.76	72.60	72.47	72.32	72.18	72.05	71.91	71.80	71.67	71.51	71.47	71.29	71.19	71.11	70.99
10-14	97.5%	70.29	70.52	70.71	70.85	71.01	71.05	71.13	71.19	71.28	71.35	71.43	71.45	71.53	71.58	71.65	71.70	71.70	71.78	71.83	71.83
	mean	69.62	69.59	69.57	69.54	69.52	69.49	69.46	69.43	69.40	69.38	69.35	69.32	69.29	69.26	69.23	69.20	69.18	69.15	69.12	69.10
	2.5%	68.92	68.60	68.36	68.18	67.99	67.82	67.65	67.52	67.38	67.24	67.11	66.97	66.86	66.74	66.57	66.54	66.36	66.25	66.18	66.07
15-19	97.5%	65.33	65.56	65.75	65.89	66.04	66.09	66.17	66.22	66.32	66.38	66.46	66.48	66.56	66.61	66.68	66.73	66.73	66.80	66.86	66.86
	mean	64.66	64.63	64.61	64.59	64.56	64.53	64.50	64.48	64.45	64.43	64.40	64.37	64.34	64.31	64.28	64.25	64.23	64.20	64.17	64.14
	2.5%	63.97	63.65	63.42	63.24	63.04	62.88	62.72	62.59	62.44	62.31	62.18	62.04	61.93	61.81	61.65	61.61	61.45	61.33	61.26	61.16
20-24	97.5%	60.47	60.69	60.88	61.01	61.15	61.21	61.29	61.34	61.43	61.49	61.57	61.60	61.67	61.72	61.79	61.84	61.84	61.91	61.96	61.96
	mean	59.82	59.79	59.77	59.75	59.72	59.69	59.66	59.64	59.61	59.59	59.56	59.53	59.50	59.47	59.45	59.42	59.39	59.37	59.34	59.31
	2.5%	59.15	58.83	58.60	58.44	58.24	58.09	57.93	57.81	57.66	57.53	57.41	57.27	57.17	57.05	56.89	56.86	56.69	56.59	56.53	56.42
25-29	97.5%	55.67	55.88	56.06	56.19	56.33	56.38	56.46	56.51	56.59	56.66	56.74	56.76	56.83	56.87	56.93	57.00	57.01	57.07	57.12	57.11
	mean	55.03	55.00	54.98	54.96	54.94	54.91	54.88	54.86	54.83	54.81	54.79	54.76	54.73	54.70	54.68	54.65	54.62	54.60	54.57	54.55
	2.5%	54.38	54.07	53.85	53.70	53.52	53.37	53.20	53.09	52.96	52.83	52.72	52.58	52.48	52.37	52.21	52.18	52.04	51.93	51.87	51.76
30-34	97.5%	50.85	51.06	51.23	51.35	51.50	51.55	51.61	51.67	51.75	51.82	51.89	51.91	51.98	52.03	52.08	52.15	52.15	52.21	52.25	52.25
	mean	50.23	50.20	50.18	50.16	50.14	50.11	50.08	50.06	50.03	50.01	49.99	49.96	49.93	49.91	49.88	49.85	49.83	49.80	49.78	49.75
	2.5%	49.60	49.30	49.08	48.93	48.75	48.60	48.45	48.34	48.21	48.07	47.97	47.84	47.75	47.63	47.49	47.45	47.31	47.20	47.15	47.05
35-39	97.5%	46.03	46.23	46.40	46.52	46.67	46.71	46.78	46.82	46.91	46.97	47.04	47.07	47.13	47.18	47.23	47.30	47.30	47.35	47.40	47.40
	mean	45.42	45.39	45.38	45.36	45.33	45.31	45.28	45.25	45.23	45.21	45.18	45.16	45.13	45.10	45.08	45.05	45.03	45.01	44.98	44.95
	2.5%	44.80	44.51	44.31	44.15	43.98	43.84	43.68	43.57	43.45	43.31	43.21	43.09	43.01	42.89	42.77	42.71	42.57	42.47	42.42	42.33
40-44	97.5%	41.24	41.44	41.60	41.71	41.85	41.89	41.96	42.01	42.08	42.14	42.22	42.25	42.30	42.35	42.41	42.46	42.46	42.53	42.58	42.57
	mean	40.65	40.63	40.61	40.59	40.57	40.54	40.51	40.49	40.47	40.45	40.42	40.40	40.37	40.35	40.32	40.29	40.27	40.25	40.23	40.20
	2.5%	40.06	39.77	39.57	39.43	39.26	39.12	38.98	38.88	38.76	38.63	38.53	38.41	38.33	38.23	38.10	38.06	37.93	37.82	37.77	37.68
45-49	97.5%	36.52	36.72	36.87	36.97	37.11	37.15	37.21	37.26	37.34	37.40	37.47	37.50	37.54	37.60	37.65	37.69	37.70	37.78	37.81	37.81
	mean	35.96	35.93	35.91	35.90	35.87	35.85	35.82	35.80	35.78	35.76	35.74	35.71	35.69	35.66	35.64	35.61	35.59	35.57	35.55	35.53
	2.5%	35.38	35.11	34.92	34.78	34.62	34.49	34.35	34.27	34.13	34.03	33.93	33.82	33.74	33.64	33.53	33.49	33.35	33.26	33.20	33.13
50-54	97.5%	31.91	32.08	32.23	32.33	32.47	32.50	32.56	32.62	32.68	32.73	32.80	32.83	32.88	32.92	32.97	33.02	33.03	33.10	33.12	33.13
	mean	31.37	31.34	31.32	31.31	31.29	31.26	31.24	31.22	31.20	31.18	31.16	31.13	31.11	31.09	31.06	31.04	31.02	31.00	30.98	30.96
	2.5%	30.81	30.57	30.38	30.24	30.10	29.98	29.85	29.76	29.63	29.54	29.45	29.34	29.27	29.17	29.07	29.02	28.90	28.82	28.75	28.68
55-59	97.5%	27.45	27.60	27.75	27.84	27.97	28.00	28.06	28.10	28.16	28.22	28.28	28.31	28.36	28.39	28.45	28.49	28.51	28.56	28.59	28.61
	mean	26.93	26.91	26.89	26.88	26.86	26.84	26.81	26.80	26.78	26.76	26.74	26.72	26.70	26.67	26.65	26.63	26.61	26.59	26.57	26.55
	2.5%	26.41	26.18	26.01	25.89	25.75	25.63	25.52	25.43	25.32	25.23	25.14	25.04	24.98	24.87	24.79	24.73	24.63	24.55	24.50	24.43
60-64	97.5%	23.17	23.32	23.44	23.54	23.65	23.69	23.74	23.78	23.82	23.88	23.93	23.97	24.01	24.06	24.09	24.12	24.14	24.19	24.22	24.26
	mean	22.69	22.67	22.66	22.64	22.62	22.60	22.58	22.57	22.52	22.53	22.51	22.49	22.48	22.45	22.44	22.41	22.40	22.38	22.36	22.34
	2.5%	22.22	22.00	21.84	21.73	21.61	21.49	21.38	21.32	21.23	21.13	21.05	20.97	20.90	20.81	20.74	20.67	20.56	20.53	20.46	20.40
65-69	97.5%	19.11	19.24	19.36	19.44	19.54	19.57	19.61	19.64	19.69	19.75	19.79	19.82	19.86	19.91	19.93	19.97	19.99	20.01	20.05	20.07
	mean	18.68	18.66	18.64	18.63	18.61	18.60	18.58	18.56	18.55	18.53	18.52	18.50	18.48	18.46	18.45	18.43	18.41	18.40	18.38	18.37
	2.5%	18.24	18.05	17.91	17.81	17.70	17.60	17.50	17.45	17.36	17.27	17.21	17.12	17.07	16.99	16.93	16.89	16.78	16.73	16.68	16.64
70-74	97.5%	15.31	15.43	15.53	15.60	15.68	15.71	15.76	15.78	15.82	15.87	15.90	15.93	15.96	15.99	16.02	16.06	16.08	16.08	16.12	16.15
	mean	14.93	14.92	14.91	14.90	14.88	14.87	14.85	14.84	14.82	14.81	14.80	14.78	14.77	14.75	14.73	14.72	14.71	14.69	14.68	14.66
	2.5%	14.55	14.39	14.28	14.19	14.09	14.00	13.91	13.87	13.80	13.72	13.66	13.60	13.55	13.47	13.44	13.36	13.30	13.27	13.21	13.17
75-79	97.5%	11.85	11.94	12.02	12.08	12.15	12.18	12.21	12.24	12.27	12.30	12.34	12.37	12.39	12.41	12.44	12.46				

Table A.11. The forecasted Canada female life expectancy by model 8

age 0	97.5%	83.77	84.11	84.41	84.69	85.02	85.28	85.54	85.80	86.11	86.37	86.66	86.90	87.17	87.45	87.72	88.02	88.24	88.53	88.75	88.99
	mean	83.10	83.31	83.51	83.72	83.93	84.15	84.36	84.57	84.79	85.01	85.23	85.44	85.66	85.88	86.09	86.32	86.54	86.77	86.99	87.20
	2.5%	82.40	82.47	82.58	82.69	82.82	82.92	83.05	83.21	83.32	83.51	83.65	83.82	83.96	84.12	84.31	84.45	84.67	84.88	85.02	85.17
1-4	97.5%	83.15	83.46	83.73	83.99	84.30	84.55	84.78	85.04	85.32	85.56	85.86	86.08	86.35	86.60	86.87	87.15	87.38	87.65	87.87	88.10
	mean	82.51	82.69	82.89	83.08	83.28	83.48	83.68	83.88	84.08	84.29	84.50	84.70	84.91	85.12	85.32	85.53	85.75	85.97	86.18	86.39
	2.5%	81.83	81.91	82.01	82.11	82.24	82.32	82.45	82.59	82.70	82.90	83.04	83.19	83.33	83.48	83.64	83.77	83.99	84.16	84.31	84.48
5-9	97.5%	79.19	79.50	79.77	80.03	80.33	80.57	80.81	81.06	81.34	81.58	81.88	82.09	82.37	82.62	82.88	83.16	83.38	83.65	83.88	84.10
	mean	78.56	78.74	78.93	79.13	79.32	79.52	79.71	79.91	80.12	80.32	80.53	80.73	80.94	81.14	81.35	81.56	81.77	81.98	82.19	82.40
	2.5%	77.89	77.96	78.07	78.17	78.29	78.38	78.51	78.64	78.75	78.95	79.09	79.23	79.37	79.52	79.69	79.81	80.02	80.20	80.36	80.51
10-14	97.5%	74.22	74.52	74.79	75.05	75.35	75.59	75.82	76.07	76.36	76.59	76.89	77.11	77.38	77.62	77.89	78.17	78.39	78.66	78.89	79.11
	mean	73.59	73.77	73.96	74.15	74.35	74.54	74.74	74.93	75.14	75.34	75.55	75.75	75.96	76.16	76.36	76.57	76.78	77.00	77.21	77.41
	2.5%	72.92	73.00	73.10	73.21	73.33	73.42	73.54	73.67	73.78	73.98	74.12	74.26	74.40	74.55	74.71	74.84	75.06	75.22	75.38	75.54
15-19	97.5%	69.25	69.55	69.82	70.08	70.37	70.61	70.84	71.09	71.38	71.60	71.90	72.13	72.39	72.64	72.90	73.18	73.41	73.67	73.89	74.12
	mean	68.62	68.81	68.99	69.19	69.38	69.57	69.77	69.96	70.17	70.37	70.57	70.77	70.98	71.18	71.38	71.59	71.80	72.02	72.23	72.43
	2.5%	67.96	68.04	68.14	68.24	68.36	68.46	68.59	68.71	68.83	69.02	69.17	69.30	69.44	69.59	69.75	69.88	70.09	70.26	70.41	70.57
20-24	97.5%	64.34	64.63	64.89	65.15	65.43	65.67	65.90	66.14	66.43	66.65	66.95	67.17	67.43	67.68	67.94	68.21	68.44	68.70	68.93	69.16
	mean	63.71	63.89	64.08	64.27	64.46	64.65	64.84	65.04	65.24	65.44	65.64	65.84	66.04	66.25	66.45	66.65	66.86	67.08	67.28	67.49
	2.5%	63.05	63.14	63.25	63.34	63.46	63.57	63.69	63.82	63.93	64.11	64.26	64.40	64.54	64.69	64.85	64.97	65.19	65.35	65.51	65.67
25-29	97.5%	59.44	59.72	59.98	60.23	60.50	60.75	60.97	61.21	61.49	61.72	62.01	62.22	62.49	62.72	62.99	63.26	63.48	63.75	63.97	64.19
	mean	58.82	59.00	59.18	59.37	59.56	59.75	59.94	60.13	60.33	60.53	60.73	60.93	61.13	61.33	61.53	61.73	61.94	62.15	62.36	62.56
	2.5%	58.17	58.25	58.37	58.47	58.59	58.70	58.82	58.94	59.06	59.25	59.38	59.53	59.66	59.82	59.97	60.09	60.31	60.47	60.62	60.79
30-34	97.5%	54.52	54.80	55.05	55.30	55.57	55.82	56.03	56.26	56.54	56.77	57.06	57.27	57.53	57.77	58.03	58.30	58.53	58.78	59.00	59.22
	mean	53.91	54.09	54.27	54.45	54.64	54.83	55.01	55.21	55.40	55.60	55.80	55.99	56.19	56.39	56.59	56.79	57.00	57.21	57.41	57.61
	2.5%	53.26	53.35	53.47	53.56	53.69	53.80	53.92	54.05	54.16	54.34	54.48	54.63	54.75	54.92	55.07	55.19	55.40	55.56	55.71	55.87
35-39	97.5%	49.63	49.90	50.15	50.38	50.66	50.90	51.12	51.34	51.61	51.85	52.13	52.33	52.59	52.82	53.09	53.35	53.57	53.82	54.05	54.26
	mean	49.02	49.20	49.38	49.56	49.74	49.93	50.12	50.30	50.50	50.69	50.89	51.08	51.28	51.48	51.67	51.87	52.08	52.29	52.49	52.69
	2.5%	48.38	48.47	48.60	48.69	48.81	48.93	49.05	49.18	49.28	49.46	49.61	49.76	49.88	50.04	50.18	50.31	50.52	50.67	50.82	50.98
40-44	97.5%	44.78	45.05	45.28	45.51	45.77	46.02	46.23	46.45	46.71	46.95	47.22	47.43	47.68	47.91	48.17	48.43	48.65	48.90	49.12	49.34
	mean	44.18	44.35	44.52	44.71	44.89	45.07	45.25	45.44	45.63	45.82	46.01	46.20	46.40	46.59	46.79	46.98	47.18	47.39	47.59	47.79
	2.5%	43.55	43.64	43.77	43.86	43.98	44.08	44.21	44.34	44.44	44.61	44.77	44.92	45.04	45.19	45.34	45.45	45.66	45.82	45.97	46.11
45-49	97.5%	39.98	40.23	40.46	40.69	40.94	41.18	41.39	41.60	41.86	42.09	42.35	42.56	42.81	43.04	43.28	43.54	43.76	44.00	44.22	44.43
	mean	39.38	39.55	39.73	39.90	40.08	40.26	40.44	40.62	40.81	40.99	41.18	41.37	41.56	41.75	41.94	42.14	42.34	42.54	42.73	42.93
	2.5%	38.78	38.85	38.99	39.09	39.21	39.31	39.43	39.57	39.69	39.84	39.99	40.11	40.25	40.40	40.55	40.66	40.87	41.03	41.17	41.32
50-54	97.5%	35.28	35.54	35.75	35.96	36.22	36.43	36.65	36.86	37.11	37.33	37.58	37.80	38.02	38.24	38.48	38.74	38.95	39.19	39.41	39.62
	mean	34.71	34.88	35.05	35.22	35.39	35.56	35.74	35.92	36.10	36.28	36.46	36.65	36.84	37.02	37.21	37.40	37.59	37.79	37.98	38.17
	2.5%	34.11	34.21	34.34	34.44	34.57	34.65	34.79	34.92	35.04	35.18	35.32	35.45	35.59	35.75	35.88	35.99	36.20	36.35	36.48	36.62
55-59	97.5%	30.70	30.93	31.15	31.35	31.59	31.79	32.00	32.21	32.43	32.67	32.90	33.11	33.34	33.55	33.79	34.02	34.23	34.46	34.68	34.87
	mean	30.15	30.31	30.47	30.63	30.80	30.97	31.14	31.31	31.48	31.66	31.84	32.01	32.19	32.37	32.55	32.73	32.92	33.11	33.29	33.48
	2.5%	29.58	29.66	29.79	29.89	30.01	30.11	30.23	30.36	30.47	30.61	30.73	30.87	31.02	31.14	31.28	31.39	31.60	31.73	31.86	32.00
60-64	97.5%	26.21	26.42	26.64	26.82	27.04	27.24	27.44	27.63	27.84	28.06	28.28	28.49	28.69	28.91	29.12	29.36	29.55	29.75	29.97	30.15
	mean	25.68	25.83	25.98	26.14	26.29	26.45	26.61	26.77	26.94	27.10	27.27	27.44	27.61	27.78	27.94	28.12	28.29	28.47	28.65	28.82
	2.5%	25.13	25.21	25.32	25.42	25.53	25.64	25.77	25.86	25.98	26.10	26.22	26.36	26.49	26.62	26.76	26.86	27.03	27.17	27.28	27.43
65-69	97.5%	21.90	22.10	22.28	22.46	22.67	22.85	23.04	23.20	23.41	23.60	23.82	23.99	24.20	24.39	24.59	24.81	25.00	25.20	25.38	25.56
	mean	21.40	21.53	21.67	21.81	21.95	22.10	22.24	22.39	22.54	22.70	22.85	23.00	23.16	23.31	23.47	23.63	23.79	23.96	24.12	24.28
	2.5%	20.87	20.94	21.06	21.14	21.24	21.33	21.46	21.54	21.63	21.77	21.85	22.00	22.12	22.23	22.34	22.44	22.59	22.70	22.82	22.94
70-74	97.5%	17.78	17.95	18.12	18.28	18.46	18.62	18.77	18.93	19.11	19.28	19.48	19.64	19.84	20.00	20.19	20.39	20.55	20.73	20.91	21.07
	mean	17.30	17.42	17.54	17.66	17.79	17.92	18.05	18.18	18.31	18.45	18.58	18.72	18.86	18.99	19.13	19.28	19.42	19.57	19.71	19.86
	2.5%	16.81	16.87	16.95	17.05	17.14	17.21	17.33	17.40	17.47	17.57	17.67	17.78	17.89	17.98	18.08	18.19	18.31	18.41	18.48	18.59
75-79	97.5%	13.91	14.06	14.20	14.34	14.49	14.63	14.76	14.91	15.05	15.21	15.36	15.51	15.69	15.82	15.99	16.14	16.28	16.46	16.61	16.7

Table A.12. The forecasted Canada male life expectancy by model 8

age 0	97.5%	79.69	80.12	80.56	80.01	81.43	81.88	82.34	82.80	83.27	83.72	84.21	84.68	85.16	85.62	86.07	86.56	87.01	87.47	87.94	88.37
	mean	79.02	79.38	79.76	80.15	80.53	80.93	81.34	81.75	82.16	82.58	83.00	83.43	83.85	84.28	84.70	85.13	85.56	85.98	86.39	86.80
	2.5%	78.31	78.61	78.89	79.26	79.58	79.88	80.23	80.61	80.93	81.33	81.66	82.06	82.41	82.79	83.18	83.58	83.92	84.32	84.68	85.04
1-4	97.5%	79.11	79.49	79.92	80.35	80.75	81.17	81.62	82.07	82.51	82.97	83.42	83.89	84.34	84.80	85.24	85.71	86.17	86.62	87.07	87.50
	mean	78.46	78.81	79.17	79.55	79.91	80.29	80.69	81.09	81.48	81.89	82.30	82.71	83.13	83.54	83.95	84.37	84.79	85.20	85.60	86.00
	2.5%	77.79	78.08	78.36	78.71	79.03	79.34	79.67	80.04	80.35	80.73	81.06	81.44	81.81	82.15	82.55	82.92	83.27	83.66	84.02	84.34
5-9	97.5%	75.17	75.55	75.96	76.40	76.79	77.21	77.66	78.10	78.54	78.99	79.44	79.91	80.37	80.83	81.26	81.73	82.18	82.64	83.09	83.52
	mean	74.53	74.87	75.23	75.61	75.97	76.35	76.74	77.13	77.53	77.93	78.34	78.75	79.16	79.58	79.99	80.40	80.82	81.23	81.63	82.03
	2.5%	73.86	74.15	74.43	74.78	75.09	75.41	75.75	76.11	76.42	76.79	77.12	77.51	77.87	78.22	78.60	78.97	79.32	79.70	80.07	80.40
10-14	97.5%	70.21	70.58	70.99	71.42	71.82	72.23	72.68	73.12	73.56	74.01	74.46	74.92	75.38	75.84	76.27	76.73	77.19	77.64	78.10	78.52
	mean	69.57	69.91	70.27	70.64	71.00	71.38	71.77	72.16	72.56	72.96	73.36	73.77	74.19	74.60	75.01	75.42	75.84	76.24	76.64	77.04
	2.5%	68.90	69.19	69.47	69.83	70.13	70.45	70.79	71.15	71.46	71.83	72.16	72.54	72.91	73.25	73.64	74.00	74.35	74.73	75.09	75.42
15-19	97.5%	65.25	65.62	66.03	66.46	66.85	67.26	67.70	68.14	68.58	69.03	69.48	69.94	70.39	70.85	71.28	71.75	72.20	72.65	73.11	73.53
	mean	64.61	64.95	65.31	65.68	66.03	66.41	66.80	67.19	67.58	67.98	68.39	68.80	69.21	69.62	70.03	70.44	70.86	71.26	71.66	72.06
	2.5%	63.94	64.24	64.52	64.87	65.17	65.49	65.82	66.19	66.50	66.86	67.20	67.57	67.93	68.28	68.67	69.04	69.38	69.75	70.12	70.43
20-24	97.5%	60.40	60.75	61.15	61.57	61.96	62.38	62.81	63.24	63.67	64.13	64.57	65.02	65.46	65.92	66.36	66.82	67.26	67.71	68.17	68.59
	mean	59.77	60.10	60.46	60.82	61.17	61.55	61.93	62.32	62.71	63.11	63.51	63.91	64.32	64.73	65.13	65.54	65.96	66.36	66.75	67.15
	2.5%	59.11	59.41	59.69	60.03	60.35	60.66	60.99	61.36	61.66	62.03	62.35	62.72	63.10	63.43	63.81	64.18	64.52	64.89	65.25	65.58
25-29	97.5%	55.61	55.96	56.36	56.76	57.14	57.56	57.97	58.40	58.83	59.27	59.71	60.16	60.60	61.06	61.48	61.94	62.38	62.83	63.28	63.69
	mean	55.00	55.33	55.68	56.04	56.39	56.76	57.13	57.52	57.91	58.30	58.70	59.10	59.50	59.90	60.30	60.71	61.12	61.51	61.91	62.30
	2.5%	54.35	54.66	54.94	55.29	55.60	55.90	56.23	56.59	56.90	57.26	57.59	57.97	58.31	58.66	59.04	59.42	59.74	60.12	60.47	60.80
30-34	97.5%	50.80	51.14	51.54	51.93	52.30	52.71	53.13	53.56	53.97	54.41	54.85	55.29	55.72	56.18	56.61	57.06	57.51	57.94	58.39	58.80
	mean	50.20	50.53	50.87	51.23	51.57	51.94	52.32	52.70	53.08	53.47	53.86	54.26	54.66	55.06	55.46	55.86	56.27	56.66	57.05	57.43
	2.5%	49.56	49.86	50.14	50.50	50.82	51.12	51.44	51.79	52.12	52.47	52.80	53.18	53.52	53.86	54.24	54.62	54.94	55.30	55.65	55.97
35-39	97.5%	45.99	46.33	46.71	47.10	47.47	47.87	48.28	48.70	49.12	49.55	49.98	50.42	50.84	51.31	51.72	52.17	52.62	53.05	53.49	53.90
	mean	45.40	45.72	46.06	46.41	46.75	47.12	47.49	47.87	48.25	48.63	49.02	49.42	49.81	50.21	50.60	51.00	51.41	51.80	52.18	52.57
	2.5%	44.76	45.06	45.35	45.70	46.01	46.32	46.65	46.99	47.31	47.67	48.00	48.36	48.71	49.04	49.41	49.79	50.11	50.47	50.81	51.15
40-44	97.5%	41.22	41.55	41.93	42.32	42.67	43.07	43.48	43.89	44.29	44.72	45.15	45.59	46.02	46.45	46.88	47.32	47.77	48.20	48.61	49.04
	mean	40.64	40.96	41.30	41.65	41.99	42.35	42.72	43.09	43.46	43.84	44.23	44.62	45.02	45.41	45.80	46.20	46.59	46.98	47.36	47.74
	2.5%	40.03	40.31	40.61	40.96	41.28	41.57	41.91	42.25	42.58	42.92	43.26	43.61	43.97	44.31	44.67	45.02	45.34	45.70	46.05	46.38
45-49	97.5%	36.53	36.86	37.23	37.59	37.93	38.34	38.72	39.12	39.52	39.93	40.34	40.79	41.20	41.63	42.04	42.50	42.90	43.35	43.77	44.18
	mean	35.97	36.29	36.62	36.95	37.28	37.64	37.99	38.36	38.73	39.10	39.48	39.86	40.24	40.63	41.01	41.40	41.79	42.17	42.55	42.92
	2.5%	35.38	35.64	35.95	36.30	36.60	36.89	37.22	37.55	37.88	38.22	38.56	38.90	39.24	39.58	39.92	40.28	40.60	40.93	41.27	41.59
50-54	97.5%	32.00	32.31	32.67	33.02	33.36	33.74	34.12	34.51	34.90	35.30	35.70	36.12	36.54	36.95	37.36	37.81	38.21	38.62	39.04	39.44
	mean	31.46	31.76	32.08	32.41	32.73	33.07	33.42	33.77	34.13	34.49	34.86	35.24	35.61	35.98	36.36	36.73	37.11	37.48	37.85	38.21
	2.5%	30.86	31.14	31.44	31.77	32.07	32.37	32.68	33.00	33.32	33.65	33.98	34.31	34.63	34.96	35.31	35.65	35.95	36.27	36.62	36.92
55-59	97.5%	27.60	27.89	28.24	28.57	28.90	29.27	29.62	30.00	30.37	30.76	31.14	31.55	31.96	32.35	32.76	33.17	33.56	33.96	34.35	34.76
	mean	27.09	27.38	27.69	28.00	28.31	28.64	28.97	29.32	29.66	30.01	30.36	30.72	31.08	31.44	31.80	32.16	32.53	32.88	33.23	33.58
	2.5%	26.53	26.80	27.08	27.39	27.70	27.98	28.28	28.59	28.91	29.21	29.55	29.86	30.17	30.49	30.82	31.14	31.45	31.73	32.06	32.35
60-64	97.5%	23.36	23.64	23.95	24.26	24.57	24.91	25.25	25.59	25.95	26.32	26.67	27.05	27.44	27.82	28.20	28.58	28.97	29.34	29.72	30.10
	mean	22.88	23.15	23.44	23.73	24.02	24.33	24.64	24.96	25.29	25.61	25.95	26.28	26.62	26.96	27.29	27.63	27.97	28.31	28.64	28.96
	2.5%	22.35	22.61	22.87	23.17	23.45	23.72	24.00	24.30	24.59	24.87	25.19	25.49	25.77	26.07	26.37	26.67	26.97	27.23	27.53	27.82
65-69	97.5%	19.33	19.57	19.86	20.13	20.41	20.71	21.00	21.32	21.64	21.97	22.30	22.65	23.00	23.35	23.69	24.03	24.39	24.73	25.08	25.43
	mean	18.87	19.12	19.37	19.63	19.89	20.17	20.44	20.73	21.02	21.31	21.61	21.91	22.21	22.52	22.82	23.13	23.44	23.74	24.03	24.33
	2.5%	18.38	18.62	18.86	19.11	19.36	19.59	19.83	20.13	20.36	20.61	20.88	21.15	21.39	21.67	21.94	22.19	22.45	22.69	22.97	23.22
70-74	97.5%	15.53	15.75	15.98	16.22	16.44	16.71	16.96	17.23	17.50	17.80	18.09	18.37	18.69	19.00	19.30	19.60	19.91	20.22	20.53	20.84
	mean	15.11	15.31	15.53	15.75	15.96	16.19	16.43	16.67	16.92	17.16	17.42	17.67	17.93	18.19	18.45	18.71	18.98	19.24	19.50	19.75
	2.5%	14.66	14.84	15.05	15.24	15.44	15.65	15.85	16.08	16.28	16.49	16.72	16.94	17.14	17.36	17.59	17.80	18.02	18.21	18.44	18.65
75-79	97.5%	12.05	12.23	12.40	12.58	12.76	12.96	13.15	13.37	13.58	13.80	14.03	14.26	14.51	14.74	14.98	15.23				

Table A.13. The forecasted Denmark female life expectancy by model 3

age 0	97.5%	82.93	83.16	83.36	83.52	83.66	83.76	83.82	83.89	83.98	84.05	84.15	84.22	84.26	84.32	84.40	84.43	84.46	84.56	84.59	84.64
	mean	82.23	82.20	82.18	82.16	82.14	82.11	82.08	82.06	82.04	82.02	81.99	81.96	81.94	81.91	81.88	81.85	81.82	81.80	81.77	81.74
	2.5%	81.47	81.14	80.86	80.68	80.46	80.27	80.06	79.91	79.73	79.55	79.41	79.27	79.17	78.97	78.82	78.70	78.54	78.39	78.25	78.18
1-4	97.5%	82.16	82.38	82.57	82.72	82.86	82.94	82.99	83.06	83.15	83.22	83.31	83.38	83.43	83.48	83.55	83.58	83.61	83.69	83.73	83.78
	mean	81.51	81.49	81.47	81.46	81.43	81.41	81.38	81.37	81.34	81.32	81.30	81.28	81.25	81.23	81.20	81.17	81.15	81.13	81.11	81.08
	2.5%	80.82	80.51	80.26	80.10	79.89	79.73	79.55	79.43	79.25	79.09	78.97	78.84	78.74	78.57	78.44	78.32	78.22	78.09	77.92	77.88
5-9	97.5%	78.19	78.41	78.59	78.74	78.88	78.95	79.02	79.08	79.17	79.23	79.32	79.40	79.44	79.49	79.56	79.59	79.62	79.71	79.74	79.79
	mean	77.55	77.53	77.51	77.50	77.48	77.46	77.43	77.41	77.39	77.37	77.35	77.33	77.31	77.28	77.26	77.23	77.21	77.19	77.17	77.14
	2.5%	76.87	76.58	76.33	76.17	75.97	75.81	75.65	75.53	75.37	75.20	75.09	74.96	74.88	74.72	74.59	74.48	74.39	74.26	74.12	74.08
10-14	97.5%	73.21	73.42	73.61	73.75	73.89	73.97	74.03	74.09	74.18	74.24	74.33	74.41	74.45	74.50	74.57	74.61	74.63	74.71	74.74	74.79
	mean	72.57	72.55	72.54	72.52	72.50	72.48	72.46	72.44	72.42	72.40	72.38	72.36	72.33	72.31	72.29	72.26	72.24	72.22	72.20	72.17
	2.5%	71.90	71.61	71.37	71.21	71.01	70.86	70.71	70.58	70.43	70.26	70.16	70.01	69.94	69.80	69.66	69.56	69.47	69.35	69.21	69.17
15-19	97.5%	68.23	68.44	68.62	68.77	68.90	68.98	69.04	69.11	69.19	69.26	69.34	69.42	69.46	69.51	69.57	69.62	69.64	69.72	69.75	69.80
	mean	67.59	67.57	67.56	67.54	67.52	67.50	67.48	67.46	67.44	67.42	67.40	67.38	67.36	67.33	67.31	67.29	67.27	67.25	67.22	67.20
	2.5%	66.93	66.64	66.40	66.24	66.05	65.90	65.74	65.62	65.47	65.31	65.21	65.06	64.99	64.85	64.73	64.62	64.53	64.41	64.27	64.23
20-24	97.5%	63.26	63.47	63.65	63.79	63.92	64.00	64.06	64.13	64.21	64.28	64.36	64.44	64.49	64.53	64.59	64.63	64.66	64.74	64.77	64.82
	mean	62.64	62.62	62.60	62.59	62.57	62.55	62.52	62.51	62.49	62.47	62.45	62.43	62.41	62.38	62.36	62.34	62.32	62.30	62.28	62.26
	2.5%	61.98	61.70	61.47	61.31	61.12	60.97	60.81	60.69	60.55	60.40	60.30	60.17	60.08	59.95	59.84	59.73	59.64	59.53	59.37	59.35
25-29	97.5%	58.31	58.52	58.70	58.83	58.96	59.04	59.10	59.17	59.25	59.31	59.39	59.47	59.52	59.56	59.63	59.66	59.69	59.77	59.80	59.85
	mean	57.70	57.68	57.67	57.65	57.64	57.61	57.59	57.58	57.56	57.54	57.52	57.50	57.48	57.46	57.44	57.41	57.40	57.38	57.36	57.34
	2.5%	57.06	56.79	56.56	56.41	56.23	56.08	55.93	55.82	55.69	55.53	55.44	55.32	55.23	55.10	54.99	54.90	54.80	54.70	54.55	54.54
30-34	97.5%	53.36	53.56	53.73	53.87	54.00	54.07	54.13	54.20	54.28	54.34	54.42	54.50	54.55	54.58	54.65	54.69	54.72	54.79	54.82	54.87
	mean	52.76	52.74	52.72	52.71	52.69	52.67	52.65	52.64	52.62	52.60	52.59	52.56	52.55	52.52	52.50	52.48	52.46	52.45	52.43	52.41
	2.5%	52.12	51.86	51.65	51.50	51.33	51.18	51.03	50.93	50.80	50.65	50.56	50.45	50.37	50.25	50.13	50.05	49.95	49.87	49.73	49.70
35-39	97.5%	48.43	48.63	48.79	48.93	49.06	49.13	49.19	49.25	49.33	49.40	49.47	49.54	49.59	49.63	49.70	49.73	49.76	49.83	49.87	49.91
	mean	47.84	47.83	47.81	47.80	47.79	47.77	47.75	47.73	47.72	47.70	47.69	47.67	47.65	47.63	47.61	47.59	47.57	47.56	47.54	47.52
	2.5%	47.24	46.99	46.77	46.63	46.46	46.32	46.19	46.09	45.97	45.84	45.75	45.64	45.57	45.46	45.34	45.26	45.18	45.07	44.97	44.94
40-44	97.5%	43.53	43.72	43.88	44.01	44.14	44.21	44.26	44.32	44.41	44.47	44.55	44.61	44.66	44.69	44.77	44.80	44.83	44.90	44.93	44.98
	mean	42.96	42.94	42.93	42.92	42.90	42.89	42.87	42.85	42.84	42.82	42.81	42.79	42.77	42.75	42.74	42.71	42.70	42.69	42.67	42.65
	2.5%	42.37	42.13	41.92	41.80	41.63	41.49	41.37	41.27	41.16	41.03	40.96	40.85	40.76	40.66	40.56	40.49	40.40	40.32	40.21	40.18
45-49	97.5%	38.70	38.88	39.05	39.17	39.28	39.36	39.41	39.47	39.55	39.62	39.69	39.75	39.80	39.84	39.91	39.93	39.96	40.03	40.07	40.11
	mean	38.15	38.13	38.12	38.11	38.09	38.08	38.06	38.04	38.03	38.01	38.00	37.98	37.97	37.95	37.93	37.91	37.90	37.88	37.86	37.85
	2.5%	37.58	37.34	37.15	37.02	36.86	36.72	36.60	36.51	36.41	36.28	36.19	36.10	36.02	35.93	35.82	35.75	35.68	35.59	35.49	35.46
50-54	97.5%	33.98	34.15	34.31	34.42	34.54	34.61	34.66	34.71	34.79	34.86	34.93	34.99	35.03	35.07	35.14	35.17	35.19	35.26	35.29	35.34
	mean	33.44	33.42	33.41	33.40	33.39	33.37	33.35	33.34	33.33	33.31	33.30	33.28	33.26	33.25	33.23	33.21	33.20	33.18	33.17	33.15
	2.5%	32.89	32.66	32.47	32.36	32.21	32.07	31.95	31.86	31.77	31.66	31.57	31.48	31.40	31.31	31.21	31.14	31.07	30.98	30.89	30.85
55-59	97.5%	29.39	29.54	29.70	29.81	29.91	29.99	30.03	30.08	30.16	30.23	30.28	30.34	30.38	30.43	30.49	30.51	30.54	30.60	30.64	30.67
	mean	28.88	28.86	28.86	28.85	28.83	28.82	28.80	28.79	28.77	28.76	28.75	28.73	28.72	28.70	28.68	28.66	28.65	28.64	28.62	28.61
	2.5%	28.36	28.16	27.97	27.87	27.73	27.59	27.49	27.40	27.30	27.20	27.13	27.03	26.96	26.87	26.80	26.72	26.64	26.56	26.49	26.46
60-64	97.5%	24.98	25.13	25.27	25.38	25.48	25.55	25.59	25.63	25.71	25.76	25.83	25.88	25.91	25.96	26.02	26.03	26.05	26.12	26.14	26.18
	mean	24.50	24.49	24.48	24.47	24.46	24.44	24.42	24.41	24.40	24.39	24.37	24.36	24.34	24.33	24.31	24.30	24.28	24.27	24.26	24.24
	2.5%	24.01	23.82	23.65	23.54	23.42	23.31	23.19	23.11	23.02	22.93	22.86	22.76	22.70	22.61	22.55	22.48	22.42	22.33	22.25	22.23
65-69	97.5%	20.75	20.89	21.01	21.11	21.21	21.27	21.31	21.35	21.42	21.47	21.53	21.58	21.61	21.65	21.70	21.71	21.74	21.81	21.82	21.85
	mean	20.32	20.30	20.30	20.29	20.27	20.26	20.25	20.24	20.22	20.21	20.20	20.18	20.17	20.16	20.14	20.13	20.12	20.10	20.09	20.08
	2.5%	19.87	19.69	19.53	19.44	19.33	19.23	19.12	19.05	18.96	18.88	18.81	18.74	18.68	18.61	18.54	18.46	18.41	18.34	18.26	18.24
70-74	97.5%	16.72	16.84	16.94	17.03	17.11	17.17	17.20	17.23	17.30	17.34	17.40	17.44	17.47	17.50	17.54	17.55	17.58	17.63	17.66	17.68
	mean	16.33	16.32	16.31	16.30	16.29	16.28	16.26	16.25	16.24	16.23	16.22	16.21	16.20	16.18	16.17	16.16	16.15	16.14	16.12	16.11
	2.5%	15.93	15.77	15.64	15.55	15.46	15.36	15.29	15.21	15.13	15.08	15.01	14.95	14.89	14.84	14.78	14.72	14.65	14.62	14.53	14.51
75-79	97.5%	12.93	13.03	13.12	13.19	13.26	13.30	13.33	13.36	13.41	13.45	13.48	13.52	13.54	13.58	13.62	13.62	13.63	13.67	13.71	13.71

Table A.14. The forecasted Denmark male life expectancy by model 3

age 0	97.5%	79.03	79.30	79.52	79.69	79.84	79.92	79.99	80.08	80.15	80.25	80.34	80.41	80.46	80.54	80.63	80.66	80.70	80.79	80.83	80.87
	mean	78.26	78.22	78.20	78.17	78.14	78.11	78.07	78.04	78.01	77.98	77.95	77.91	77.88	77.84	77.80	77.76	77.72	77.69	77.65	77.62
	2.5%	77.43	77.01	76.68	76.46	76.19	75.96	75.72	75.57	75.38	75.12	74.92	74.74	74.56	74.33	74.17	74.00	73.78	73.57	73.40	73.31
1-4	97.5%	78.31	78.56	78.77	78.93	79.06	79.14	79.20	79.28	79.36	79.45	79.54	79.61	79.65	79.71	79.80	79.84	79.88	79.96	80.00	80.02
	mean	77.59	77.56	77.54	77.51	77.49	77.45	77.42	77.39	77.36	77.34	77.31	77.27	77.24	77.20	77.17	77.13	77.10	77.07	77.04	77.00
	2.5%	76.84	76.44	76.15	75.94	75.70	75.49	75.27	75.13	74.93	74.73	74.55	74.37	74.23	73.99	73.87	73.69	73.49	73.32	73.15	73.05
5-9	97.5%	74.34	74.59	74.80	74.95	75.09	75.16	75.23	75.30	75.38	75.47	75.57	75.63	75.67	75.73	75.83	75.85	75.90	75.98	76.02	76.04
	mean	73.64	73.61	73.59	73.57	73.54	73.51	73.47	73.45	73.42	73.39	73.37	73.33	73.30	73.27	73.23	73.20	73.17	73.14	73.10	73.07
	2.5%	72.90	72.52	72.23	72.03	71.78	71.59	71.38	71.24	71.05	70.86	70.70	70.51	70.37	70.14	70.03	69.85	69.65	69.50	69.37	69.26
10-14	97.5%	69.36	69.61	69.81	69.97	70.10	70.17	70.24	70.31	70.39	70.48	70.57	70.64	70.67	70.74	70.83	70.86	70.91	70.98	71.02	71.04
	mean	68.66	68.63	68.61	68.59	68.56	68.53	68.50	68.48	68.45	68.42	68.39	68.36	68.33	68.30	68.27	68.23	68.20	68.17	68.14	68.11
	2.5%	67.93	67.55	67.27	67.07	66.84	66.63	66.43	66.31	66.11	65.92	65.76	65.59	65.44	65.22	65.12	64.94	64.74	64.59	64.46	64.37
15-19	97.5%	64.39	64.63	64.83	64.99	65.12	65.19	65.26	65.33	65.41	65.49	65.59	65.65	65.69	65.75	65.84	65.88	65.92	65.99	66.03	66.06
	mean	63.69	63.66	63.64	63.62	63.59	63.56	63.53	63.51	63.48	63.45	63.43	63.39	63.36	63.33	63.30	63.26	63.23	63.21	63.17	63.14
	2.5%	62.96	62.59	62.31	62.12	61.88	61.69	61.48	61.36	61.17	60.97	60.82	60.65	60.50	60.30	60.20	60.01	59.82	59.67	59.53	59.45
20-24	97.5%	59.45	59.69	59.89	59.80	60.03	60.17	60.23	60.30	60.37	60.45	60.54	60.63	60.69	60.73	60.78	60.88	60.91	60.96	61.03	61.07
	mean	58.77	58.74	58.72	58.70	58.67	58.64	58.61	58.59	58.56	58.54	58.51	58.48	58.45	58.42	58.39	58.36	58.33	58.30	58.27	58.24
	2.5%	58.06	57.69	57.42	57.23	57.01	56.82	56.61	56.50	56.31	56.12	55.98	55.81	55.68	55.47	55.39	55.21	55.04	54.88	54.75	54.65
25-29	97.5%	54.52	54.76	54.94	55.09	55.23	55.28	55.35	55.43	55.50	55.59	55.68	55.74	55.77	55.83	55.93	55.95	56.00	56.08	56.11	56.13
	mean	53.87	53.85	53.83	53.81	53.79	53.77	53.74	53.72	53.69	53.67	53.65	53.62	53.60	53.57	53.54	53.51	53.49	53.47	53.44	53.41
	2.5%	53.20	52.87	52.61	52.44	52.23	52.08	51.88	51.77	51.60	51.43	51.30	51.17	51.04	50.88	50.76	50.64	50.46	50.34	50.24	50.15
30-34	97.5%	49.62	49.85	50.03	50.17	50.30	50.35	50.42	50.50	50.57	50.65	50.73	50.80	50.83	50.89	50.97	51.00	51.05	51.12	51.17	51.19
	mean	49.01	48.99	48.97	48.96	48.94	48.92	48.89	48.88	48.86	48.84	48.82	48.80	48.78	48.75	48.73	48.70	48.69	48.67	48.65	48.62
	2.5%	48.38	48.08	47.84	47.68	47.50	47.36	47.19	47.10	46.97	46.81	46.69	46.56	46.46	46.34	46.24	46.15	45.98	45.90	45.82	45.75
35-39	97.5%	44.75	44.97	45.14	45.27	45.40	45.45	45.52	45.58	45.66	45.75	45.82	45.88	45.91	45.97	46.05	46.09	46.13	46.20	46.25	46.26
	mean	44.17	44.15	44.14	44.13	44.11	44.10	44.07	44.06	44.04	44.03	44.01	43.99	43.98	43.96	43.94	43.92	43.90	43.88	43.87	43.85
	2.5%	43.58	43.30	43.10	42.95	42.79	42.66	42.52	42.44	42.31	42.18	42.06	41.97	41.87	41.77	41.69	41.60	41.49	41.39	41.33	41.29
40-44	97.5%	39.93	40.14	40.30	40.43	40.55	40.60	40.67	40.72	40.80	40.89	40.96	41.01	41.04	41.11	41.18	41.21	41.26	41.32	41.37	41.38
	mean	39.37	39.36	39.35	39.34	39.32	39.30	39.28	39.27	39.26	39.24	39.23	39.21	39.19	39.17	39.16	39.14	39.12	39.11	39.09	39.07
	2.5%	38.81	38.54	38.35	38.22	38.07	37.94	37.81	37.73	37.62	37.48	37.38	37.30	37.20	37.12	37.03	36.96	36.84	36.77	36.71	36.66
45-49	97.5%	35.19	35.39	35.54	35.66	35.79	35.84	35.90	35.96	36.03	36.10	36.17	36.22	36.26	36.33	36.40	36.42	36.46	36.53	36.57	36.58
	mean	34.66	34.64	34.63	34.62	34.61	34.59	34.57	34.56	34.54	34.53	34.52	34.50	34.48	34.47	34.45	34.43	34.42	34.40	34.39	34.37
	2.5%	34.12	33.86	33.68	33.55	33.42	33.29	33.18	33.08	32.97	32.86	32.76	32.67	32.59	32.50	32.44	32.36	32.26	32.19	32.12	32.08
50-54	97.5%	30.59	30.77	30.91	31.04	31.15	31.20	31.26	31.32	31.37	31.45	31.52	31.56	31.60	31.65	31.73	31.75	31.79	31.86	31.89	31.91
	mean	30.08	30.06	30.05	30.04	30.03	30.01	29.99	29.98	29.97	29.96	29.94	29.93	29.91	29.89	29.88	29.86	29.85	29.83	29.82	29.80
	2.5%	29.56	29.33	29.15	29.04	28.91	28.78	28.67	28.59	28.49	28.38	28.28	28.20	28.12	28.04	27.98	27.91	27.81	27.74	27.68	27.64
55-59	97.5%	26.19	26.36	26.49	26.60	26.71	26.77	26.81	26.87	26.92	26.99	27.06	27.10	27.13	27.19	27.25	27.28	27.31	27.37	27.41	27.42
	mean	25.71	25.70	25.69	25.68	25.67	25.65	25.63	25.62	25.61	25.60	25.58	25.57	25.55	25.54	25.52	25.51	25.49	25.48	25.47	25.45
	2.5%	25.23	25.01	24.85	24.75	24.62	24.50	24.39	24.32	24.22	24.14	24.04	23.96	23.89	23.82	23.75	23.68	23.60	23.53	23.49	23.45
60-64	97.5%	22.05	22.20	22.33	22.44	22.53	22.58	22.62	22.68	22.73	22.80	22.85	22.90	22.92	22.98	23.03	23.05	23.07	23.14	23.17	23.18
	mean	21.60	21.59	21.58	21.57	21.56	21.54	21.53	21.52	21.50	21.49	21.48	21.47	21.45	21.44	21.42	21.41	21.40	21.38	21.37	21.36
	2.5%	21.16	20.95	20.81	20.71	20.59	20.48	20.38	20.32	20.22	20.14	20.05	19.99	19.93	19.85	19.79	19.72	19.65	19.57	19.54	19.49
65-69	97.5%	18.15	18.29	18.40	18.50	18.59	18.62	18.66	18.71	18.76	18.82	18.87	18.91	18.93	18.98	19.02	19.05	19.07	19.12	19.15	19.17
	mean	17.74	17.73	17.72	17.71	17.70	17.69	17.68	17.67	17.65	17.64	17.63	17.62	17.61	17.59	17.58	17.56	17.55	17.54	17.53	17.52
	2.5%	17.34	17.16	17.02	16.94	16.82	16.73	16.64	16.59	16.51	16.43	16.35	16.29	16.22	16.17	16.12	16.05	15.98	15.93	15.88	15.85
70-74	97.5%	14.46	14.58	14.68	14.77	14.84	14.88	14.91	14.95	14.99	15.05	15.08	15.12	15.14	15.18	15.22	15.24	15.25	15.30	15.33	15.35
	mean	14.11	14.09	14.09	14.08	14.07	14.06	14.04	14.03	14.02	14.01	14.00	13.99	13.98	13.97	13.96	13.94	13.93	13.92	13.91	13.90
	2.5%	13.75	13.59	13.47	13.39	13.30	13.22	13.14	13.09	13.02	12.96	12.88	12.84	12.77	12.72	12.67	12.61	12.55	12.52	12.47	12.45
75-79	97.5%	11.07	11.16	11.24	11.31	11.37	11.40	11.43	11.46	11.50	11.54	11.58	11.61	11.63	11.66	11.69	11.70				

Table A.15. The forecasted Denmark female life expectancy by model 8

age 0	97.5%	83.62	83.85	84.05	84.24	84.42	84.62	84.79	84.97	85.13	85.31	85.50	85.62	85.79	85.98	86.11	86.23	86.40	86.52	86.64	86.75
	mean	81.56	81.79	81.90	82.00	82.11	82.22	82.33	82.45	82.55	82.66	82.76	82.87	82.98	83.07	83.17	83.27	83.38	83.47	83.58	83.67
	2.5%	79.38	79.65	79.67	79.66	79.74	79.79	79.84	79.89	79.91	80.03	80.04	80.06	80.14	80.17	80.27	80.31	80.39	80.40	80.50	80.53
1-4	97.5%	82.89	83.10	83.30	83.46	83.64	83.82	83.97	84.15	84.32	84.48	84.65	84.75	84.92	85.09	85.24	85.35	85.51	85.62	85.75	85.85
	mean	80.84	81.06	81.16	81.26	81.36	81.46	81.56	81.67	81.76	81.87	81.96	82.06	82.17	82.25	82.34	82.44	82.54	82.63	82.73	82.82
	2.5%	78.67	78.95	78.97	78.96	79.02	79.05	79.12	79.16	79.18	79.27	79.30	79.31	79.38	79.41	79.49	79.52	79.59	79.61	79.70	79.71
5-9	97.5%	78.93	79.13	79.32	79.49	79.67	79.85	79.99	80.18	80.34	80.50	80.67	80.78	80.94	81.10	81.26	81.36	81.52	81.63	81.76	81.86
	mean	76.89	77.11	77.20	77.30	77.40	77.50	77.60	77.71	77.80	77.90	77.99	78.09	78.20	78.28	78.37	78.47	78.57	78.66	78.75	78.84
	2.5%	74.72	74.99	75.01	75.01	75.07	75.10	75.17	75.20	75.23	75.31	75.34	75.36	75.44	75.46	75.54	75.56	75.64	75.65	75.74	75.78
10-14	97.5%	73.96	74.15	74.35	74.51	74.68	74.86	75.01	75.20	75.35	75.51	75.68	75.79	75.94	76.11	76.27	76.38	76.52	76.64	76.77	76.87
	mean	71.91	72.13	72.23	72.32	72.42	72.52	72.62	72.73	72.82	72.92	73.01	73.11	73.21	73.30	73.39	73.48	73.58	73.67	73.77	73.86
	2.5%	69.75	70.02	70.04	70.03	70.11	70.13	70.19	70.23	70.27	70.34	70.36	70.39	70.46	70.48	70.56	70.59	70.68	70.67	70.77	70.81
15-19	97.5%	68.98	69.17	69.36	69.52	69.70	69.87	70.02	70.20	70.36	70.52	70.69	70.79	70.95	71.11	71.27	71.38	71.53	71.64	71.77	71.87
	mean	66.93	67.15	67.24	67.34	67.43	67.53	67.64	67.74	67.83	67.93	68.02	68.12	68.22	68.31	68.40	68.49	68.59	68.68	68.78	68.86
	2.5%	64.77	65.04	65.06	65.05	65.13	65.16	65.21	65.25	65.28	65.37	65.38	65.40	65.48	65.50	65.57	65.60	65.69	65.68	65.78	65.82
20-24	97.5%	64.01	64.20	64.39	64.55	64.72	64.90	65.03	65.22	65.38	65.54	65.71	65.81	65.96	66.13	66.29	66.39	66.54	66.65	66.77	66.88
	mean	61.97	62.18	62.28	62.37	62.46	62.56	62.66	62.77	62.86	62.96	63.05	63.14	63.25	63.33	63.42	63.51	63.61	63.70	63.79	63.88
	2.5%	59.80	60.08	60.09	60.09	60.16	60.19	60.25	60.28	60.32	60.41	60.41	60.43	60.50	60.52	60.59	60.64	60.73	60.72	60.81	60.85
25-29	97.5%	59.06	59.25	59.43	59.59	59.77	59.93	60.06	60.25	60.41	60.56	60.74	60.83	60.98	61.16	61.31	61.41	61.57	61.66	61.79	61.90
	mean	57.02	57.23	57.33	57.42	57.51	57.61	57.71	57.81	57.90	58.00	58.09	58.19	58.29	58.37	58.46	58.55	58.65	58.74	58.83	58.92
	2.5%	54.85	55.13	55.15	55.15	55.22	55.24	55.30	55.35	55.40	55.46	55.47	55.51	55.56	55.59	55.64	55.72	55.78	55.77	55.86	55.91
30-34	97.5%	54.11	54.30	54.47	54.63	54.80	54.97	55.09	55.28	55.44	55.59	55.76	55.84	56.01	56.18	56.33	56.42	56.59	56.68	56.80	56.92
	mean	52.07	52.28	52.38	52.47	52.56	52.65	52.75	52.85	52.94	53.04	53.13	53.22	53.32	53.41	53.49	53.58	53.68	53.76	53.86	53.94
	2.5%	49.91	50.18	50.21	50.21	50.27	50.30	50.35	50.40	50.45	50.52	50.51	50.56	50.60	50.62	50.69	50.77	50.83	50.82	50.91	50.95
35-39	97.5%	49.19	49.36	49.54	49.70	49.86	50.02	50.15	50.33	50.48	50.64	50.80	50.89	51.04	51.22	51.37	51.46	51.62	51.71	51.83	51.95
	mean	47.15	47.36	47.45	47.54	47.63	47.72	47.82	47.92	48.00	48.10	48.18	48.28	48.37	48.46	48.54	48.63	48.72	48.81	48.90	48.99
	2.5%	44.98	45.26	45.29	45.28	45.35	45.38	45.43	45.47	45.51	45.59	45.59	45.62	45.67	45.69	45.75	45.83	45.89	45.89	45.98	46.01
40-44	97.5%	44.28	44.45	44.61	44.77	44.94	45.09	45.21	45.39	45.54	45.69	45.85	45.93	46.09	46.26	46.41	46.51	46.65	46.75	46.86	46.98
	mean	42.24	42.45	42.54	42.62	42.71	42.80	42.89	42.99	43.07	43.16	43.25	43.34	43.44	43.52	43.60	43.68	43.78	43.86	43.95	44.03
	2.5%	40.08	40.36	40.38	40.37	40.44	40.47	40.52	40.55	40.59	40.66	40.66	40.69	40.75	40.76	40.83	40.90	40.95	40.95	41.05	41.06
45-49	97.5%	39.45	39.61	39.76	39.92	40.07	40.23	40.34	40.50	40.66	40.80	40.96	41.04	41.19	41.35	41.50	41.59	41.72	41.83	41.95	42.05
	mean	37.40	37.61	37.69	37.77	37.85	37.94	38.03	38.12	38.19	38.28	38.36	38.45	38.54	38.62	38.70	38.78	38.87	38.95	39.04	39.12
	2.5%	35.24	35.51	35.52	35.53	35.59	35.61	35.66	35.68	35.71	35.79	35.78	35.82	35.86	35.89	35.94	36.00	36.05	36.06	36.14	36.16
50-54	97.5%	34.74	34.88	35.04	35.17	35.32	35.46	35.58	35.73	35.88	36.01	36.16	36.24	36.39	36.53	36.67	36.74	36.88	36.99	37.10	37.20
	mean	32.68	32.88	32.95	33.02	33.10	33.18	33.26	33.35	33.42	33.50	33.58	33.66	33.75	33.82	33.89	33.97	34.05	34.12	34.21	34.28
	2.5%	30.50	30.77	30.79	30.78	30.84	30.85	30.90	30.91	30.94	31.02	31.00	31.04	31.08	31.10	31.14	31.20	31.23	31.24	31.32	31.31
55-59	97.5%	30.19	30.33	30.48	30.59	30.72	30.85	30.95	31.12	31.23	31.35	31.48	31.57	31.72	31.84	31.97	32.04	32.15	32.26	32.39	32.46
	mean	28.12	28.31	28.38	28.44	28.51	28.58	28.66	28.74	28.80	28.88	28.94	29.02	29.09	29.16	29.22	29.29	29.37	29.44	29.51	29.58
	2.5%	25.92	26.19	26.21	26.19	26.26	26.26	26.30	26.30	26.33	26.39	26.38	26.40	26.45	26.46	26.49	26.56	26.56	26.57	26.65	26.64
60-64	97.5%	25.85	25.98	26.11	26.21	26.33	26.45	26.54	26.68	26.79	26.91	27.02	27.11	27.22	27.32	27.45	27.51	27.61	27.70	27.81	27.91
	mean	23.76	23.94	24.00	24.06	24.12	24.18	24.25	24.31	24.37	24.43	24.49	24.56	24.62	24.68	24.73	24.80	24.86	24.92	24.99	25.04
	2.5%	21.53	21.79	21.80	21.78	21.83	21.84	21.86	21.86	21.88	21.91	21.92	21.94	21.98	21.98	22.00	22.04	22.08	22.06	22.12	22.11
65-69	97.5%	21.64	21.76	21.85	21.94	22.03	22.15	22.23	22.36	22.44	22.53	22.64	22.70	22.81	22.91	22.99	23.05	23.15	23.23	23.33	23.40
	mean	19.50	19.68	19.73	19.77	19.81	19.86	19.92	19.97	20.02	20.07	20.11	20.17	20.22	20.26	20.30	20.36	20.41	20.46	20.51	20.55
	2.5%	17.20	17.48	17.46	17.45	17.49	17.48	17.49	17.50	17.50	17.51	17.54	17.54	17.55	17.53	17.56	17.60	17.62	17.61	17.64	17.64
70-74	97.5%	17.60	17.69	17.75	17.82	17.90	17.98	18.05	18.15	18.21	18.28	18.36	18.42	18.47	18.56	18.66	18.68	18.74	18.81	18.89	18.95
	mean	15.38	15.54	15.57	15.60	15.63	15.66	15.70	15.73	15.76	15.80	15.83	15.86	15.90	15.93	15.96	15.99	16.03	16.06	16.09	16.13
	2.5%	12.98	13.25	13.21	13.23	13.23	13.22	13.19	13.21	13.20	13.21	13.18	13.17	13.19	13.17	13.19	13.21	13.18	13.19	13.21	13.19
75-79	97.5%	13.05	13.17	13.29	13.39	13.49	13.60	13.70	13.80	13.89	14.00	14.09	14.19	14.29	14.37	14.46	14.53	14.62	14.70	14.79	14.8

Table A.16. The forecasted Denmark male life expectancy by model 8

age 0	97.5%	78.99	79.36	79.71	80.01	80.30	80.57	80.85	81.10	81.37	81.65	81.90	82.19	82.48	82.70	82.94	83.20	83.43	83.69	83.92	84.15
	mean	78.22	78.39	78.56	78.73	78.90	79.08	79.26	79.44	79.61	79.79	79.97	80.15	80.33	80.51	80.69	80.87	81.05	81.23	81.42	81.60
	2.5%	77.40	77.39	77.40	77.38	77.38	77.45	77.53	77.62	77.68	77.76	77.78	77.87	77.95	78.09	78.18	78.28	78.36	78.50	78.64	78.72
1-4	97.5%	78.26	78.61	78.93	79.22	79.50	79.77	80.02	80.27	80.53	80.81	81.04	81.32	81.61	81.82	82.05	82.29	82.52	82.79	83.00	83.23
	mean	77.52	77.69	77.84	78.00	78.16	78.33	78.50	78.67	78.84	79.01	79.19	79.36	79.52	79.70	79.87	80.04	80.22	80.39	80.57	80.75
	2.5%	76.74	76.73	76.74	76.72	76.72	76.78	76.86	76.94	76.99	77.06	77.11	77.18	77.25	77.38	77.48	77.55	77.64	77.76	77.85	77.98
5-9	97.5%	74.30	74.64	74.96	75.25	75.53	75.79	76.05	76.29	76.54	76.83	77.06	77.34	77.62	77.83	78.06	78.30	78.53	78.80	79.00	79.24
	mean	73.57	73.73	73.88	74.04	74.20	74.37	74.54	74.71	74.87	75.04	75.21	75.38	75.55	75.72	75.89	76.06	76.24	76.41	76.59	76.76
	2.5%	72.80	72.78	72.79	72.78	72.76	72.83	72.91	72.99	73.04	73.10	73.15	73.23	73.30	73.42	73.52	73.59	73.69	73.79	73.89	74.01
10-14	97.5%	69.32	69.66	69.98	70.26	70.53	70.80	71.06	71.30	71.55	71.83	72.06	72.34	72.62	72.84	73.07	73.30	73.53	73.80	74.01	74.24
	mean	68.59	68.75	68.91	69.06	69.22	69.39	69.56	69.72	69.88	70.06	70.23	70.40	70.56	70.73	70.91	71.08	71.25	71.42	71.60	71.77
	2.5%	67.83	67.81	67.81	67.81	67.80	67.87	67.94	68.02	68.08	68.14	68.19	68.26	68.32	68.45	68.54	68.62	68.71	68.82	68.93	69.03
15-19	97.5%	64.34	64.68	64.99	65.27	65.55	65.81	66.06	66.30	66.56	66.84	67.07	67.35	67.63	67.84	68.07	68.31	68.54	68.81	69.01	69.25
	mean	63.61	63.77	63.93	64.08	64.24	64.40	64.57	64.74	64.90	65.07	65.24	65.41	65.57	65.74	65.92	66.09	66.26	66.43	66.61	66.78
	2.5%	62.85	62.84	62.84	62.84	62.82	62.90	62.96	63.05	63.10	63.17	63.21	63.29	63.35	63.47	63.56	63.64	63.73	63.84	63.94	64.06
20-24	97.5%	59.39	59.73	60.04	60.32	60.58	60.85	61.09	61.34	61.59	61.87	62.09	62.37	62.65	62.87	63.09	63.32	63.55	63.82	64.03	64.26
	mean	58.68	58.84	58.99	59.14	59.29	59.46	59.62	59.79	59.95	60.12	60.29	60.45	60.62	60.79	60.96	61.13	61.29	61.47	61.65	61.82
	2.5%	57.93	57.92	57.92	57.91	57.92	57.98	58.06	58.12	58.17	58.24	58.28	58.36	58.43	58.55	58.65	58.70	58.81	58.91	59.02	59.13
25-29	97.5%	54.49	54.82	55.13	55.40	55.66	55.93	56.17	56.40	56.65	56.94	57.14	57.43	57.70	57.92	58.15	58.37	58.59	58.85	59.07	59.30
	mean	53.80	53.95	54.10	54.25	54.41	54.57	54.73	54.89	55.04	55.21	55.38	55.54	55.70	55.87	56.04	56.21	56.37	56.54	56.72	56.89
	2.5%	53.07	53.06	53.07	53.06	53.07	53.13	53.20	53.27	53.33	53.39	53.43	53.49	53.58	53.71	53.79	53.84	53.93	54.06	54.18	54.24
30-34	97.5%	49.59	49.91	50.20	50.48	50.73	51.00	51.23	51.47	51.71	52.00	52.19	52.48	52.74	52.97	53.18	53.41	53.63	53.89	54.11	54.34
	mean	48.91	49.06	49.21	49.36	49.51	49.67	49.83	49.98	50.14	50.30	50.47	50.63	50.79	50.95	51.12	51.28	51.45	51.62	51.79	51.96
	2.5%	48.20	48.20	48.20	48.19	48.21	48.27	48.34	48.41	48.46	48.52	48.57	48.64	48.72	48.83	48.93	48.96	49.07	49.17	49.31	49.36
35-39	97.5%	44.73	45.03	45.31	45.60	45.84	46.10	46.33	46.56	46.80	47.08	47.29	47.55	47.82	48.03	48.26	48.48	48.70	48.95	49.17	49.39
	mean	44.06	44.21	44.36	44.51	44.66	44.81	44.97	45.13	45.28	45.44	45.61	45.77	45.93	46.09	46.26	46.42	46.58	46.75	46.92	47.09
	2.5%	43.38	43.38	43.40	43.41	43.43	43.50	43.56	43.62	43.67	43.75	43.80	43.89	43.95	44.07	44.17	44.23	44.31	44.44	44.55	44.64
40-44	97.5%	39.92	40.22	40.50	40.76	41.00	41.25	41.46	41.70	41.94	42.21	42.41	42.68	42.93	43.15	43.37	43.59	43.82	44.06	44.26	44.50
	mean	39.28	39.43	39.57	39.72	39.87	40.02	40.18	40.33	40.48	40.64	40.80	40.96	41.12	41.28	41.44	41.60	41.76	41.93	42.10	42.26
	2.5%	38.62	38.64	38.66	38.68	38.70	38.76	38.82	38.90	38.96	39.04	39.10	39.18	39.24	39.34	39.46	39.54	39.61	39.73	39.85	39.96
45-49	97.5%	35.17	35.47	35.74	35.99	36.21	36.45	36.66	36.91	37.12	37.38	37.58	37.84	38.10	38.31	38.53	38.75	38.95	39.19	39.39	39.63
	mean	34.55	34.69	34.84	34.98	35.12	35.27	35.42	35.57	35.72	35.87	36.03	36.19	36.34	36.49	36.65	36.81	36.96	37.13	37.29	37.45
	2.5%	33.92	33.93	33.94	33.97	34.00	34.05	34.12	34.19	34.26	34.32	34.39	34.45	34.52	34.63	34.71	34.81	34.89	35.00	35.12	35.19
50-54	97.5%	30.57	30.84	31.10	31.34	31.56	31.80	31.98	32.22	32.44	32.68	32.89	33.14	33.38	33.59	33.80	33.99	34.20	34.43	34.63	34.87
	mean	29.97	30.10	30.24	30.38	30.52	30.66	30.81	30.95	31.09	31.24	31.39	31.54	31.69	31.84	31.99	32.14	32.30	32.45	32.61	32.77
	2.5%	29.36	29.36	29.39	29.41	29.44	29.50	29.56	29.62	29.69	29.77	29.82	29.89	29.95	30.06	30.14	30.22	30.31	30.43	30.52	30.61
55-59	97.5%	26.21	26.46	26.70	26.93	27.15	27.36	27.56	27.78	27.98	28.22	28.41	28.63	28.86	29.07	29.27	29.45	29.66	29.88	30.08	30.30
	mean	25.64	25.77	25.90	26.03	26.16	26.29	26.43	26.57	26.70	26.84	26.99	27.13	27.26	27.41	27.56	27.70	27.84	27.99	28.14	28.29
	2.5%	25.06	25.06	25.09	25.11	25.14	25.19	25.26	25.30	25.38	25.44	25.50	25.56	25.63	25.73	25.83	25.87	25.96	26.08	26.18	26.24
60-64	97.5%	22.12	22.35	22.57	22.79	23.00	23.20	23.38	23.58	23.77	23.99	24.19	24.37	24.59	24.79	24.97	25.14	25.36	25.54	25.74	25.94
	mean	21.58	21.69	21.82	21.94	22.06	22.18	22.31	22.44	22.56	22.69	22.83	22.96	23.09	23.22	23.36	23.49	23.62	23.76	23.90	24.04
	2.5%	21.02	21.03	21.04	21.06	21.09	21.15	21.21	21.22	21.32	21.36	21.41	21.48	21.53	21.62	21.72	21.77	21.84	21.91	22.05	22.11
65-69	97.5%	18.22	18.43	18.62	18.82	19.02	19.18	19.36	19.52	19.70	19.90	20.08	20.24	20.45	20.62	20.79	20.96	21.13	21.30	21.47	21.64
	mean	17.72	17.82	17.93	18.03	18.14	18.25	18.37	18.48	18.59	18.71	18.83	18.95	19.06	19.18	19.30	19.42	19.54	19.66	19.79	19.91
	2.5%	17.21	17.21	17.23	17.24	17.26	17.29	17.36	17.38	17.46	17.50	17.53	17.59	17.65	17.73	17.80	17.83	17.89	17.98	18.06	18.12
70-74	97.5%	14.51	14.69	14.87	15.03	15.20	15.36	15.52	15.65	15.80	15.98	16.13	16.28	16.45	16.59	16.75	16.90	17.04	17.19	17.34	17.49
	mean	14.07	14.15	14.24	14.33	14.42	14.52	14.62	14.71	14.80	14.90	15.01	15.10	15.20	15.30	15.40	15.51	15.61	15.71	15.82	15.92
	2.5%	13.61	13.62	13.64	13.62	13.63	13.67	13.72	13.74	13.79	13.82	13.84	13.89	13.94	14.01	14.05	14.09	14.12	14.18	14.25	14.29
75-79	97.5%	11.05	11.19	11.33	11.48	11.60	11.72	11.84	11.97	12.08	12.22	12.33	12.46	12.59	12.71	12.83	12.9				

Table A.17. The forecasted Finland female life expectancy by model 3

age 0	97.5%	84.37	84.61	84.85	85.00	85.17	85.24	85.31	85.41	85.50	85.56	85.68	85.73	85.79	85.83	85.94	85.97	86.02	86.07	86.14	86.15
	mean	83.59	83.56	83.54	83.52	83.49	83.46	83.42	83.40	83.37	83.34	83.31	83.28	83.24	83.21	83.18	83.14	83.11	83.08	83.05	83.02
	2.5%	82.78	82.39	82.10	81.89	81.64	81.42	81.21	81.04	80.85	80.66	80.50	80.33	80.18	80.00	79.80	79.68	79.54	79.41	79.21	79.10
1-4	97.5%	83.49	83.73	83.97	84.11	84.27	84.34	84.41	84.50	84.59	84.65	84.76	84.82	84.87	84.91	85.02	85.06	85.10	85.14	85.22	85.22
	mean	82.75	82.72	82.70	82.67	82.65	82.62	82.58	82.56	82.53	82.50	82.48	82.44	82.41	82.38	82.35	82.31	82.29	82.26	82.23	82.20
	2.5%	81.97	81.59	81.31	81.11	80.88	80.68	80.46	80.32	80.14	79.96	79.82	79.64	79.49	79.30	79.14	79.03	78.89	78.75	78.57	78.47
5-9	97.5%	79.51	79.75	79.98	80.12	80.28	80.35	80.42	80.51	80.60	80.67	80.76	80.82	80.88	80.92	81.02	81.06	81.10	81.15	81.22	81.23
	mean	78.78	78.75	78.73	78.70	78.68	78.65	78.62	78.59	78.57	78.54	78.51	78.48	78.45	78.42	78.39	78.36	78.33	78.31	78.28	78.25
	2.5%	78.00	77.64	77.37	77.18	76.95	76.74	76.54	76.41	76.24	76.05	75.92	75.75	75.62	75.43	75.28	75.17	75.03	74.91	74.74	74.64
10-14	97.5%	74.53	74.76	75.00	75.13	75.30	75.37	75.43	75.52	75.61	75.67	75.77	75.84	75.89	75.93	76.03	76.07	76.11	76.16	76.23	76.24
	mean	73.81	73.78	73.76	73.74	73.71	73.68	73.65	73.63	73.60	73.58	73.55	73.52	73.49	73.46	73.43	73.40	73.38	73.35	73.32	73.30
	2.5%	73.05	72.69	72.42	72.24	72.01	71.82	71.62	71.48	71.30	71.13	71.02	70.85	70.73	70.55	70.42	70.31	70.18	70.06	69.91	69.80
15-19	97.5%	69.56	69.79	70.02	70.15	70.31	70.39	70.45	70.54	70.62	70.69	70.79	70.85	70.91	70.94	71.05	71.08	71.13	71.17	71.25	71.25
	mean	68.84	68.81	68.79	68.77	68.75	68.72	68.69	68.67	68.64	68.62	68.59	68.56	68.54	68.50	68.48	68.44	68.42	68.40	68.37	68.34
	2.5%	68.09	67.73	67.48	67.29	67.08	66.88	66.69	66.56	66.39	66.22	66.10	65.94	65.82	65.65	65.51	65.42	65.29	65.18	65.03	64.92
20-24	97.5%	64.61	64.84	65.06	65.20	65.35	65.42	65.49	65.58	65.66	65.72	65.82	65.89	65.94	65.97	66.08	66.11	66.15	66.20	66.28	66.27
	mean	63.91	63.88	63.86	63.84	63.82	63.79	63.76	63.74	63.72	63.69	63.67	63.64	63.62	63.59	63.56	63.53	63.51	63.48	63.46	63.43
	2.5%	63.18	62.83	62.58	62.40	62.19	62.01	61.83	61.70	61.54	61.37	61.25	61.10	60.98	60.84	60.69	60.61	60.49	60.38	60.25	60.16
25-29	97.5%	59.67	59.89	60.11	60.24	60.40	60.47	60.53	60.62	60.70	60.76	60.86	60.92	60.97	61.01	61.11	61.14	61.19	61.23	61.31	61.31
	mean	58.98	58.95	58.93	58.91	58.89	58.87	58.84	58.82	58.79	58.77	58.75	58.72	58.70	58.67	58.64	58.61	58.59	58.57	58.55	58.52
	2.5%	58.27	57.93	57.68	57.51	57.30	57.12	56.96	56.83	56.69	56.53	56.42	56.25	56.16	56.00	55.87	55.77	55.68	55.57	55.45	55.35
30-34	97.5%	54.73	54.94	55.17	55.29	55.45	55.52	55.58	55.67	55.74	55.80	55.91	55.97	56.01	56.05	56.16	56.18	56.22	56.27	56.35	56.34
	mean	54.06	54.03	54.01	54.00	53.98	53.95	53.92	53.90	53.88	53.86	53.84	53.81	53.79	53.76	53.74	53.71	53.69	53.67	53.65	53.62
	2.5%	53.36	53.04	52.81	52.64	52.43	52.28	52.11	51.99	51.85	51.70	51.60	51.44	51.35	51.19	51.08	50.96	50.90	50.79	50.67	50.58
35-39	97.5%	49.82	50.03	50.25	50.37	50.52	50.59	50.65	50.73	50.81	50.87	50.97	51.03	51.07	51.11	51.22	51.24	51.28	51.32	51.40	51.40
	mean	49.17	49.14	49.12	49.11	49.09	49.06	49.04	49.02	49.00	48.98	48.96	48.93	48.91	48.89	48.86	48.84	48.82	48.80	48.77	48.75
	2.5%	48.49	48.18	47.96	47.80	47.59	47.45	47.28	47.17	47.04	46.89	46.80	46.65	46.56	46.41	46.30	46.20	46.12	46.04	45.92	45.84
40-44	97.5%	44.92	45.13	45.33	45.46	45.61	45.67	45.72	45.81	45.89	45.94	46.04	46.10	46.14	46.18	46.28	46.31	46.35	46.39	46.46	46.46
	mean	44.28	44.26	44.24	44.23	44.21	44.18	44.16	44.14	44.12	44.10	44.08	44.06	44.04	44.01	43.99	43.97	43.95	43.93	43.91	43.89
	2.5%	43.63	43.32	43.11	42.95	42.76	42.62	42.45	42.36	42.22	42.09	41.98	41.85	41.78	41.63	41.54	41.43	41.36	41.27	41.19	41.09
45-49	97.5%	40.07	40.27	40.47	40.60	40.74	40.79	40.85	40.94	41.00	41.06	41.16	41.22	41.26	41.29	41.40	41.43	41.47	41.50	41.57	41.57
	mean	39.45	39.42	39.41	39.40	39.38	39.35	39.33	39.31	39.29	39.27	39.26	39.23	39.21	39.19	39.17	39.14	39.13	39.11	39.09	39.07
	2.5%	38.82	38.52	38.30	38.16	37.98	37.84	37.68	37.59	37.46	37.34	37.24	37.11	37.03	36.91	36.78	36.69	36.64	36.55	36.47	36.36
50-54	97.5%	35.31	35.50	35.68	35.81	35.95	36.01	36.05	36.14	36.20	36.26	36.36	36.41	36.45	36.49	36.59	36.61	36.65	36.68	36.75	36.74
	mean	34.70	34.68	34.66	34.65	34.63	34.61	34.59	34.57	34.55	34.53	34.52	34.49	34.47	34.45	34.43	34.41	34.39	34.37	34.35	34.33
	2.5%	34.09	33.81	33.60	33.47	33.29	33.15	33.00	32.92	32.78	32.67	32.59	32.46	32.37	32.25	32.17	32.06	32.00	31.92	31.84	31.75
55-59	97.5%	30.65	30.83	31.00	31.14	31.26	31.32	31.36	31.45	31.51	31.56	31.66	31.71	31.73	31.78	31.88	31.90	31.94	31.98	32.03	32.03
	mean	30.07	30.05	30.03	30.02	30.00	29.98	29.96	29.94	29.92	29.91	29.89	29.87	29.85	29.83	29.81	29.79	29.77	29.75	29.73	29.71
	2.5%	29.47	29.21	29.01	28.89	28.72	28.58	28.43	28.35	28.23	28.13	28.04	27.92	27.84	27.71	27.63	27.56	27.49	27.40	27.34	27.23
60-64	97.5%	26.09	26.27	26.42	26.55	26.67	26.73	26.76	26.85	26.91	26.96	27.06	27.11	27.14	27.17	27.26	27.28	27.31	27.35	27.39	27.40
	mean	25.54	25.52	25.51	25.49	25.48	25.46	25.43	25.42	25.40	25.38	25.37	25.35	25.33	25.31	25.29	25.27	25.25	25.24	25.22	25.20
	2.5%	24.98	24.73	24.54	24.42	24.25	24.13	24.00	23.93	23.79	23.70	23.62	23.51	23.42	23.30	23.23	23.16	23.09	23.01	22.95	22.86
65-69	97.5%	21.67	21.83	21.98	22.10	22.21	22.26	22.31	22.38	22.43	22.49	22.58	22.61	22.65	22.68	22.77	22.77	22.82	22.85	22.90	22.89
	mean	21.15	21.13	21.12	21.11	21.09	21.07	21.05	21.04	21.02	21.00	20.99	20.97	20.95	20.93	20.91	20.89	20.88	20.86	20.85	20.83
	2.5%	20.62	20.39	20.22	20.10	19.95	19.83	19.71	19.64	19.53	19.44	19.35	19.25	19.16	19.04	18.99	18.94	18.84	18.78	18.72	18.63
70-74	97.5%	17.47	17.62	17.76	17.86	17.97	18.01	18.05	18.11	18.16	18.22	18.29	18.32	18.36	18.38	18.46	18.46	18.50	18.51	18.58	18.57
	mean	17.02	17.00	16.99	16.97	16.96	16.94	16.92	16.91	16.89	16.88	16.86	16.84	16.83	16.81	16.79	16.78	16.76	16.75	16.73	16.72
	2.5%	16.54	16.33	16.18	16.07	15.94	15.83	15.72	15.66	15.55	15.47	15.41	15.31	15.23	15.14	15.09	15.03	14.94	14.91	14.84	14.75
75-79	97.5%	13.37	13.51	13.62	13.71	13.80	13.84	13.88	13.94	13.98	14.03	14.08	14.10	14.15	14.18	14.23	14.22	14.27	14.28	14.33	14.

Table A.18. The forecasted Finland male life expectancy by model 3

age 0	97.5%	79.79	80.16	80.48	80.68	80.81	80.93	80.95	81.00	81.09	81.18	81.29	81.35	81.39	81.42	81.51	81.53	81.53	81.66	81.65	81.65
	mean	78.15	77.92	77.73	77.55	77.36	77.16	76.94	76.76	76.56	76.37	76.17	75.95	75.75	75.52	75.32	75.10	74.90	74.71	74.50	74.31
	2.5%	76.18	75.00	73.91	73.15	72.14	71.28	70.35	69.55	68.58	67.70	66.77	65.74	65.20	64.06	63.05	62.10	61.37	60.30	59.90	58.82
1-4	97.5%	78.91	79.27	79.57	79.78	79.91	80.02	80.04	80.09	80.17	80.26	80.37	80.43	80.46	80.49	80.59	80.59	80.60	80.72	80.73	80.71
	mean	77.29	77.07	76.88	76.70	76.50	76.30	76.09	75.91	75.71	75.52	75.32	75.11	74.90	74.68	74.47	74.25	74.06	73.87	73.66	73.46
	2.5%	75.35	74.18	73.13	72.35	71.35	70.49	69.56	68.79	67.79	66.91	66.02	65.01	64.46	63.30	62.35	61.38	60.65	59.56	59.23	58.10
5-9	97.5%	74.94	75.29	75.59	75.80	75.92	76.04	76.05	76.11	76.19	76.28	76.39	76.45	76.47	76.51	76.60	76.61	76.61	76.73	76.74	76.72
	mean	73.34	73.11	72.92	72.75	72.55	72.35	72.14	71.96	71.76	71.57	71.37	71.16	70.95	70.74	70.53	70.31	70.12	69.93	69.72	69.53
	2.5%	71.42	70.26	69.20	68.43	67.44	66.59	65.66	64.90	63.89	63.05	62.12	61.14	60.60	59.46	58.50	57.54	56.78	55.72	55.43	54.28
10-14	97.5%	69.96	70.30	70.61	70.81	70.94	71.05	71.06	71.11	71.20	71.29	71.40	71.46	71.48	71.52	71.61	71.61	71.62	71.74	71.75	71.73
	mean	68.36	68.14	67.95	67.77	67.58	67.38	67.17	66.99	66.79	66.60	66.40	66.19	65.99	65.77	65.56	65.35	65.15	64.97	64.76	64.57
	2.5%	66.45	65.30	64.25	63.47	62.49	61.65	60.72	59.97	58.96	58.13	57.20	56.22	55.68	54.56	53.56	52.65	51.86	50.81	50.50	49.41
15-19	97.5%	64.99	65.33	65.63	65.83	65.95	66.06	66.08	66.13	66.22	66.31	66.41	66.47	66.49	66.53	66.62	66.63	66.63	66.75	66.76	66.74
	mean	63.39	63.17	62.98	62.81	62.61	62.41	62.20	62.03	61.83	61.64	61.44	61.23	61.03	60.81	60.61	60.39	60.20	60.01	59.80	59.61
	2.5%	61.50	60.34	59.30	58.53	57.57	56.71	55.80	55.04	54.04	53.20	52.29	51.30	50.75	49.64	48.63	47.75	46.98	45.89	45.57	44.51
20-24	97.5%	60.05	60.39	60.67	60.87	61.00	61.10	61.13	61.17	61.26	61.34	61.44	61.51	61.55	61.57	61.66	61.66	61.67	61.79	61.80	61.79
	mean	58.50	58.29	58.11	57.94	57.76	57.57	57.37	57.20	57.01	56.83	56.64	56.44	56.25	56.04	55.85	55.65	55.47	55.29	55.09	54.92
	2.5%	56.67	55.55	54.54	53.84	52.92	52.11	51.22	50.52	49.58	48.79	47.89	46.93	46.43	45.31	44.51	43.68	42.85	41.84	41.52	40.47
25-29	97.5%	55.17	55.49	55.76	55.94	56.06	56.17	56.21	56.24	56.34	56.45	56.51	56.58	56.61	56.65	56.73	56.76	56.76	56.87	56.87	56.88
	mean	53.74	53.56	53.42	53.28	53.14	52.99	52.83	52.70	52.55	52.41	52.27	52.11	51.96	51.80	51.64	51.48	51.34	51.20	51.05	50.91
	2.5%	52.12	51.18	50.38	49.81	49.08	48.42	47.77	47.28	46.51	46.07	45.28	44.52	44.15	43.23	42.56	41.94	41.46	40.70	40.12	39.67
30-34	97.5%	50.30	50.60	50.85	51.04	51.17	51.26	51.28	51.33	51.42	51.51	51.59	51.65	51.70	51.76	51.81	51.87	51.87	51.97	51.97	51.99
	mean	48.98	48.83	48.72	48.61	48.50	48.38	48.26	48.16	48.04	47.94	47.83	47.71	47.59	47.47	47.36	47.23	47.12	47.02	46.90	46.79
	2.5%	47.52	46.75	46.14	45.67	45.09	44.59	44.09	43.77	43.22	42.79	42.28	41.74	41.43	40.72	40.19	40.04	39.49	38.95	38.66	38.27
35-39	97.5%	45.44	45.71	45.97	46.15	46.24	46.36	46.39	46.42	46.52	46.61	46.68	46.77	46.79	46.86	46.92	46.94	46.98	47.06	47.10	47.13
	mean	44.23	44.10	44.01	43.92	43.83	43.74	43.65	43.57	43.49	43.41	43.33	43.24	43.15	43.06	42.98	42.88	42.80	42.72	42.63	42.55
	2.5%	42.92	42.27	41.78	41.38	41.00	40.58	40.18	39.95	39.58	39.22	38.84	38.48	38.23	37.72	37.42	37.25	36.84	36.46	36.27	35.90
40-44	97.5%	40.62	40.86	41.12	41.29	41.38	41.47	41.53	41.54	41.64	41.72	41.81	41.87	41.92	41.99	42.07	42.08	42.10	42.20	42.24	42.24
	mean	39.50	39.39	39.32	39.26	39.19	39.13	39.06	39.01	38.95	38.89	38.83	38.77	38.71	38.65	38.58	38.52	38.46	38.41	38.35	38.29
	2.5%	38.34	37.79	37.38	37.08	36.78	36.49	36.22	36.05	35.77	35.51	35.19	35.01	34.83	34.47	34.27	34.11	33.84	33.67	33.38	33.15
45-49	97.5%	35.84	36.09	36.31	36.48	36.56	36.66	36.69	36.71	36.80	36.90	36.98	37.04	37.08	37.15	37.22	37.23	37.26	37.36	37.38	37.39
	mean	34.82	34.73	34.67	34.63	34.57	34.52	34.47	34.43	34.39	34.35	34.31	34.26	34.22	34.17	34.13	34.08	34.04	34.00	33.96	33.92
	2.5%	33.78	33.31	32.98	32.74	32.49	32.31	32.08	31.93	31.71	31.56	31.32	31.20	31.04	30.77	30.67	30.52	30.40	30.18	30.04	29.88
50-54	97.5%	31.23	31.45	31.65	31.81	31.87	31.95	31.99	32.02	32.11	32.19	32.25	32.33	32.38	32.44	32.49	32.51	32.52	32.62	32.66	32.68
	mean	30.30	30.22	30.17	30.13	30.09	30.05	30.01	29.98	29.95	29.92	29.89	29.85	29.82	29.79	29.75	29.72	29.69	29.66	29.63	29.60
	2.5%	29.35	28.95	28.68	28.50	28.27	28.12	27.94	27.79	27.66	27.53	27.38	27.26	27.13	26.98	26.84	26.78	26.63	26.51	26.40	26.28
55-59	97.5%	26.77	26.96	27.14	27.29	27.35	27.42	27.45	27.48	27.56	27.63	27.69	27.77	27.81	27.89	27.92	27.94	27.97	28.04	28.06	28.08
	mean	25.90	25.83	25.78	25.75	25.71	25.68	25.64	25.62	25.60	25.57	25.55	25.52	25.49	25.46	25.44	25.41	25.39	25.36	25.34	25.31
	2.5%	25.04	24.68	24.45	24.27	24.09	23.97	23.82	23.70	23.59	23.46	23.35	23.23	23.12	23.00	22.89	22.84	22.71	22.63	22.52	22.47
60-64	97.5%	22.58	22.75	22.90	23.03	23.08	23.15	23.15	23.20	23.27	23.35	23.39	23.44	23.49	23.56	23.59	23.61	23.62	23.69	23.71	23.76
	mean	21.78	21.70	21.66	21.63	21.60	21.57	21.54	21.52	21.49	21.47	21.45	21.43	21.41	21.38	21.36	21.33	21.32	21.30	21.28	21.25
	2.5%	20.97	20.67	20.43	20.29	20.15	20.03	19.91	19.81	19.72	19.62	19.50	19.40	19.34	19.23	19.14	19.07	18.96	18.89	18.80	18.76
65-69	97.5%	18.72	18.86	19.00	19.11	19.14	19.19	19.22	19.25	19.32	19.36	19.44	19.48	19.52	19.56	19.61	19.60	19.63	19.68	19.71	19.74
	mean	17.98	17.90	17.86	17.83	17.80	17.77	17.74	17.73	17.71	17.69	17.67	17.65	17.63	17.61	17.59	17.57	17.56	17.54	17.52	17.50
	2.5%	17.24	16.95	16.76	16.63	16.51	16.41	16.28	16.20	16.11	16.04	15.93	15.87	15.81	15.72	15.62	15.57	15.47	15.43	15.35	15.30
70-74	97.5%	15.05	15.18	15.30	15.37	15.40	15.45	15.45	15.50	15.52	15.57	15.65	15.68	15.71	15.74	15.79	15.76	15.80	15.85	15.87	15.88
	mean	14.36	14.28	14.24	14.21	14.18	14.15	14.13	14.11	14.10	14.08	14.07	14.05	14.03	14.01	14.00	13.98	13.96	13.95	13.93	13.92
	2.5%	13.67	13.41	13.23	13.13	13.03	12.94	12.84	12.76	12.68	12.62	12.53	12.49	12.44	12.35	12.29	12.25	12.16	12.09	12.06	12.02
75-79	97.5%	11.62	11.71	11.80	11.87	11.89	11.91	11.90	11.93	11.97	12.00	12.06	12.09	12.12	12.13	12.15	12.1				

Table A.19. The forecasted Finland female life expectancy by model 8

age 0	97.5%	84.69	85.09	85.44	85.75	86.04	86.33	86.60	86.86	87.12	87.41	87.61	87.86	88.12	88.39	88.60	88.80	89.03	89.24	89.43	89.65
	mean	83.92	84.12	84.31	84.51	84.70	84.89	85.08	85.27	85.46	85.64	85.82	86.00	86.19	86.36	86.53	86.71	86.88	87.04	87.21	87.37
	2.5%	83.07	83.08	83.11	83.15	83.19	83.25	83.35	83.43	83.51	83.61	83.67	83.77	83.89	84.00	84.05	84.18	84.26	84.42	84.50	84.60
1-4	97.5%	83.84	84.22	84.55	84.86	85.14	85.42	85.69	85.94	86.19	86.47	86.67	86.91	87.17	87.44	87.64	87.84	88.08	88.28	88.46	88.67
	mean	83.09	83.28	83.46	83.65	83.83	84.02	84.20	84.39	84.56	84.75	84.92	85.10	85.27	85.44	85.60	85.78	85.94	86.10	86.27	86.43
	2.5%	82.27	82.28	82.30	82.33	82.36	82.43	82.53	82.60	82.69	82.77	82.83	82.92	83.02	83.12	83.19	83.30	83.38	83.55	83.61	83.71
5-9	97.5%	79.88	80.25	80.58	80.89	81.16	81.44	81.71	81.96	82.21	82.49	82.68	82.93	83.18	83.44	83.65	83.85	84.08	84.28	84.46	84.67
	mean	79.15	79.33	79.51	79.70	79.88	80.06	80.25	80.43	80.61	80.79	80.96	81.13	81.31	81.48	81.64	81.81	81.98	82.14	82.30	82.46
	2.5%	78.33	78.35	78.39	78.43	78.47	78.52	78.61	78.70	78.77	78.87	78.93	79.05	79.12	79.22	79.29	79.43	79.50	79.64	79.74	79.82
10-14	97.5%	74.91	75.27	75.60	75.91	76.18	76.46	76.72	76.97	77.22	77.49	77.69	77.93	78.19	78.45	78.65	78.86	79.08	79.29	79.47	79.67
	mean	74.18	74.37	74.55	74.73	74.91	75.09	75.28	75.46	75.64	75.82	75.99	76.16	76.34	76.51	76.67	76.84	77.00	77.16	77.33	77.48
	2.5%	73.37	73.41	73.45	73.48	73.52	73.59	73.68	73.78	73.85	73.94	74.00	74.12	74.22	74.31	74.39	74.50	74.59	74.71	74.83	74.90
15-19	97.5%	69.93	70.29	70.62	70.92	71.19	71.47	71.73	71.98	72.22	72.50	72.70	72.94	73.20	73.46	73.66	73.87	74.09	74.29	74.47	74.68
	mean	69.20	69.39	69.57	69.75	69.93	70.11	70.30	70.48	70.65	70.84	71.01	71.18	71.36	71.52	71.68	71.86	72.02	72.18	72.34	72.50
	2.5%	68.40	68.45	68.47	68.51	68.55	68.62	68.71	68.81	68.89	68.99	69.04	69.15	69.25	69.34	69.43	69.54	69.62	69.74	69.87	69.94
20-24	97.5%	64.98	65.34	65.66	65.96	66.23	66.50	66.76	67.00	67.25	67.53	67.72	67.97	68.21	68.48	68.68	68.89	69.10	69.31	69.49	69.71
	mean	64.27	64.45	64.63	64.81	64.99	65.17	65.35	65.54	65.71	65.89	66.06	66.23	66.41	66.57	66.73	66.90	67.06	67.22	67.39	67.54
	2.5%	63.47	63.53	63.55	63.60	63.64	63.71	63.81	63.90	63.98	64.10	64.14	64.26	64.34	64.44	64.51	64.65	64.73	64.83	64.96	65.06
25-29	97.5%	60.06	60.41	60.72	61.01	61.28	61.55	61.81	62.05	62.30	62.57	62.77	63.00	63.25	63.51	63.71	63.92	64.13	64.33	64.52	64.74
	mean	59.36	59.55	59.72	59.90	60.08	60.26	60.44	60.62	60.79	60.97	61.13	61.31	61.48	61.64	61.80	61.97	62.13	62.29	62.45	62.61
	2.5%	58.59	58.63	58.67	58.73	58.77	58.84	58.96	59.03	59.12	59.22	59.28	59.39	59.48	59.59	59.65	59.79	59.89	59.99	60.11	60.19
30-34	97.5%	55.15	55.49	55.80	56.09	56.35	56.61	56.86	57.11	57.35	57.61	57.81	58.05	58.29	58.55	58.75	58.95	59.16	59.36	59.55	59.77
	mean	54.47	54.65	54.83	55.00	55.18	55.36	55.53	55.71	55.88	56.05	56.22	56.39	56.56	56.72	56.88	57.05	57.21	57.36	57.52	57.67
	2.5%	53.72	53.77	53.81	53.86	53.91	53.99	54.09	54.18	54.26	54.38	54.43	54.54	54.65	54.73	54.80	54.94	55.04	55.15	55.25	55.31
35-39	97.5%	50.23	50.57	50.87	51.15	51.40	51.67	51.92	52.16	52.40	52.66	52.85	53.09	53.33	53.59	53.78	53.98	54.19	54.38	54.58	54.79
	mean	49.56	49.74	49.91	50.09	50.26	50.44	50.61	50.79	50.95	51.13	51.29	51.46	51.63	51.79	51.95	52.11	52.27	52.42	52.58	52.73
	2.5%	48.82	48.87	48.92	48.97	49.03	49.10	49.19	49.28	49.38	49.48	49.54	49.65	49.77	49.84	49.91	50.06	50.15	50.25	50.35	50.40
40-44	97.5%	45.35	45.67	45.97	46.24	46.49	46.75	46.99	47.23	47.47	47.72	47.91	48.15	48.38	48.64	48.84	49.04	49.23	49.43	49.63	49.83
	mean	44.69	44.87	45.04	45.21	45.38	45.55	45.72	45.89	46.06	46.23	46.39	46.55	46.72	46.88	47.03	47.19	47.35	47.50	47.66	47.80
	2.5%	43.96	44.01	44.06	44.11	44.19	44.24	44.33	44.42	44.52	44.63	44.67	44.79	44.91	44.99	45.07	45.19	45.30	45.38	45.50	45.53
45-49	97.5%	40.51	40.81	41.11	41.37	41.62	41.86	42.10	42.34	42.57	42.82	43.01	43.23	43.46	43.71	43.90	44.11	44.31	44.50	44.69	44.89
	mean	39.86	40.03	40.20	40.36	40.53	40.69	40.86	41.03	41.19	41.36	41.51	41.67	41.84	41.99	42.14	42.30	42.45	42.60	42.75	42.90
	2.5%	39.14	39.19	39.24	39.31	39.38	39.43	39.53	39.59	39.69	39.80	39.85	39.94	40.08	40.13	40.21	40.34	40.44	40.52	40.62	40.67
50-54	97.5%	35.76	36.05	36.34	36.59	36.82	37.06	37.30	37.52	37.74	37.99	38.18	38.40	38.62	38.87	39.04	39.24	39.44	39.63	39.81	40.00
	mean	35.13	35.30	35.45	35.62	35.78	35.94	36.10	36.26	36.42	36.58	36.73	36.88	37.04	37.19	37.34	37.49	37.64	37.78	37.93	38.07
	2.5%	34.43	34.48	34.54	34.59	34.67	34.73	34.80	34.88	34.99	35.07	35.13	35.23	35.35	35.39	35.47	35.59	35.69	35.75	35.85	35.91
55-59	97.5%	31.15	31.42	31.69	31.94	32.16	32.39	32.62	32.84	33.05	33.29	33.46	33.66	33.88	34.13	34.30	34.48	34.66	34.85	35.03	35.20
	mean	30.53	30.69	30.84	31.00	31.15	31.30	31.46	31.61	31.76	31.92	32.06	32.21	32.36	32.50	32.64	32.79	32.93	33.07	33.21	33.34
	2.5%	29.85	29.90	29.96	30.02	30.08	30.13	30.20	30.28	30.37	30.46	30.52	30.61	30.73	30.77	30.85	30.97	31.05	31.11	31.21	31.22
60-64	97.5%	26.65	26.91	27.15	27.39	27.60	27.82	28.03	28.23	28.45	28.65	28.82	29.03	29.24	29.46	29.63	29.80	29.99	30.15	30.32	30.50
	mean	26.05	26.20	26.34	26.49	26.63	26.78	26.92	27.07	27.21	27.36	27.49	27.63	27.77	27.91	28.04	28.18	28.31	28.44	28.57	28.69
	2.5%	25.38	25.45	25.48	25.55	25.61	25.66	25.72	25.81	25.88	25.96	26.02	26.11	26.18	26.25	26.33	26.40	26.50	26.58	26.66	26.67
65-69	97.5%	21.72	21.97	22.22	22.45	22.67	22.89	23.11	23.30	23.52	23.74	23.92	24.13	24.32	24.55	24.71	24.90	25.07	25.26	25.43	25.59
	mean	21.15	21.29	21.44	21.59	21.74	21.88	22.03	22.18	22.33	22.47	22.61	22.76	22.90	23.04	23.18	23.32	23.46	23.59	23.73	23.86
	2.5%	20.57	20.61	20.68	20.73	20.80	20.85	20.94	21.02	21.10	21.19	21.26	21.34	21.44	21.49	21.59	21.69	21.78	21.91	21.95	22.04
70-74	97.5%	17.51	17.75	17.98	18.18	18.39	18.58	18.76	18.95	19.13	19.35	19.50	19.69	19.85	20.06	20.22	20.39	20.52	20.69	20.83	20.99
	mean	16.98	17.10	17.23	17.36	17.49	17.61	17.74	17.87	18.00	18.13	18.25	18.37	18.50	18.62	18.74	18.86	18.98	19.09	19.21	19.32
	2.5%	16.43	16.47	16.50	16.54	16.60	16.65	16.70	16.76	16.83	16.90	16.95	17.03	17.10	17.15	17.23	17.30	17.38	17.46	17.54	17.60
75-79	97.5%	13.46	13.66	13.84	14.01	14.19	14.36	14.50	14.64	14.79	14.99	15.11	15.28	15.42	15.59	15.72	15.86	15.98	16.14	16.25	16.38</

Table A.20. The forecasted Finland male life expectancy by model 8

age	97.5%	78.87	79.46	79.89	80.34	80.72	81.08	81.50	81.83	82.16	82.49	82.79	83.12	83.43	83.71	84.01	84.31	84.61	84.86	85.08	85.34
	mean	77.28	77.46	77.59	77.77	77.93	78.09	78.27	78.43	78.58	78.75	78.88	79.06	79.21	79.36	79.46	79.64	79.77	79.92	80.04	80.18
	2.5%	75.10	74.77	74.38	74.09	73.73	73.20	73.10	72.64	72.58	72.16	71.89	71.50	71.18	71.07	70.35	69.90	69.90	69.85	68.78	67.94
1-4	97.5%	78.04	78.61	79.02	79.46	79.83	80.18	80.60	80.92	81.25	81.55	81.85	82.18	82.49	82.75	83.05	83.35	83.64	83.89	84.11	84.36
	mean	76.47	76.64	76.75	76.92	77.07	77.22	77.40	77.55	77.69	77.86	77.98	78.15	78.29	78.44	78.53	78.71	78.83	78.98	79.09	79.23
	2.5%	74.30	73.96	73.59	73.28	72.89	72.35	72.29	71.78	71.72	71.33	71.04	70.59	70.29	70.19	69.46	69.02	69.00	68.95	67.86	67.00
5-9	97.5%	74.08	74.64	75.04	75.49	75.85	76.20	76.62	76.94	77.26	77.56	77.86	78.19	78.50	78.77	79.06	79.35	79.65	79.90	80.12	80.36
	mean	72.51	72.68	72.80	72.97	73.11	73.26	73.44	73.59	73.72	73.89	74.02	74.19	74.33	74.47	74.57	74.74	74.86	75.01	75.12	75.25
	2.5%	70.37	70.03	69.66	69.37	68.95	68.42	68.35	67.89	67.80	67.38	67.10	66.68	66.37	66.28	65.54	65.09	65.09	65.04	63.95	63.13
10-14	97.5%	69.10	69.65	70.06	70.50	70.86	71.21	71.63	71.94	72.27	72.57	72.86	73.20	73.50	73.77	74.07	74.35	74.65	74.90	75.12	75.36
	mean	67.54	67.71	67.82	67.99	68.14	68.28	68.46	68.60	68.74	68.91	69.03	69.20	69.34	69.49	69.58	69.76	69.87	70.02	70.13	70.27
	2.5%	65.39	65.05	64.68	64.40	63.98	63.47	63.38	62.90	62.85	62.41	62.14	61.71	61.40	61.30	60.59	60.15	60.11	60.06	59.00	58.14
15-19	97.5%	64.12	64.67	65.07	65.51	65.88	66.22	66.64	66.95	67.28	67.58	67.87	68.21	68.51	68.78	69.07	69.36	69.66	69.91	70.13	70.37
	mean	62.57	62.73	62.85	63.01	63.16	63.31	63.48	63.63	63.76	63.93	64.05	64.22	64.36	64.50	64.60	64.77	64.89	65.03	65.14	65.28
	2.5%	60.42	60.08	59.73	59.43	59.01	58.51	58.41	57.95	57.87	57.44	57.16	56.73	56.42	56.32	55.63	55.18	55.13	55.12	54.02	53.14
20-24	97.5%	59.20	59.74	60.14	60.56	60.93	61.27	61.67	62.00	62.31	62.63	62.91	63.23	63.53	63.80	64.09	64.38	64.68	64.93	65.15	65.39
	mean	57.69	57.86	57.98	58.14	58.29	58.44	58.61	58.76	58.90	59.07	59.19	59.36	59.50	59.65	59.75	59.93	60.04	60.19	60.30	60.44
	2.5%	55.60	55.26	54.94	54.67	54.28	53.82	53.63	53.28	53.27	52.78	52.49	52.20	51.92	51.78	51.04	50.74	50.52	50.65	49.57	48.65
25-29	97.5%	54.38	54.86	55.27	55.67	56.03	56.38	56.76	57.10	57.40	57.70	57.99	58.32	58.61	58.89	59.17	59.45	59.75	60.02	60.24	60.50
	mean	52.97	53.16	53.29	53.49	53.66	53.84	54.04	54.22	54.39	54.59	54.75	54.95	55.13	55.31	55.44	55.66	55.82	56.00	56.16	56.34
	2.5%	51.02	50.82	50.59	50.51	50.23	50.04	50.01	49.72	49.79	49.72	49.30	49.38	49.19	49.13	48.83	48.67	48.40	48.75	48.24	47.98
30-34	97.5%	49.66	50.14	50.52	50.95	51.31	51.65	52.04	52.37	52.69	53.05	53.31	53.67	53.99	54.27	54.57	54.84	55.15	55.41	55.66	55.90
	mean	48.54	48.76	48.95	49.19	49.42	49.62	49.84	50.07	50.30	50.52	50.73	50.96	51.18	51.39	51.60	51.83	52.04	52.25	52.46	52.66
	2.5%	47.30	47.27	47.10	47.13	47.06	47.05	47.10	47.12	47.11	47.06	47.11	47.14	47.20	47.23	47.19	47.29	47.38	47.36	47.32	47.39
35-39	97.5%	44.80	45.26	45.64	46.05	46.40	46.73	47.12	47.44	47.76	48.10	48.36	48.71	49.03	49.32	49.62	49.88	50.19	50.44	50.69	50.92
	mean	43.75	43.97	44.16	44.39	44.62	44.83	45.06	45.29	45.51	45.74	45.95	46.18	46.41	46.62	46.83	47.06	47.28	47.49	47.70	47.91
	2.5%	42.61	42.59	42.48	42.50	42.50	42.52	42.56	42.62	42.65	42.73	42.72	42.86	42.97	42.98	43.03	43.15	43.24	43.25	43.30	43.34
40-44	97.5%	40.03	40.45	40.84	41.22	41.56	41.89	42.28	42.57	42.89	43.24	43.47	43.83	44.15	44.41	44.70	44.98	45.28	45.53	45.78	46.02
	mean	39.08	39.30	39.51	39.75	39.97	40.19	40.43	40.66	40.88	41.12	41.35	41.58	41.81	42.04	42.25	42.49	42.72	42.94	43.16	43.37
	2.5%	38.06	38.10	38.06	38.13	38.19	38.29	38.34	38.44	38.53	38.66	38.71	38.86	39.01	39.15	39.21	39.40	39.54	39.65	39.80	39.86
45-49	97.5%	35.32	35.72	36.08	36.45	36.77	37.10	37.46	37.76	38.04	38.38	38.63	38.96	39.27	39.54	39.82	40.10	40.39	40.64	40.89	41.13
	mean	34.44	34.66	34.88	35.11	35.33	35.56	35.79	36.02	36.25	36.48	36.71	36.94	37.17	37.40	37.62	37.85	38.08	38.30	38.53	38.74
	2.5%	33.51	33.60	33.60	33.71	33.77	33.88	33.98	34.13	34.22	34.42	34.51	34.63	34.81	34.94	35.07	35.24	35.43	35.56	35.70	35.84
50-54	97.5%	30.79	31.15	31.48	31.82	32.14	32.46	32.78	33.06	33.34	33.63	33.91	34.23	34.52	34.77	35.05	35.33	35.58	35.83	36.09	36.31
	mean	29.97	30.19	30.40	30.62	30.84	31.06	31.28	31.51	31.73	31.96	32.18	32.41	32.63	32.85	33.07	33.30	33.52	33.73	33.95	34.16
	2.5%	29.14	29.23	29.28	29.40	29.50	29.61	29.72	29.87	29.98	30.17	30.30	30.44	30.62	30.78	30.92	31.09	31.27	31.40	31.55	31.74
55-59	97.5%	26.38	26.71	27.00	27.31	27.61	27.89	28.20	28.46	28.73	29.00	29.24	29.56	29.84	30.08	30.34	30.58	30.86	31.11	31.32	31.52
	mean	25.61	25.81	26.01	26.22	26.43	26.63	26.85	27.06	27.27	27.48	27.69	27.91	28.12	28.33	28.54	28.75	28.96	29.17	29.38	29.58
	2.5%	24.81	24.94	25.00	25.11	25.23	25.34	25.45	25.59	25.72	25.92	26.04	26.18	26.35	26.48	26.61	26.79	26.96	27.11	27.22	27.39
60-64	97.5%	22.23	22.54	22.81	23.10	23.36	23.62	23.90	24.16	24.41	24.65	24.88	25.15	25.42	25.66	25.88	26.13	26.38	26.60	26.81	27.01
	mean	21.52	21.71	21.90	22.10	22.30	22.49	22.69	22.89	23.09	23.29	23.49	23.69	23.89	24.09	24.28	24.49	24.68	24.87	25.07	25.25
	2.5%	20.80	20.91	21.00	21.11	21.23	21.35	21.45	21.61	21.73	21.91	22.03	22.15	22.33	22.46	22.58	22.77	22.91	23.05	23.18	23.34
65-69	97.5%	18.40	18.64	18.89	19.15	19.38	19.61	19.84	20.08	20.29	20.49	20.72	20.95	21.19	21.36	21.60	21.81	22.02	22.23	22.40	22.58
	mean	17.71	17.87	18.03	18.21	18.38	18.55	18.72	18.89	19.06	19.24	19.41	19.58	19.76	19.93	20.09	20.27	20.44	20.60	20.77	20.93
	2.5%	17.06	17.13	17.20	17.29	17.37	17.50	17.58	17.68	17.82	17.95	18.05	18.16	18.31	18.43	18.52	18.66	18.78	18.91	19.04	19.13
70-74	97.5%	14.75	14.94	15.14	15.35	15.55	15.73	15.93	16.12	16.32	16.46	16.67	16.84	17.03	17.20	17.38	17.54	17.73	17.87	18.03	18.18
	mean	14.10	14.23	14.36	14.50	14.64	14.78	14.92	15.07	15.20	15.35	15.49	15.63	15.77	15.90	16.04	16.18	16.32	16.45	16.59	16.72
	2.5%	13.49	13.54	13.58	13.67	13.75	13.84	13.91	14.00	14.11	14.21	14.28	14.38	14.47	14.59	14.66	14.76	14.88	14.97	15.05	15.16
75-79	97.5%	11.25	11.40	11.52	11.70	11.84	11.97	12.13	12.26	12.39	12.52	12.63	12.78	12.93	13.05	13.20	13.31				

Table A.21. The forecasted France female life expectancy by model 3

age 0	97.5%	85.77	86.06	86.31	86.49	86.65	86.74	86.84	86.92	86.98	87.08	87.20	87.26	87.30	87.39	87.48	87.53	87.57	87.67	87.71	87.75
	mean	84.90	84.85	84.82	84.80	84.76	84.72	84.67	84.64	84.60	84.57	84.53	84.49	84.44	84.40	84.36	84.31	84.27	84.23	84.19	84.14
	2.5%	83.93	83.48	83.12	82.87	82.55	82.29	82.04	81.86	81.63	81.35	81.11	80.93	80.72	80.47	80.20	80.06	79.79	79.61	79.40	79.25
1-4	97.5%	84.99	85.27	85.52	85.68	85.82	85.90	86.00	86.08	86.15	86.24	86.35	86.40	86.45	86.53	86.62	86.66	86.71	86.79	86.84	86.87
	mean	84.17	84.13	84.11	84.08	84.05	84.01	83.97	83.94	83.90	83.87	83.83	83.79	83.76	83.71	83.68	83.63	83.60	83.56	83.52	83.48
	2.5%	83.27	82.86	82.52	82.28	82.00	81.76	81.53	81.35	81.14	80.91	80.69	80.51	80.32	80.07	79.86	79.70	79.52	79.33	79.13	79.00
5-9	97.5%	81.04	81.31	81.55	81.71	81.85	81.92	82.03	82.11	82.18	82.26	82.36	82.43	82.47	82.55	82.64	82.68	82.73	82.81	82.86	82.89
	mean	80.22	80.18	80.16	80.13	80.10	80.06	80.02	79.99	79.96	79.92	79.89	79.85	79.81	79.77	79.74	79.69	79.66	79.62	79.58	79.55
	2.5%	79.33	78.93	78.59	78.36	78.09	77.85	77.62	77.45	77.25	77.03	76.80	76.63	76.43	76.22	76.00	75.85	75.65	75.46	75.29	75.16
10-14	97.5%	76.06	76.33	76.57	76.72	76.87	76.94	77.05	77.12	77.19	77.27	77.38	77.44	77.48	77.56	77.65	77.69	77.74	77.82	77.87	77.90
	mean	75.25	75.21	75.19	75.16	75.13	75.09	75.05	75.02	74.99	74.96	74.92	74.89	74.85	74.81	74.77	74.73	74.70	74.66	74.62	74.58
	2.5%	74.37	73.97	73.65	73.41	73.15	72.91	72.69	72.51	72.31	72.10	71.87	71.71	71.51	71.30	71.09	70.94	70.73	70.55	70.39	70.24
15-19	97.5%	71.08	71.35	71.58	71.74	71.88	71.96	72.06	72.14	72.20	72.29	72.39	72.45	72.49	72.58	72.66	72.70	72.75	72.83	72.88	72.91
	mean	70.28	70.24	70.22	70.19	70.16	70.12	70.08	70.06	70.02	69.99	69.96	69.92	69.88	69.84	69.81	69.76	69.73	69.70	69.66	69.62
	2.5%	69.41	69.01	68.69	68.45	68.19	67.97	67.74	67.56	67.37	67.16	66.93	66.78	66.58	66.37	66.17	66.02	65.82	65.65	65.49	65.35
20-24	97.5%	66.12	66.38	66.62	66.77	66.91	66.99	67.09	67.16	67.23	67.31	67.41	67.48	67.52	67.60	67.68	67.73	67.77	67.85	67.90	67.93
	mean	65.33	65.29	65.27	65.24	65.21	65.18	65.14	65.11	65.08	65.05	65.02	64.98	64.94	64.91	64.87	64.83	64.80	64.76	64.73	64.69
	2.5%	64.47	64.08	63.77	63.54	63.28	63.07	62.83	62.66	62.48	62.28	62.06	61.90	61.72	61.51	61.31	61.15	60.98	60.82	60.64	60.51
25-29	97.5%	61.17	61.42	61.66	61.81	61.95	62.02	62.12	62.19	62.25	62.34	62.45	62.50	62.54	62.63	62.71	62.76	62.80	62.87	62.92	62.95
	mean	60.39	60.36	60.33	60.31	60.28	60.25	60.21	60.19	60.15	60.12	60.09	60.06	60.03	59.99	59.96	59.92	59.89	59.86	59.82	59.79
	2.5%	59.56	59.18	58.87	58.66	58.41	58.20	57.97	57.82	57.64	57.45	57.24	57.10	56.90	56.70	56.52	56.37	56.19	56.05	55.91	55.77
30-34	97.5%	56.23	56.48	56.70	56.85	56.99	57.06	57.16	57.23	57.29	57.38	57.48	57.54	57.58	57.65	57.75	57.79	57.82	57.90	57.95	57.98
	mean	55.47	55.44	55.42	55.40	55.37	55.34	55.31	55.28	55.25	55.23	55.20	55.16	55.14	55.10	55.07	55.03	55.01	54.98	54.95	54.92
	2.5%	54.66	54.30	54.01	53.81	53.56	53.39	53.18	53.02	52.85	52.68	52.49	52.34	52.18	51.98	51.84	51.73	51.53	51.41	51.24	51.14
35-39	97.5%	51.30	51.54	51.76	51.91	52.04	52.11	52.21	52.28	52.34	52.42	52.52	52.58	52.62	52.69	52.79	52.82	52.87	52.94	52.99	53.02
	mean	50.56	50.53	50.51	50.49	50.47	50.44	50.41	50.38	50.36	50.33	50.31	50.28	50.25	50.22	50.19	50.15	50.13	50.10	50.07	50.04
	2.5%	49.78	49.43	49.15	48.97	48.73	48.56	48.36	48.21	48.06	47.89	47.71	47.58	47.42	47.23	47.11	46.99	46.82	46.71	46.57	46.48
40-44	97.5%	46.42	46.65	46.86	47.01	47.14	47.21	47.30	47.36	47.43	47.50	47.61	47.67	47.70	47.77	47.86	47.90	47.94	48.01	48.06	48.10
	mean	45.71	45.68	45.66	45.64	45.62	45.59	45.56	45.54	45.52	45.50	45.47	45.44	45.42	45.39	45.36	45.33	45.31	45.28	45.26	45.23
	2.5%	44.96	44.63	44.35	44.19	43.97	43.81	43.62	43.49	43.34	43.17	43.00	42.90	42.75	42.59	42.45	42.38	42.21	42.09	41.94	41.89
45-49	97.5%	41.60	41.81	42.02	42.16	42.29	42.36	42.44	42.50	42.56	42.64	42.74	42.78	42.83	42.89	42.98	43.01	43.07	43.13	43.18	43.20
	mean	40.92	40.90	40.88	40.86	40.84	40.82	40.79	40.77	40.75	40.73	40.70	40.68	40.66	40.63	40.60	40.58	40.55	40.53	40.51	40.49
	2.5%	40.21	39.90	39.65	39.48	39.29	39.13	38.96	38.85	38.71	38.55	38.40	38.28	38.17	38.01	37.90	37.84	37.67	37.56	37.43	37.37
50-54	97.5%	36.88	37.08	37.27	37.41	37.53	37.58	37.66	37.73	37.78	37.86	37.95	37.98	38.04	38.10	38.18	38.22	38.27	38.31	38.36	38.40
	mean	36.24	36.21	36.20	36.18	36.16	36.14	36.12	36.10	36.08	36.06	36.04	36.02	35.99	35.97	35.95	35.92	35.90	35.88	35.86	35.84
	2.5%	35.57	35.28	35.04	34.90	34.71	34.58	34.41	34.31	34.19	34.06	33.90	33.79	33.67	33.56	33.48	33.39	33.23	33.13	33.03	32.97
55-59	97.5%	32.26	32.45	32.62	32.76	32.87	32.92	33.00	33.06	33.11	33.17	33.26	33.29	33.34	33.41	33.48	33.52	33.56	33.60	33.66	33.69
	mean	31.66	31.64	31.63	31.62	31.60	31.58	31.55	31.54	31.52	31.50	31.48	31.46	31.44	31.42	31.40	31.38	31.36	31.34	31.32	31.30
	2.5%	31.04	30.78	30.56	30.43	30.27	30.14	29.99	29.90	29.77	29.65	29.53	29.40	29.31	29.21	29.12	29.04	28.91	28.81	28.73	28.67
60-64	97.5%	27.75	27.92	28.09	28.21	28.31	28.35	28.42	28.48	28.53	28.59	28.66	28.71	28.75	28.80	28.87	28.91	28.95	28.97	29.03	29.07
	mean	27.20	27.18	27.16	27.15	27.14	27.12	27.10	27.08	27.07	27.05	27.04	27.02	27.00	26.98	26.96	26.94	26.92	26.91	26.89	26.87
	2.5%	26.62	26.39	26.18	26.07	25.91	25.80	25.68	25.58	25.47	25.38	25.25	25.15	25.08	24.97	24.89	24.80	24.72	24.61	24.54	24.50
65-69	97.5%	23.33	23.49	23.62	23.74	23.84	23.87	23.93	23.99	24.04	24.09	24.15	24.20	24.24	24.29	24.34	24.38	24.41	24.44	24.50	24.52
	mean	22.82	22.81	22.79	22.78	22.77	22.75	22.73	22.72	22.71	22.69	22.68	22.66	22.64	22.63	22.61	22.59	22.58	22.56	22.55	22.53
	2.5%	22.29	22.09	21.91	21.80	21.66	21.56	21.45	21.38	21.27	21.17	21.08	20.97	20.92	20.82	20.75	20.67	20.60	20.51	20.44	20.38
70-74	97.5%	19.00	19.13	19.25	19.35	19.44	19.48	19.53	19.57	19.62	19.66	19.71	19.76	19.79	19.83	19.88	19.93	19.94	19.98	20.01	20.04
	mean	18.53	18.52	18.51	18.50	18.49	18.47	18.46	18.44	18.43	18.42	18.41	18.39	18.38	18.36	18.35	18.33	18.32	18.31	18.29	18.28
	2.5%	18.06	17.87	17.72	17.63	17.51	17.42	17.33	17.25	17.17	17.09	17.00	16.91	16.86	16.78	16.72	16.66	16.59	16.51	16.46	16.41
75-79	97.5%	14.87	14.97	15.07	15.17	15.24	15.27	15.33	15.35	15.40	15.44	15.48	15.51	15.55	15.57	15.62	15.66	15.68	15.72	15.75	15.

Table A.22. The forecasted France male life expectancy by model 3

age 0	97.5%	79.97	80.35	80.67	80.85	81.04	81.13	81.18	81.25	81.40	81.45	81.61	81.68	81.66	81.74	81.83	81.87	81.85	81.98	82.00	82.04
	mean	78.63	78.48	78.34	78.20	78.04	77.86	77.67	77.51	77.32	77.15	76.96	76.76	76.56	76.35	76.14	75.93	75.74	75.54	75.33	75.13
	2.5%	77.02	76.05	75.14	74.44	73.56	72.69	71.65	71.05	70.14	69.20	68.35	67.34	66.67	65.42	64.26	63.54	62.95	62.00	60.94	60.64
1-4	97.5%	79.22	79.58	79.88	80.03	80.23	80.31	80.37	80.44	80.59	80.62	80.78	80.84	80.81	80.90	81.00	81.02	81.00	81.11	81.15	81.17
	mean	77.93	77.78	77.64	77.50	77.34	77.17	76.98	76.82	76.63	76.46	76.27	76.07	75.88	75.67	75.46	75.25	75.06	74.87	74.66	74.46
	2.5%	76.36	75.43	74.55	73.84	72.99	72.16	71.17	70.56	69.63	68.71	67.83	66.88	66.18	64.92	63.88	63.06	62.50	61.58	60.51	60.17
5-9	97.5%	75.26	75.62	75.92	76.07	76.27	76.34	76.40	76.46	76.62	76.65	76.80	76.87	76.83	76.92	77.02	77.05	77.03	77.13	77.17	77.19
	mean	73.98	73.83	73.69	73.55	73.39	73.22	73.04	72.88	72.69	72.52	72.33	72.13	71.94	71.73	71.53	71.32	71.12	70.93	70.72	70.53
	2.5%	72.43	71.51	70.62	69.93	69.07	68.26	67.25	66.67	65.72	64.82	63.96	63.01	62.30	61.05	59.99	59.19	58.62	57.69	56.64	56.31
10-14	97.5%	70.28	70.64	70.94	71.09	71.28	71.36	71.41	71.48	71.63	71.67	71.82	71.88	71.85	71.94	72.04	72.06	72.05	72.14	72.18	72.20
	mean	69.01	68.86	68.73	68.59	68.43	68.26	68.07	67.91	67.73	67.55	67.37	67.17	66.97	66.77	66.56	66.36	66.16	65.97	65.76	65.57
	2.5%	67.47	66.55	65.67	64.99	64.13	63.32	62.32	61.74	60.79	59.88	59.03	58.09	57.37	56.14	55.09	54.27	53.70	52.78	51.73	51.39
15-19	97.5%	65.31	65.67	65.96	66.11	66.30	66.37	66.43	66.50	66.65	66.68	66.83	66.89	66.87	66.95	67.05	67.07	67.06	67.16	67.19	67.22
	mean	64.05	63.90	63.76	63.62	63.46	63.29	63.11	62.95	62.76	62.59	62.41	62.21	62.01	61.80	61.60	61.40	61.20	61.01	60.80	60.61
	2.5%	62.52	61.59	60.72	60.05	59.19	58.38	57.38	56.80	55.85	54.96	54.11	53.17	52.44	51.21	50.15	49.35	48.80	47.87	46.81	46.45
20-24	97.5%	60.38	60.73	61.01	61.16	61.34	61.41	61.48	61.55	61.69	61.73	61.86	61.94	61.90	61.99	62.08	62.11	62.10	62.21	62.22	62.26
	mean	59.15	59.00	58.87	58.74	58.59	58.42	58.25	58.09	57.92	57.75	57.57	57.38	57.19	57.00	56.80	56.60	56.42	56.24	56.04	55.85
	2.5%	57.67	56.78	55.94	55.28	54.47	53.69	52.68	52.13	51.26	50.42	49.58	48.66	47.92	46.81	45.73	45.07	44.42	43.55	42.53	42.17
25-29	97.5%	55.48	55.81	56.08	56.23	56.40	56.47	56.56	56.62	56.75	56.80	56.93	56.99	57.00	57.08	57.17	57.19	57.20	57.29	57.32	57.38
	mean	54.35	54.24	54.14	54.04	53.93	53.81	53.67	53.56	53.44	53.32	53.19	53.06	52.92	52.78	52.64	52.50	52.36	52.23	52.09	51.96
	2.5%	53.05	52.32	51.65	51.11	50.54	49.95	49.19	48.81	48.10	47.65	47.00	46.39	45.88	45.24	44.55	43.91	43.48	42.78	42.35	41.97
30-34	97.5%	50.59	50.90	51.17	51.32	51.47	51.56	51.63	51.70	51.83	51.88	52.02	52.07	52.07	52.17	52.24	52.27	52.30	52.38	52.42	52.46
	mean	49.56	49.48	49.41	49.35	49.27	49.18	49.09	49.01	48.93	48.84	48.76	48.67	48.58	48.48	48.39	48.29	48.20	48.11	48.01	47.92
	2.5%	48.42	47.84	47.33	46.92	46.53	46.06	45.59	45.31	44.82	44.43	44.03	43.64	43.35	42.88	42.49	42.04	41.83	41.36	41.02	40.81
35-39	97.5%	45.73	46.02	46.28	46.43	46.57	46.66	46.73	46.80	46.92	46.97	47.10	47.16	47.18	47.28	47.32	47.37	47.42	47.48	47.51	47.57
	mean	44.80	44.74	44.69	44.65	44.59	44.53	44.46	44.41	44.35	44.30	44.24	44.18	44.12	44.05	43.99	43.92	43.86	43.80	43.73	43.67
	2.5%	43.79	43.32	42.92	42.63	42.30	41.99	41.67	41.45	41.13	40.86	40.57	40.30	40.08	39.77	39.55	39.23	39.06	38.82	38.55	38.30
40-44	97.5%	40.93	41.19	41.43	41.56	41.72	41.80	41.88	41.93	42.05	42.11	42.22	42.30	42.31	42.39	42.48	42.50	42.53	42.61	42.67	42.70
	mean	40.07	40.02	39.99	39.96	39.92	39.88	39.83	39.79	39.75	39.71	39.67	39.63	39.59	39.54	39.50	39.45	39.41	39.37	39.32	39.28
	2.5%	39.18	38.79	38.45	38.23	37.97	37.69	37.47	37.34	37.10	36.88	36.68	36.50	36.36	36.10	35.93	35.75	35.59	35.48	35.25	35.09
45-49	97.5%	36.20	36.44	36.67	36.79	36.93	37.02	37.09	37.15	37.25	37.30	37.43	37.48	37.50	37.58	37.67	37.71	37.74	37.78	37.84	37.86
	mean	35.42	35.39	35.36	35.34	35.31	35.27	35.24	35.21	35.18	35.15	35.12	35.09	35.06	35.02	34.99	34.95	34.92	34.89	34.86	34.83
	2.5%	34.62	34.27	34.01	33.81	33.61	33.37	33.20	33.07	32.92	32.73	32.56	32.43	32.30	32.11	32.00	31.85	31.72	31.59	31.45	31.39
50-54	97.5%	31.65	31.86	32.07	32.17	32.31	32.38	32.45	32.52	32.61	32.66	32.77	32.81	32.84	32.91	32.98	33.01	33.05	33.11	33.16	33.19
	mean	30.94	30.91	30.89	30.88	30.85	30.83	30.80	30.78	30.75	30.73	30.71	30.68	30.66	30.63	30.61	30.58	30.56	30.53	30.51	30.48
	2.5%	30.21	29.93	29.72	29.53	29.37	29.18	29.03	28.92	28.80	28.65	28.52	28.39	28.32	28.16	28.07	27.98	27.84	27.75	27.66	27.59
55-59	97.5%	27.32	27.49	27.69	27.77	27.91	27.96	28.02	28.09	28.18	28.21	28.32	28.36	28.38	28.45	28.51	28.53	28.56	28.64	28.69	28.72
	mean	26.67	26.64	26.63	26.62	26.60	26.58	26.55	26.54	26.52	26.50	26.48	26.46	26.44	26.42	26.40	26.38	26.36	26.34	26.32	26.30
	2.5%	26.01	25.77	25.59	25.44	25.29	25.13	25.02	24.93	24.82	24.69	24.57	24.47	24.40	24.28	24.19	24.12	24.03	23.96	23.85	23.81
60-64	97.5%	23.30	23.46	23.63	23.72	23.83	23.88	23.94	24.00	24.08	24.13	24.21	24.23	24.27	24.34	24.39	24.42	24.45	24.50	24.54	24.57
	mean	22.70	22.68	22.66	22.65	22.63	22.61	22.59	22.58	22.56	22.55	22.53	22.51	22.49	22.47	22.45	22.43	22.41	22.40	22.38	22.36
	2.5%	22.09	21.87	21.70	21.56	21.43	21.30	21.18	21.09	21.00	20.87	20.76	20.67	20.62	20.52	20.45	20.38	20.25	20.22	20.14	20.07
65-69	97.5%	19.46	19.61	19.75	19.84	19.95	20.01	20.06	20.09	20.15	20.22	20.27	20.31	20.32	20.41	20.45	20.47	20.51	20.56	20.58	20.58
	mean	18.91	18.89	18.88	18.87	18.85	18.84	18.82	18.80	18.79	18.77	18.76	18.74	18.72	18.70	18.69	18.67	18.65	18.64	18.62	18.60
	2.5%	18.35	18.16	18.00	17.88	17.76	17.62	17.53	17.47	17.38	17.28	17.16	17.08	17.02	16.95	16.86	16.83	16.75	16.69	16.61	16.53
70-74	97.5%	15.75	15.87	16.00	16.10	16.17	16.23	16.25	16.31	16.37	16.42	16.46	16.49	16.51	16.58	16.60	16.64	16.64	16.71	16.73	16.74
	mean	15.26	15.24	15.23	15.22	15.21	15.19	15.17	15.16	15.15	15.13	15.12	15.11	15.09	15.07	15.06	15.04	15.03	15.02	15.00	14.99
	2.5%	14.77	14.60	14.45	14.36	14.25	14.15	14.05	13.99	13.90	13.81	13.72	13.65	13.60	13.52	13.45	13.42	13.33	13.28	13.24	13.17
75-79	97.5%	12.25	12.36	12.46	12.53	12.60	12.65	12.68	12.73	12.78	12.81	12.84	12.87	12.89	12.95	12.97	12.9				

Table A.23. The forecasted France male life expectancy by model 3

age	97.5%	85.57	85.83	86.08	86.33	86.49	86.79	86.99	87.18	87.37	87.52	87.77	87.94	88.11	88.29	88.49	88.65	88.81	89.01	89.13	89.30
0	mean	83.97	84.15	84.32	84.48	84.63	84.79	84.94	85.10	85.25	85.41	85.56	85.71	85.86	86.00	86.14	86.28	86.42	86.56	86.70	86.84
	2.5%	82.76	82.83	82.88	82.90	82.96	83.02	83.11	83.18	83.29	83.37	83.44	83.51	83.61	83.67	83.73	83.82	83.92	83.96	84.13	84.15
1-4	97.5%	84.88	85.10	85.33	85.55	85.71	86.00	86.19	86.37	86.57	86.72	86.93	87.09	87.26	87.44	87.64	87.79	87.94	88.13	88.28	88.41
	mean	83.27	83.45	83.61	83.76	83.91	84.06	84.21	84.36	84.50	84.66	84.80	84.94	85.08	85.22	85.35	85.49	85.63	85.76	85.89	86.03
	2.5%	82.10	82.18	82.23	82.27	82.33	82.38	82.46	82.55	82.65	82.72	82.79	82.87	82.97	83.03	83.09	83.19	83.26	83.35	83.47	83.50
5-9	97.5%	80.92	81.14	81.37	81.59	81.74	82.04	82.22	82.39	82.59	82.75	82.96	83.11	83.28	83.46	83.65	83.80	83.95	84.14	84.29	84.42
	mean	79.32	79.49	79.65	79.80	79.95	80.10	80.24	80.40	80.54	80.69	80.83	80.97	81.11	81.24	81.38	81.51	81.65	81.78	81.91	82.04
	2.5%	78.15	78.24	78.28	78.32	78.38	78.42	78.52	78.60	78.70	78.77	78.85	78.92	79.01	79.07	79.14	79.24	79.30	79.39	79.50	79.55
10-14	97.5%	75.95	76.16	76.40	76.61	76.75	77.06	77.24	77.40	77.60	77.76	77.97	78.12	78.29	78.46	78.66	78.81	78.96	79.15	79.29	79.43
	mean	74.35	74.52	74.67	74.83	74.97	75.12	75.26	75.42	75.56	75.71	75.84	75.98	76.13	76.26	76.40	76.53	76.67	76.79	76.93	77.06
	2.5%	73.19	73.27	73.31	73.35	73.41	73.46	73.55	73.63	73.73	73.80	73.87	73.94	74.05	74.10	74.16	74.26	74.33	74.42	74.54	74.58
15-19	97.5%	70.98	71.18	71.42	71.63	71.78	72.07	72.25	72.42	72.61	72.76	72.98	73.13	73.31	73.47	73.67	73.82	73.97	74.16	74.29	74.43
	mean	69.37	69.55	69.70	69.85	70.00	70.14	70.29	70.44	70.58	70.73	70.86	71.00	71.14	71.28	71.41	71.55	71.68	71.81	71.94	72.07
	2.5%	68.21	68.31	68.34	68.39	68.44	68.50	68.59	68.67	68.77	68.84	68.91	68.97	69.08	69.14	69.20	69.29	69.37	69.46	69.58	69.61
20-24	97.5%	66.03	66.23	66.45	66.67	66.81	67.10	67.27	67.44	67.65	67.78	68.01	68.15	68.33	68.49	68.69	68.84	68.99	69.17	69.32	69.45
	mean	64.42	64.59	64.75	64.90	65.04	65.18	65.33	65.48	65.62	65.76	65.90	66.04	66.18	66.31	66.45	66.58	66.71	66.84	66.97	67.10
	2.5%	63.27	63.36	63.40	63.46	63.51	63.56	63.65	63.73	63.83	63.90	63.98	64.05	64.14	64.20	64.28	64.37	64.45	64.54	64.63	64.68
25-29	97.5%	61.08	61.29	61.50	61.71	61.86	62.14	62.31	62.49	62.67	62.82	63.04	63.18	63.36	63.51	63.71	63.86	64.02	64.19	64.34	64.48
	mean	59.48	59.66	59.81	59.96	60.10	60.25	60.39	60.54	60.68	60.83	60.96	61.10	61.24	61.37	61.51	61.64	61.77	61.90	62.03	62.16
	2.5%	58.35	58.44	58.50	58.55	58.61	58.67	58.76	58.84	58.94	59.02	59.09	59.18	59.26	59.31	59.42	59.53	59.59	59.69	59.79	59.84
30-34	97.5%	56.16	56.36	56.56	56.77	56.91	57.19	57.38	57.53	57.72	57.86	58.07	58.22	58.39	58.56	58.75	58.89	59.06	59.21	59.37	59.51
	mean	54.56	54.73	54.89	55.03	55.18	55.32	55.46	55.61	55.75	55.90	56.04	56.17	56.31	56.45	56.58	56.71	56.85	56.97	57.10	57.23
	2.5%	53.45	53.54	53.60	53.65	53.73	53.80	53.90	53.97	54.08	54.16	54.23	54.33	54.41	54.48	54.59	54.69	54.75	54.85	54.96	55.04
35-39	97.5%	51.24	51.45	51.64	51.85	51.98	52.26	52.44	52.60	52.78	52.92	53.12	53.28	53.43	53.62	53.80	53.92	54.11	54.26	54.40	54.55
	mean	49.66	49.83	49.98	50.13	50.27	50.41	50.55	50.70	50.83	50.98	51.11	51.25	51.39	51.52	51.65	51.78	51.92	52.04	52.17	52.30
	2.5%	48.56	48.66	48.71	48.77	48.85	48.92	49.01	49.11	49.20	49.29	49.36	49.46	49.55	49.61	49.71	49.81	49.88	49.97	50.07	50.15
40-44	97.5%	46.38	46.59	46.76	46.96	47.10	47.36	47.54	47.70	47.88	48.02	48.22	48.36	48.53	48.69	48.88	49.02	49.19	49.34	49.46	49.61
	mean	44.80	44.97	45.11	45.26	45.40	45.54	45.68	45.82	45.96	46.10	46.23	46.37	46.51	46.64	46.77	46.89	47.03	47.15	47.28	47.41
	2.5%	43.72	43.81	43.87	43.94	44.01	44.10	44.18	44.27	44.36	44.46	44.53	44.64	44.72	44.76	44.89	44.98	45.04	45.16	45.24	45.33
45-49	97.5%	41.60	41.80	41.97	42.14	42.28	42.54	42.71	42.88	43.04	43.20	43.38	43.52	43.68	43.84	44.02	44.15	44.33	44.46	44.57	44.74
	mean	40.01	40.18	40.33	40.47	40.60	40.74	40.87	41.02	41.15	41.29	41.42	41.55	41.68	41.81	41.94	42.06	42.19	42.31	42.44	42.56
	2.5%	38.96	39.05	39.13	39.17	39.28	39.34	39.42	39.52	39.60	39.70	39.77	39.88	39.94	40.00	40.14	40.21	40.27	40.37	40.46	40.54
50-54	97.5%	36.92	37.10	37.26	37.43	37.56	37.81	37.98	38.14	38.28	38.43	38.63	38.77	38.91	39.06	39.23	39.37	39.53	39.65	39.76	39.92
	mean	35.33	35.49	35.63	35.77	35.90	36.03	36.16	36.30	36.43	36.56	36.69	36.81	36.94	37.07	37.19	37.31	37.43	37.55	37.67	37.79
	2.5%	34.30	34.38	34.46	34.51	34.61	34.69	34.77	34.85	34.94	35.02	35.10	35.21	35.25	35.33	35.45	35.53	35.59	35.68	35.75	35.85
55-59	97.5%	32.34	32.52	32.67	32.84	32.97	33.20	33.32	33.49	33.63	33.79	33.96	34.11	34.23	34.38	34.54	34.66	34.81	34.95	35.05	35.18
	mean	30.76	30.91	31.05	31.18	31.30	31.43	31.56	31.68	31.81	31.94	32.06	32.18	32.30	32.42	32.53	32.65	32.77	32.88	32.99	33.11
	2.5%	29.73	29.83	29.91	29.96	30.05	30.12	30.21	30.29	30.37	30.45	30.51	30.62	30.68	30.74	30.84	30.92	30.98	31.07	31.15	31.24
60-64	97.5%	27.92	28.06	28.21	28.36	28.48	28.69	28.82	28.97	29.11	29.24	29.42	29.57	29.67	29.79	29.94	30.06	30.21	30.30	30.43	30.56
	mean	26.30	26.45	26.58	26.70	26.82	26.95	27.07	27.19	27.30	27.43	27.54	27.66	27.78	27.89	28.00	28.11	28.22	28.33	28.43	28.54
	2.5%	25.29	25.38	25.46	25.53	25.62	25.68	25.78	25.87	25.94	26.02	26.10	26.19	26.27	26.31	26.40	26.46	26.54	26.63	26.69	26.78
65-69	97.5%	23.57	23.69	23.83	23.97	24.08	24.26	24.36	24.51	24.64	24.77	24.92	25.03	25.14	25.27	25.40	25.49	25.62	25.73	25.84	25.98
	mean	21.92	22.06	22.18	22.29	22.40	22.52	22.63	22.74	22.85	22.96	23.07	23.17	23.28	23.38	23.48	23.58	23.69	23.78	23.88	23.98
	2.5%	20.90	21.00	21.08	21.15	21.23	21.30	21.38	21.47	21.52	21.61	21.70	21.76	21.84	21.87	21.95	22.03	22.09	22.16	22.22	22.28
70-74	97.5%	19.35	19.44	19.56	19.70	19.79	19.94	20.03	20.18	20.25	20.36	20.50	20.58	20.69	20.78	20.90	21.02	21.12	21.19	21.31	21.41
	mean	17.64	17.77	17.87	17.97	18.07	18.17	18.27	18.37	18.46	18.56	18.65	18.75	18.84	18.93	19.02	19.11	19.20	19.28	19.37	19.46
	2.5%	16.59	16.71	16.78	16.85	16.90	16.96	17.04	17.14	17.17	17.25	17.34	17.39	17.44	17.50	17.57	17.62	17.68	17.72	17.78	17.85
75-79	97.5%	15.28	15.35	15.43	15.53	15.61	15.76	15.80	15.91	15.98	16.07	16.16	16.22	16.30	16.37	16.45	16.54	16.64	16.71	16.77	16.88
	mean	13.49	13.60	13.68	13.76	13.83	13.91	13.99	14.06	14.13	14.21	14.28	14.35	14.43	14.49	14.56	14.63	14.70	14.77	14.83	14.90
	2.5%	12.42	12.52	12.57	12.62	12.66	12.72	12.76	12.83	12.85	12.91	12.97	13.00	13.03	13.05	13.11	13.16	13.20	13.24	13.27	13.32
80-84	97.5%	11.58	11.61	11.66	11.74	11.81	11.86	11.87	11.95	11.99	12.04	12.08	12.14	12.18	12.23	12.27	12.32	12.39	12.43	12.47	12.53
	mean	9.63	9.71	9.77	9.82	9.86	9.91	9.96	10.01	10.06	10.11	10.15	10.20	10.25	10.29	10.34	10.38	10.42	10.47	10.51	10.55
	2.5%	8.49	8.57	8.62	8.64	8.67	8.71	8.73	8.77	8.78	8.83	8.85	8.89	8.93	8.94	8.97	9.00	9.01	9.06	9.06	9.09
85-89	97.5%	8.47	8.45	8.46	8.49	8.43	8.48	8.42	8.44	8.42	8.41	8.43	8.41	8.40	8.42	8.40	8.38	8.38	8.40	8.38	8.40
	mean	6.10	6.16	6.17	6.18	6.18	6.19	6.20	6.21	6.21	6.23	6.23	6.24	6.25	6.25	6.26	6.27	6.28	6.28	6.29	6.30

Table A.24. The forecasted France male life expectancy by model 8

age 0	97.5%	79.31	79.74	80.06	80.31	80.52	80.77	80.94	81.11	81.30	81.51	81.66	81.86	82.05	82.20	82.39	82.54	82.65	82.84	82.91	83.05
	mean	77.78	77.72	77.63	77.60	77.54	77.50	77.47	77.45	77.42	77.44	77.42	77.42	77.43	77.44	77.42	77.45	77.48	77.49	77.52	77.55
	2.5%	75.92	75.17	74.43	73.80	73.29	72.77	72.29	71.95	71.51	71.16	70.88	70.71	70.33	70.10	69.95	69.71	69.46	69.20	69.15	
1-4	97.5%	78.58	78.98	79.27	79.52	79.72	79.93	80.12	80.27	80.44	80.66	80.79	80.98	81.17	81.31	81.47	81.64	81.75	81.93	81.98	82.13
	mean	77.09	77.02	76.92	76.88	76.81	76.76	76.72	76.69	76.66	76.66	76.63	76.63	76.63	76.63	76.61	76.63	76.66	76.66	76.68	76.70
	2.5%	75.27	74.53	73.77	73.15	72.64	72.08	71.59	71.22	70.87	70.51	70.23	70.00	69.65	69.39	69.29	68.94	68.78	68.88	68.47	68.39
5-9	97.5%	74.62	75.02	75.32	75.56	75.76	75.97	76.15	76.29	76.46	76.68	76.80	77.00	77.19	77.32	77.49	77.65	77.76	77.94	77.99	78.14
	mean	73.15	73.07	72.97	72.92	72.86	72.81	72.76	72.73	72.69	72.70	72.67	72.66	72.66	72.66	72.64	72.66	72.68	72.68	72.70	72.72
	2.5%	71.33	70.59	69.82	69.21	68.70	68.15	67.66	67.28	66.92	66.55	66.29	66.05	65.71	65.43	65.32	64.98	64.82	64.91	64.51	64.46
10-14	97.5%	69.65	70.04	70.34	70.58	70.78	70.99	71.16	71.31	71.47	71.69	71.81	72.01	72.20	72.33	72.50	72.66	72.77	72.95	73.00	73.15
	mean	68.18	68.10	68.00	67.95	67.89	67.83	67.79	67.76	67.72	67.72	67.69	67.68	67.68	67.68	67.66	67.68	67.70	67.70	67.72	67.74
	2.5%	66.36	65.64	64.87	64.25	63.75	63.20	62.71	62.32	61.95	61.62	61.33	61.09	60.75	60.46	60.35	60.02	59.88	59.93	59.54	59.49
15-19	97.5%	64.68	65.07	65.36	65.60	65.80	66.01	66.17	66.32	66.48	66.70	66.82	67.03	67.21	67.34	67.51	67.66	67.78	67.96	68.00	68.15
	mean	63.21	63.13	63.03	62.98	62.91	62.86	62.81	62.78	62.74	62.74	62.71	62.70	62.70	62.70	62.68	62.70	62.72	62.72	62.74	62.76
	2.5%	61.41	60.68	59.92	59.29	58.79	58.26	57.75	57.36	57.00	56.65	56.36	56.14	55.79	55.50	55.39	55.07	54.92	54.98	54.57	54.51
20-24	97.5%	59.77	60.13	60.44	60.66	60.86	61.08	61.24	61.39	61.55	61.77	61.88	62.08	62.25	62.38	62.55	62.71	62.83	62.99	63.05	63.19
	mean	58.33	58.25	58.16	58.12	58.06	58.01	57.97	57.95	57.92	57.93	57.90	57.90	57.91	57.91	57.90	57.93	57.96	57.96	57.99	58.02
	2.5%	56.57	55.87	55.19	54.57	54.09	53.60	53.10	52.73	52.42	52.05	51.89	51.74	51.40	51.08	51.05	50.77	50.59	50.67	50.31	50.32
25-29	97.5%	54.92	55.27	55.58	55.78	56.00	56.22	56.38	56.57	56.72	56.93	57.08	57.26	57.44	57.56	57.71	57.90	58.01	58.18	58.26	58.43
	mean	53.62	53.59	53.54	53.54	53.52	53.51	53.50	53.50	53.50	53.53	53.53	53.55	53.58	53.60	53.60	53.64	53.68	53.70	53.74	53.78
	2.5%	52.06	51.58	51.04	50.60	50.22	49.85	49.50	49.32	48.97	48.83	48.63	48.28	48.26	48.02	47.99	47.69	47.73	47.61	47.42	47.40
30-34	97.5%	50.11	50.45	50.76	50.95	51.17	51.39	51.55	51.79	51.90	52.13	52.30	52.46	52.62	52.78	52.97	53.13	53.24	53.39	53.51	53.68
	mean	48.94	48.94	48.94	48.98	48.99	49.01	49.03	49.06	49.08	49.13	49.16	49.20	49.24	49.28	49.30	49.35	49.40	49.44	49.49	49.53
	2.5%	47.59	47.21	46.89	46.58	46.29	46.04	45.81	45.75	45.48	45.31	45.21	45.03	44.94	44.86	44.72	44.46	44.56	44.40	44.30	44.31
35-39	97.5%	45.32	45.65	45.93	46.17	46.36	46.60	46.77	46.98	47.14	47.33	47.55	47.72	47.88	48.04	48.24	48.37	48.51	48.69	48.80	49.01
	mean	44.26	44.30	44.34	44.40	44.45	44.50	44.55	44.60	44.65	44.72	44.77	44.82	44.88	44.93	44.97	45.04	45.10	45.15	45.20	45.26
	2.5%	43.07	42.84	42.60	42.44	42.25	42.09	41.92	41.92	41.69	41.63	41.53	41.42	41.29	41.30	41.11	41.05	40.98	40.96	40.95	40.87
40-44	97.5%	40.59	40.90	41.15	41.40	41.60	41.82	42.01	42.21	42.41	42.59	42.78	42.95	43.12	43.26	43.46	43.62	43.81	43.97	44.08	44.30
	mean	39.61	39.68	39.74	39.82	39.89	39.96	40.03	40.10	40.16	40.24	40.31	40.38	40.45	40.51	40.58	40.65	40.72	40.78	40.85	40.92
	2.5%	38.55	38.40	38.25	38.15	38.03	37.92	37.79	37.76	37.65	37.62	37.56	37.47	37.43	37.40	37.31	37.27	37.25	37.28	37.20	37.24
45-49	97.5%	35.94	36.23	36.45	36.69	36.90	37.12	37.32	37.51	37.69	37.87	38.07	38.24	38.40	38.55	38.74	38.89	39.07	39.25	39.38	39.57
	mean	35.04	35.11	35.18	35.27	35.35	35.43	35.51	35.59	35.67	35.75	35.83	35.91	35.99	36.06	36.14	36.22	36.30	36.37	36.44	36.52
	2.5%	34.05	33.96	33.87	33.78	33.71	33.65	33.53	33.54	33.48	33.44	33.43	33.36	33.35	33.37	33.32	33.36	33.31	33.29	33.28	33.28
50-54	97.5%	31.43	31.69	31.89	32.14	32.32	32.51	32.70	32.89	33.05	33.22	33.40	33.58	33.73	33.87	34.05	34.18	34.37	34.52	34.64	34.82
	mean	30.57	30.63	30.70	30.79	30.87	30.94	31.02	31.09	31.16	31.25	31.32	31.39	31.47	31.54	31.60	31.68	31.76	31.82	31.89	31.96
	2.5%	29.64	29.55	29.48	29.39	29.34	29.28	29.20	29.19	29.14	29.11	29.11	29.06	29.05	29.05	29.00	29.06	29.02	29.02	29.04	29.02
55-59	97.5%	27.16	27.39	27.59	27.81	27.99	28.16	28.33	28.50	28.66	28.84	29.01	29.19	29.34	29.45	29.62	29.77	29.94	30.09	30.23	30.36
	mean	26.32	26.40	26.47	26.56	26.63	26.71	26.79	26.86	26.94	27.02	27.09	27.17	27.25	27.32	27.39	27.46	27.54	27.61	27.68	27.75
	2.5%	25.44	25.38	25.29	25.26	25.22	25.18	25.11	25.09	25.06	25.06	25.07	25.02	25.03	25.03	24.99	25.03	25.03	25.04	25.02	25.05
60-64	97.5%	23.09	23.32	23.48	23.69	23.84	23.99	24.16	24.31	24.43	24.61	24.77	24.91	25.04	25.14	25.31	25.48	25.58	25.72	25.84	25.99
	mean	22.31	22.38	22.43	22.50	22.56	22.63	22.69	22.75	22.81	22.88	22.94	23.00	23.07	23.13	23.18	23.25	23.31	23.36	23.42	23.48
	2.5%	21.48	21.41	21.34	21.25	21.21	21.20	21.10	21.09	21.06	21.03	21.05	21.03	21.01	20.97	20.98	20.99	20.96	20.97	20.92	20.96
65-69	97.5%	19.23	19.43	19.57	19.73	19.85	19.98	20.12	20.25	20.38	20.50	20.60	20.73	20.84	20.95	21.07	21.17	21.28	21.37	21.48	21.58
	mean	18.46	18.50	18.53	18.57	18.61	18.65	18.69	18.72	18.75	18.79	18.83	18.87	18.91	18.94	18.97	19.01	19.05	19.08	19.11	19.15
	2.5%	17.64	17.56	17.45	17.37	17.31	17.25	17.14	17.09	17.02	17.01	16.96	16.91	16.87	16.82	16.80	16.78	16.75	16.71	16.69	16.66
70-74	97.5%	15.61	15.76	15.90	16.02	16.12	16.24	16.35	16.43	16.52	16.63	16.72	16.83	16.93	17.04	17.12	17.17	17.27	17.34	17.41	17.49
	mean	14.88	14.91	14.92	14.95	14.97	15.00	15.02	15.05	15.07	15.09	15.12	15.14	15.16	15.18	15.20	15.22	15.25	15.27	15.29	15.31
	2.5%	14.13	14.04	13.93	13.86	13.74	13.69	13.61	13.58	13.51	13.46	13.46	13.36	13.35	13.31	13.26	13.25	13.21	13.15	13.10	13.12
75-79	97.5%	12.11	12.23	12.32	12.42	12.48	12.54	12.61	12.68	12.72	12.82	12.87	12.93	12.99	13.06	13.10	13.15				

Table A.25. The forecasted Iceland female life expectancy by model 3

age 0	97.5%	84.42	84.78	85.09	85.27	85.46	85.54	85.65	85.74	85.86	85.94	86.03	86.09	86.14	86.24	86.32	86.36	86.42	86.46	86.54	86.57
	mean	83.11	83.01	82.92	82.82	82.72	82.60	82.47	82.36	82.23	82.10	81.96	81.81	81.65	81.48	81.32	81.14	80.95	80.78	80.59	80.39
	2.5%	81.58	80.75	80.02	79.50	78.83	78.05	77.44	76.75	76.08	75.06	74.19	73.25	72.44	71.46	70.04	69.28	68.30	67.23	65.94	65.33
1-4	97.5%	83.52	83.88	84.16	84.34	84.51	84.60	84.70	84.80	84.90	84.98	85.09	85.13	85.18	85.27	85.37	85.40	85.47	85.50	85.58	85.61
	mean	82.31	82.22	82.15	82.07	81.99	81.89	81.78	81.69	81.59	81.49	81.37	81.25	81.12	80.99	80.86	80.71	80.56	80.43	80.27	80.11
	2.5%	80.92	80.24	79.61	79.15	78.65	78.03	77.48	76.97	76.40	75.59	74.89	74.15	73.51	72.87	71.81	70.97	70.39	69.46	68.42	67.92
5-9	97.5%	79.55	79.90	80.17	80.35	80.53	80.61	80.71	80.82	80.91	80.99	81.09	81.14	81.19	81.28	81.38	81.41	81.47	81.50	81.58	81.61
	mean	78.37	78.29	78.23	78.17	78.10	78.02	77.94	77.87	77.79	77.72	77.64	77.55	77.46	77.37	77.28	77.18	77.09	77.00	76.90	76.81
	2.5%	77.04	76.44	75.92	75.53	75.09	74.65	74.17	73.89	73.35	72.99	72.55	71.93	71.68	71.22	70.59	70.26	69.75	69.17	68.60	68.39
10-14	97.5%	74.57	74.91	75.18	75.36	75.54	75.61	75.72	75.82	75.91	75.99	76.10	76.15	76.19	76.29	76.38	76.41	76.47	76.51	76.60	76.61
	mean	73.40	73.33	73.27	73.21	73.14	73.07	72.99	72.93	72.86	72.79	72.72	72.63	72.55	72.47	72.39	72.30	72.22	72.14	72.06	71.97
	2.5%	72.09	71.50	71.01	70.65	70.24	69.81	69.40	69.11	68.67	68.32	67.90	67.42	67.23	66.79	66.24	65.93	65.56	65.14	64.63	64.38
15-19	97.5%	69.60	69.93	70.20	70.38	70.56	70.63	70.74	70.84	70.93	71.01	71.11	71.17	71.21	71.30	71.39	71.42	71.50	71.53	71.60	71.63
	mean	68.44	68.37	68.31	68.26	68.19	68.12	68.04	67.98	67.91	67.84	67.77	67.70	67.62	67.54	67.46	67.37	67.30	67.22	67.14	67.05
	2.5%	67.16	66.58	66.09	65.75	65.34	64.95	64.55	64.25	63.83	63.49	63.08	62.60	62.44	61.97	61.50	61.17	60.79	60.35	59.96	59.66
20-24	97.5%	64.64	64.97	65.24	65.41	65.58	65.67	65.76	65.86	65.95	66.03	66.14	66.19	66.23	66.31	66.41	66.45	66.53	66.55	66.63	66.65
	mean	63.50	63.43	63.38	63.33	63.26	63.20	63.12	63.06	63.00	62.93	62.87	62.79	62.72	62.64	62.57	62.49	62.41	62.34	62.26	62.18
	2.5%	62.24	61.68	61.23	60.89	60.52	60.11	59.74	59.45	59.06	58.74	58.33	57.95	57.74	57.31	56.86	56.53	56.21	55.83	55.39	55.17
25-29	97.5%	59.70	60.03	60.28	60.45	60.61	60.70	60.81	60.89	60.99	61.06	61.16	61.21	61.25	61.34	61.43	61.48	61.54	61.56	61.65	61.67
	mean	58.59	58.52	58.47	58.43	58.37	58.30	58.24	58.18	58.12	58.06	58.00	57.93	57.87	57.80	57.73	57.66	57.59	57.53	57.46	57.39
	2.5%	57.37	56.84	56.41	56.11	55.74	55.39	55.07	54.78	54.44	54.12	53.80	53.47	53.22	52.86	52.50	52.18	51.91	51.48	51.23	50.95
30-34	97.5%	54.73	55.05	55.31	55.47	55.63	55.71	55.83	55.91	56.00	56.07	56.17	56.23	56.26	56.36	56.44	56.49	56.55	56.58	56.66	56.68
	mean	53.65	53.59	53.55	53.51	53.45	53.40	53.34	53.29	53.24	53.19	53.14	53.08	53.03	52.97	52.92	52.86	52.81	52.75	52.70	52.64
	2.5%	52.47	51.99	51.59	51.31	51.00	50.69	50.42	50.21	49.90	49.64	49.37	49.12	48.95	48.62	48.36	48.22	47.90	47.67	47.51	47.32
35-39	97.5%	49.78	50.09	50.34	50.51	50.65	50.75	50.85	50.94	51.03	51.10	51.20	51.26	51.29	51.39	51.47	51.51	51.57	51.61	51.68	51.71
	mean	48.72	48.67	48.62	48.58	48.54	48.48	48.43	48.38	48.33	48.29	48.24	48.19	48.14	48.08	48.03	47.98	47.93	47.88	47.83	47.78
	2.5%	47.57	47.11	46.71	46.50	46.17	45.88	45.62	45.40	45.11	44.90	44.66	44.41	44.27	43.97	43.76	43.61	43.36	43.09	42.95	42.80
40-44	97.5%	44.89	45.19	45.42	45.60	45.73	45.83	45.92	46.01	46.09	46.18	46.27	46.32	46.36	46.45	46.53	46.57	46.62	46.67	46.73	46.77
	mean	43.84	43.79	43.75	43.71	43.66	43.61	43.56	43.52	43.47	43.42	43.38	43.33	43.28	43.23	43.18	43.12	43.08	43.04	42.99	42.94
	2.5%	42.72	42.27	41.89	41.66	41.36	41.10	40.83	40.65	40.37	40.15	39.92	39.68	39.52	39.25	39.05	38.92	38.67	38.45	38.31	38.15
45-49	97.5%	40.04	40.33	40.55	40.72	40.86	40.95	41.03	41.10	41.20	41.27	41.37	41.41	41.45	41.56	41.62	41.65	41.72	41.75	41.81	41.85
	mean	39.03	38.98	38.95	38.91	38.87	38.82	38.77	38.73	38.69	38.65	38.60	38.56	38.51	38.46	38.42	38.37	38.33	38.29	38.24	38.20
	2.5%	37.97	37.54	37.21	36.97	36.70	36.45	36.22	36.05	35.78	35.59	35.36	35.15	35.02	34.78	34.58	34.45	34.24	34.02	33.90	33.79
50-54	97.5%	35.21	35.49	35.71	35.86	35.99	36.09	36.17	36.24	36.33	36.41	36.49	36.53	36.59	36.68	36.75	36.77	36.83	36.86	36.93	36.96
	mean	34.24	34.20	34.16	34.13	34.09	34.05	34.00	33.96	33.92	33.88	33.85	33.80	33.76	33.72	33.67	33.63	33.59	33.55	33.51	33.47
	2.5%	33.23	32.83	32.51	32.29	32.05	31.79	31.60	31.44	31.19	31.00	30.76	30.64	30.47	30.27	30.09	29.98	29.79	29.60	29.50	29.33
55-59	97.5%	30.48	30.74	30.93	31.09	31.22	31.29	31.37	31.43	31.54	31.60	31.67	31.74	31.76	31.85	31.92	31.95	31.99	32.04	32.11	32.13
	mean	29.55	29.51	29.48	29.45	29.41	29.37	29.33	29.29	29.26	29.22	29.19	29.15	29.11	29.07	29.03	28.99	28.96	28.93	28.89	28.85
	2.5%	28.58	28.22	27.92	27.71	27.49	27.27	27.07	26.94	26.71	26.58	26.37	26.23	26.07	25.92	25.77	25.64	25.46	25.32	25.25	25.04
60-64	97.5%	25.86	26.10	26.29	26.43	26.55	26.63	26.69	26.76	26.85	26.92	26.97	27.05	27.07	27.14	27.21	27.24	27.26	27.33	27.37	27.40
	mean	25.00	24.96	24.93	24.90	24.87	24.83	24.79	24.76	24.73	24.70	24.66	24.63	24.59	24.56	24.52	24.48	24.46	24.42	24.39	24.36
	2.5%	24.09	23.76	23.49	23.32	23.10	22.92	22.73	22.61	22.40	22.25	22.09	21.97	21.86	21.68	21.54	21.46	21.31	21.17	21.06	20.94
65-69	97.5%	21.50	21.71	21.86	21.98	22.11	22.17	22.23	22.29	22.37	22.43	22.47	22.54	22.56	22.62	22.69	22.74	22.74	22.76	22.83	22.87
	mean	20.69	20.66	20.63	20.60	20.57	20.54	20.50	20.48	20.45	20.42	20.39	20.36	20.33	20.29	20.26	20.23	20.20	20.17	20.14	20.12
	2.5%	19.87	19.56	19.32	19.18	18.97	18.81	18.67	18.56	18.34	18.24	18.07	17.99	17.87	17.69	17.57	17.53	17.40	17.29	17.21	17.11
70-74	97.5%	17.35	17.52	17.66	17.77	17.88	17.93	18.00	18.03	18.09	18.16	18.22	18.26	18.28	18.35	18.40	18.39	18.45	18.46	18.50	18.54
	mean	16.62	16.59	16.56	16.54	16.51	16.48	16.45	16.42	16.39	16.37	16.34	16.31	16.28	16.25	16.23	16.20	16.17	16.15	16.12	16.10
	2.5%	15.88	15.61	15.42	15.26	15.10	14.94	14.82	14.72	14.57	14.45	14.32	14.23	14.11	14.00	13.93	13.81	13.71	13.61	13.52	13.42
75-79	97.5%	13.47	13.61	13.73	13.82	13.90	13.95	13.99	14.05	14.10	14.15	14.20	14.23	14.24	14.29	14.36	14.36	14.38	14.42	14.44	14.47

Table A.26. The forecasted Iceland male life expectancy by model 3

age	97.5%	81.33	81.70	82.03	82.27	82.47	82.55	82.64	82.76	82.93	82.99	83.10	83.17	83.25	83.34	83.41	83.49	83.47	83.57	83.59	83.69
	mean	79.94	79.84	79.76	79.67	79.58	79.47	79.35	79.26	79.14	79.03	78.92	78.78	78.65	78.50	78.37	78.21	78.07	77.92	77.75	77.60
	2.5%	78.36	77.57	76.83	76.35	75.67	75.12	74.41	73.93	73.31	72.56	71.97	71.20	70.43	69.64	68.69	68.24	67.42	66.37	65.75	64.81
1-4	97.5%	80.39	80.75	81.06	81.31	81.50	81.59	81.68	81.78	81.95	82.02	82.12	82.18	82.29	82.37	82.44	82.52	82.50	82.59	82.62	82.71
	mean	79.04	78.95	78.87	78.79	78.70	78.61	78.50	78.41	78.30	78.20	78.10	77.98	77.85	77.72	77.60	77.46	77.33	77.19	77.04	76.90
	2.5%	77.50	76.76	76.10	75.67	75.00	74.50	73.86	73.45	72.82	72.26	71.68	70.94	70.46	69.55	68.79	68.29	67.41	66.68	66.05	65.00
5-9	97.5%	76.42	76.77	77.08	77.33	77.52	77.60	77.69	77.79	77.96	78.02	78.13	78.19	78.30	78.39	78.44	78.52	78.52	78.60	78.63	78.72
	mean	75.10	75.02	74.96	74.89	74.82	74.74	74.65	74.58	74.50	74.42	74.35	74.26	74.17	74.08	73.99	73.90	73.81	73.73	73.63	73.54
	2.5%	73.64	73.00	72.38	71.99	71.48	71.08	70.54	70.29	69.86	69.36	68.95	68.57	68.27	67.74	67.29	66.83	66.52	66.24	65.75	65.10
10-14	97.5%	71.45	71.79	72.10	72.34	72.53	72.61	72.71	72.81	72.97	73.04	73.15	73.21	73.31	73.40	73.45	73.53	73.53	73.62	73.63	73.73
	mean	70.15	70.08	70.02	69.96	69.89	69.81	69.73	69.67	69.59	69.53	69.46	69.38	69.30	69.22	69.14	69.06	68.99	68.91	68.83	68.76
	2.5%	68.72	68.11	67.52	67.15	66.68	66.32	65.85	65.64	65.25	64.82	64.44	64.11	63.84	63.47	63.09	62.68	62.45	62.16	61.83	61.37
15-19	97.5%	66.48	66.82	67.13	67.38	67.54	67.63	67.72	67.83	67.99	68.06	68.17	68.23	68.32	68.41	68.47	68.54	68.55	68.64	68.66	68.74
	mean	65.20	65.13	65.07	65.01	64.94	64.87	64.79	64.73	64.66	64.59	64.52	64.45	64.37	64.29	64.22	64.14	64.07	64.00	63.92	63.84
	2.5%	63.79	63.19	62.62	62.27	61.79	61.44	60.97	60.76	60.37	59.97	59.56	59.27	59.02	58.67	58.31	57.90	57.64	57.37	57.06	56.68
20-24	97.5%	61.58	61.91	62.21	62.44	62.63	62.71	62.79	62.90	63.07	63.12	63.23	63.28	63.38	63.47	63.54	63.60	63.61	63.69	63.71	63.81
	mean	60.32	60.25	60.19	60.14	60.07	60.00	59.92	59.86	59.79	59.73	59.66	59.59	59.51	59.44	59.37	59.28	59.22	59.15	59.07	59.00
	2.5%	58.94	58.35	57.79	57.45	57.00	56.64	56.20	55.99	55.58	55.25	54.81	54.56	54.33	53.96	53.62	53.21	52.99	52.68	52.37	51.99
25-29	97.5%	56.75	57.04	57.35	57.58	57.76	57.84	57.91	58.02	58.18	58.23	58.35	58.40	58.48	58.58	58.65	58.71	58.71	58.79	58.81	58.89
	mean	55.51	55.44	55.39	55.33	55.27	55.20	55.12	55.07	55.00	54.94	54.88	54.80	54.73	54.66	54.59	54.51	54.45	54.38	54.30	54.24
	2.5%	54.18	53.59	53.06	52.73	52.30	51.93	51.52	51.32	50.96	50.60	50.18	49.98	49.70	49.35	49.01	48.67	48.43	48.16	47.86	47.50
30-34	97.5%	51.89	52.18	52.48	52.67	52.87	52.94	53.02	53.14	53.28	53.32	53.45	53.50	53.58	53.68	53.74	53.78	53.83	53.86	53.90	53.98
	mean	50.70	50.64	50.58	50.53	50.47	50.41	50.34	50.28	50.22	50.16	50.10	50.03	49.96	49.89	49.83	49.75	49.69	49.63	49.56	49.50
	2.5%	49.42	48.88	48.38	48.05	47.62	47.31	46.91	46.71	46.35	46.03	45.68	45.41	45.19	44.82	44.53	44.25	43.99	43.75	43.48	43.05
35-39	97.5%	47.11	47.38	47.64	47.85	48.03	48.10	48.17	48.27	48.42	48.45	48.57	48.62	48.69	48.78	48.85	48.89	48.92	49.00	49.01	49.09
	mean	46.00	45.95	45.90	45.86	45.81	45.76	45.70	45.65	45.60	45.56	45.51	45.45	45.40	45.34	45.29	45.23	45.18	45.13	45.07	45.02
	2.5%	44.84	44.35	43.93	43.66	43.31	43.02	42.75	42.56	42.31	42.04	41.76	41.59	41.33	41.05	40.83	40.61	40.39	40.31	40.04	39.76
40-44	97.5%	42.27	42.52	42.76	42.97	43.14	43.22	43.27	43.38	43.53	43.56	43.67	43.72	43.77	43.88	43.94	43.99	44.01	44.08	44.11	44.17
	mean	41.19	41.14	41.10	41.06	41.02	40.97	40.91	40.87	40.82	40.78	40.73	40.68	40.63	40.58	40.53	40.48	40.43	40.39	40.33	40.29
	2.5%	40.07	39.61	39.24	38.99	38.62	38.41	38.13	37.94	37.68	37.45	37.23	37.02	36.80	36.53	36.34	36.16	35.99	35.87	35.58	35.36
45-49	97.5%	37.49	37.74	37.96	38.16	38.32	38.40	38.46	38.54	38.70	38.72	38.85	38.89	38.96	39.05	39.08	39.14	39.16	39.22	39.26	39.35
	mean	36.46	36.41	36.37	36.33	36.29	36.24	36.19	36.15	36.10	36.06	36.02	35.97	35.92	35.87	35.83	35.77	35.73	35.69	35.64	35.59
	2.5%	35.37	34.94	34.58	34.35	34.00	33.79	33.55	33.37	33.12	32.92	32.68	32.49	32.30	32.03	31.85	31.65	31.51	31.40	31.14	30.94
50-54	97.5%	32.78	33.00	33.23	33.41	33.54	33.63	33.69	33.77	33.90	33.94	34.04	34.09	34.17	34.24	34.30	34.34	34.36	34.40	34.46	34.55
	mean	31.79	31.75	31.71	31.68	31.64	31.59	31.54	31.51	31.47	31.43	31.39	31.34	31.30	31.26	31.21	31.16	31.12	31.09	31.04	31.00
	2.5%	30.77	30.38	30.04	29.82	29.51	29.30	29.07	28.95	28.69	28.52	28.31	28.13	27.95	27.73	27.55	27.39	27.24	27.15	26.96	26.77
55-59	97.5%	28.15	28.36	28.56	28.72	28.86	28.94	28.99	29.08	29.19	29.24	29.33	29.36	29.44	29.52	29.57	29.59	29.61	29.67	29.70	29.79
	mean	27.23	27.19	27.16	27.12	27.09	27.05	27.00	26.97	26.93	26.90	26.86	26.82	26.78	26.74	26.70	26.66	26.62	26.59	26.55	26.51
	2.5%	26.27	25.90	25.61	25.41	25.11	24.95	24.76	24.60	24.38	24.23	24.03	23.87	23.73	23.52	23.36	23.23	23.07	22.99	22.80	22.68
60-64	97.5%	23.67	23.86	24.04	24.18	24.30	24.36	24.41	24.50	24.60	24.64	24.72	24.76	24.82	24.90	24.95	24.95	24.96	25.04	25.08	25.15
	mean	22.82	22.78	22.75	22.73	22.69	22.66	22.62	22.59	22.55	22.52	22.49	22.46	22.42	22.39	22.35	22.31	22.28	22.25	22.22	22.18
	2.5%	21.94	21.62	21.37	21.20	20.93	20.78	20.59	20.48	20.31	20.18	19.98	19.85	19.71	19.52	19.38	19.29	19.14	19.10	18.92	18.79
65-69	97.5%	19.37	19.55	19.69	19.80	19.91	19.95	20.01	20.07	20.16	20.19	20.26	20.32	20.35	20.41	20.49	20.48	20.50	20.58	20.58	20.66
	mean	18.60	18.57	18.54	18.52	18.49	18.46	18.42	18.40	18.37	18.34	18.32	18.28	18.25	18.22	18.19	18.16	18.14	18.11	18.08	18.05
	2.5%	17.82	17.54	17.33	17.18	16.96	16.81	16.68	16.57	16.39	16.28	16.17	16.05	15.93	15.80	15.66	15.57	15.48	15.38	15.27	15.15
70-74	97.5%	15.39	15.54	15.65	15.75	15.83	15.86	15.90	15.96	16.02	16.07	16.12	16.16	16.23	16.25	16.30	16.30	16.31	16.35	16.41	16.43
	mean	14.75	14.72	14.70	14.68	14.66	14.64	14.61	14.59	14.57	14.55	14.53	14.51	14.48	14.46	14.44	14.41	14.39	14.37	14.35	14.33
	2.5%	14.10	13.90	13.74	13.60	13.44	13.34	13.22	13.15	13.04	12.92	12.80	12.74	12.67	12.53	12.45	12.36	12.30	12.24	12.15	12.09
75-79	97.5%	11.69	11.80	11.90	11.97	12.04	12.05	12.08	12.12	12.16	12.20	12.24	12.28	12.34	12.35	12.39	12.39				

Table A.27. The forecasted Iceland female life expectancy by model 8

age	97.5%	82.83	82.93	83.05	83.05	83.07	83.10	83.10	83.07	83.09	83.12	83.17	83.09	83.18	83.17	83.20	83.26	83.21	83.20	83.19	83.14
0	mean	80.22	80.13	80.02	79.95	79.86	79.79	79.75	79.73	79.66	79.65	79.60	79.56	79.53	79.50	79.44	79.40	79.40	79.39	79.36	79.29
	2.5%	76.80	76.51	76.22	75.98	75.80	75.48	75.34	75.20	74.94	74.95	74.75	74.59	74.52	74.31	74.18	73.98	74.11	73.81	73.69	73.47
1-4	97.5%	81.96	82.05	82.14	82.13	82.16	82.19	82.20	82.18	82.19	82.22	82.25	82.19	82.28	82.27	82.29	82.31	82.30	82.30	82.29	82.26
	mean	79.43	79.35	79.25	79.19	79.11	79.05	79.02	79.02	78.95	78.96	78.92	78.89	78.87	78.85	78.81	78.78	78.79	78.80	78.77	78.72
	2.5%	76.17	75.91	75.62	75.42	75.28	75.08	74.94	74.74	74.75	74.72	74.64	74.50	74.45	74.33	74.26	74.18	74.29	74.01	73.96	73.94
5-9	97.5%	78.01	78.08	78.17	78.15	78.18	78.22	78.22	78.20	78.21	78.24	78.26	78.20	78.29	78.28	78.31	78.33	78.32	78.32	78.31	78.28
	mean	75.49	75.41	75.31	75.25	75.17	75.11	75.08	75.08	75.01	75.02	74.98	74.95	74.93	74.91	74.88	74.84	74.85	74.86	74.83	74.78
	2.5%	72.29	71.99	71.77	71.55	71.41	71.24	71.08	70.90	70.96	70.85	70.83	70.67	70.58	70.52	70.38	70.29	70.51	70.18	70.15	70.16
10-14	97.5%	73.02	73.08	73.18	73.16	73.19	73.23	73.23	73.20	73.21	73.25	73.26	73.21	73.29	73.28	73.32	73.33	73.32	73.32	73.31	73.28
	mean	70.50	70.42	70.33	70.26	70.18	70.13	70.10	70.09	70.02	70.03	69.99	69.96	69.94	69.92	69.89	69.85	69.86	69.87	69.84	69.79
	2.5%	67.31	67.02	66.79	66.58	66.44	66.26	66.10	65.92	65.98	65.89	65.85	65.69	65.60	65.53	65.43	65.33	65.52	65.22	65.18	65.18
15-19	97.5%	68.06	68.12	68.22	68.22	68.26	68.28	68.27	68.24	68.26	68.29	68.33	68.28	68.38	68.34	68.37	68.40	68.38	68.44	68.37	68.34
	mean	65.59	65.50	65.41	65.36	65.28	65.23	65.21	65.20	65.15	65.16	65.13	65.10	65.09	65.09	65.08	65.03	65.06	65.09	65.06	65.04
	2.5%	62.49	62.24	61.95	61.84	61.57	61.43	61.37	61.25	61.25	61.19	61.22	61.01	61.09	60.92	60.89	60.81	61.02	60.81	60.76	60.81
20-24	97.5%	63.10	63.16	63.26	63.26	63.27	63.29	63.28	63.26	63.29	63.33	63.35	63.30	63.40	63.36	63.38	63.43	63.40	63.44	63.39	63.35
	mean	60.64	60.56	60.47	60.41	60.34	60.29	60.26	60.26	60.21	60.21	60.19	60.16	60.15	60.15	60.14	60.10	60.13	60.15	60.12	60.10
	2.5%	57.56	57.34	57.06	56.95	56.71	56.56	56.46	56.40	56.39	56.33	56.35	56.17	56.22	56.10	56.10	55.98	56.18	55.98	55.99	55.94
25-29	97.5%	58.14	58.19	58.30	58.28	58.30	58.33	58.31	58.29	58.30	58.35	58.38	58.32	58.42	58.38	58.40	58.46	58.41	58.46	58.41	58.37
	mean	55.71	55.63	55.54	55.48	55.40	55.35	55.33	55.33	55.28	55.28	55.25	55.23	55.21	55.21	55.21	55.16	55.19	55.22	55.19	55.17
	2.5%	52.67	52.44	52.19	52.11	51.84	51.72	51.57	51.55	51.53	51.48	51.49	51.37	51.39	51.24	51.25	51.14	51.35	51.20	51.15	51.22
30-34	97.5%	53.20	53.24	53.34	53.33	53.34	53.38	53.35	53.32	53.35	53.39	53.40	53.35	53.45	53.40	53.43	53.48	53.44	53.49	53.44	53.40
	mean	50.80	50.71	50.62	50.56	50.48	50.43	50.41	50.40	50.35	50.36	50.33	50.31	50.29	50.29	50.28	50.24	50.27	50.29	50.27	50.25
	2.5%	47.83	47.56	47.37	47.22	46.98	46.89	46.75	46.70	46.67	46.64	46.65	46.53	46.53	46.39	46.46	46.34	46.54	46.42	46.29	46.39
35-39	97.5%	48.22	48.28	48.36	48.35	48.38	48.40	48.37	48.34	48.37	48.39	48.42	48.36	48.46	48.42	48.44	48.49	48.46	48.50	48.44	48.42
	mean	45.85	45.78	45.67	45.61	45.54	45.48	45.46	45.46	45.40	45.40	45.38	45.35	45.33	45.34	45.32	45.28	45.31	45.33	45.30	45.29
	2.5%	42.91	42.67	42.44	42.33	42.09	41.98	41.85	41.80	41.75	41.72	41.77	41.62	41.61	41.53	41.57	41.43	41.64	41.50	41.39	41.51
40-44	97.5%	43.28	43.36	43.43	43.42	43.44	43.46	43.43	43.40	43.41	43.46	43.46	43.41	43.51	43.46	43.48	43.51	43.50	43.54	43.48	43.46
	mean	40.95	40.88	40.76	40.70	40.63	40.57	40.55	40.54	40.48	40.48	40.46	40.43	40.41	40.41	40.40	40.35	40.38	40.40	40.37	40.35
	2.5%	38.04	37.81	37.56	37.43	37.18	37.13	36.98	36.92	36.87	36.84	36.88	36.73	36.73	36.65	36.70	36.52	36.73	36.64	36.50	36.61
45-49	97.5%	38.57	38.61	38.65	38.66	38.67	38.68	38.67	38.65	38.67	38.71	38.71	38.66	38.76	38.71	38.74	38.76	38.75	38.76	38.75	38.70
	mean	36.39	36.31	36.18	36.15	36.09	36.01	35.98	36.00	35.94	35.93	35.91	35.89	35.87	35.86	35.86	35.82	35.84	35.86	35.83	35.82
	2.5%	33.71	33.53	33.31	33.20	33.03	32.91	32.75	32.79	32.70	32.60	32.64	32.56	32.56	32.43	32.53	32.35	32.55	32.37	32.36	32.40
50-54	97.5%	33.69	33.72	33.74	33.74	33.76	33.76	33.74	33.72	33.72	33.77	33.76	33.71	33.82	33.75	33.78	33.81	33.80	33.79	33.76	33.73
	mean	31.56	31.47	31.34	31.30	31.23	31.14	31.11	31.12	31.06	31.04	31.02	30.99	30.97	30.95	30.94	30.91	30.92	30.93	30.91	30.89
	2.5%	28.95	28.76	28.52	28.42	28.28	28.09	27.95	27.98	27.88	27.78	27.81	27.70	27.74	27.61	27.68	27.51	27.69	27.50	27.50	27.54
55-59	97.5%	29.02	29.06	29.11	29.10	29.09	29.12	29.10	29.07	29.12	29.08	29.13	29.10	29.15	29.12	29.17	29.15	29.18	29.15	29.14	29.12
	mean	27.05	27.00	26.90	26.85	26.80	26.75	26.71	26.71	26.67	26.66	26.62	26.61	26.60	26.58	26.56	26.55	26.56	26.55	26.54	26.53
	2.5%	24.66	24.55	24.38	24.27	24.21	24.04	23.87	23.94	23.85	23.84	23.76	23.71	23.74	23.58	23.62	23.57	23.66	23.50	23.52	23.59
60-64	97.5%	24.48	24.51	24.56	24.58	24.57	24.60	24.61	24.61	24.63	24.60	24.65	24.66	24.65	24.65	24.68	24.66	24.69	24.67	24.67	24.67
	mean	22.77	22.72	22.66	22.61	22.58	22.54	22.51	22.48	22.47	22.44	22.44	22.44	22.42	22.41	22.39	22.38	22.38	22.39	22.38	22.37
	2.5%	20.78	20.63	20.52	20.45	20.45	20.26	20.12	20.19	20.13	20.12	20.07	20.00	20.00	19.88	19.88	19.84	19.93	19.80	19.81	19.87
65-69	97.5%	20.14	20.21	20.21	20.23	20.22	20.29	20.29	20.31	20.30	20.31	20.34	20.35	20.35	20.35	20.38	20.39	20.41	20.39	20.41	20.40
	mean	18.70	18.65	18.62	18.56	18.52	18.50	18.48	18.47	18.45	18.44	18.41	18.40	18.39	18.38	18.36	18.36	18.36	18.35	18.35	18.35
	2.5%	17.13	17.01	16.88	16.81	16.77	16.68	16.57	16.55	16.49	16.48	16.42	16.36	16.38	16.31	16.24	16.21	16.27	16.17	16.16	16.18
70-74	97.5%	16.14	16.14	16.16	16.18	16.20	16.28	16.30	16.32	16.34	16.37	16.37	16.37	16.37	16.37	16.40	16.40	16.43	16.43	16.45	16.43
	mean	14.95	14.91	14.87	14.85	14.83	14.80	14.79	14.78	14.76	14.76	14.74	14.73	14.72	14.71	14.70	14.70	14.69	14.69	14.69	14.68
	2.5%	13.83	13.70	13.63	13.54	13.50	13.41	13.35	13.34	13.27	13.26	13.18	13.16	13.13	13.09	13.03	13.02	13.03	12.95	12.93	12.90
75-79	97.5%	12.67	12.66	12.73	12.73	12.77	12.82	12.83	12.85	12.85	12.90	12.94	12.93	12.96	12.96	12.99	13.00	13.02	13.03	13.06	13.07
	mean	11.70	11.67	11.65	11.63	11.61	11.60	11.59	11.58	11.57	11.57	11.56	11.55	11.55	11.54	11.53	11.53	11.53	11.52	11.53	11.52
	2.5%	10.72	10.63	10.57	10.48	10.45	10.40	10.33	10.28	10.24	10.19	10.17	10.13	10.10	10.06	9.98	9.98	9.97	9.95	9.93	9.91
80-84	97.5%	9.15	9.14	9.16	9.20	9.22	9.24	9.25	9.27	9.29	9.29	9.31	9.30	9.31	9.31	9.36	9.33	9.35	9.36	9.40	9.40
	mean	8.25	8.22	8.20	8.18	8.18	8.16	8.16	8.15	8.14	8.14	8.13	8.13	8.13	8.12	8.11	8.11	8.11	8.10	8.11	8.11
	2.5%	7.33	7.27	7.21	7.16	7.15	7.06	7.03	6.99	6.96	6.98	6.92	6.91	6.90	6.87	6.86	6.84	6.82	6.82	6.80	6.79
85-89	97.5%	6.16	6.13	6.11	6.13	6.10	6.12	6.12	6.11	6.12	6.11	6.13	6.09	6.10	6.09	6.10	6.08	6.09	6.09	6.10	6.10
	mean	5.52	5.47	5.46	5.44	5.44	5.43	5.42	5.42	5.41	5.41	5.40	5.40	5.39	5.39	5.38	5.38	5.38	5.38	5.38	5.39
	2.5%	4.95	4.91	4.87																	

Table A.28. The forecasted Iceland male life expectancy by model 8

age 0	97.5%	81.52	81.64	81.88	82.06	82.09	82.06	82.25	82.22	82.26	82.47	82.48	82.51	82.61	82.56	82.76	82.75	82.64	82.77	82.73	82.65
	mean	78.39	78.22	78.17	78.14	77.92	77.81	77.80	77.77	77.63	77.60	77.56	77.49	77.43	77.37	77.32	77.27	77.25	77.21	77.17	77.15
	2.5%	74.44	74.07	73.67	73.33	72.90	72.50	72.24	72.05	71.77	71.64	71.50	71.12	71.08	70.73	70.65	70.35	70.15	70.09	69.84	69.66
1-4	97.5%	80.60	80.72	80.95	81.12	81.16	81.11	81.32	81.28	81.32	81.56	81.53	81.57	81.66	81.62	81.84	81.80	81.72	81.83	81.78	81.72
	mean	77.58	77.38	77.33	77.30	77.09	76.98	76.97	76.95	76.82	76.79	76.75	76.69	76.63	76.57	76.53	76.48	76.47	76.43	76.40	76.38
	2.5%	73.90	73.32	72.93	72.59	72.22	71.85	71.57	71.40	71.15	70.98	70.84	70.57	70.52	70.16	70.14	69.89	69.65	69.58	69.32	69.15
5-9	97.5%	76.66	76.79	76.98	77.16	77.19	77.15	77.34	77.32	77.34	77.58	77.57	77.59	77.69	77.63	77.84	77.85	77.74	77.86	77.80	77.73
	mean	73.68	73.47	73.42	73.40	73.19	73.08	73.07	73.05	72.92	72.89	72.86	72.80	72.74	72.69	72.65	72.60	72.59	72.55	72.52	72.51
	2.5%	70.05	69.50	69.14	68.83	68.45	68.11	67.87	67.67	67.45	67.23	67.16	66.97	67.00	66.53	66.45	66.34	66.14	65.95	65.73	65.60
10-14	97.5%	71.67	71.81	72.02	72.17	72.20	72.16	72.35	72.33	72.35	72.60	72.58	72.60	72.69	72.64	72.85	72.85	72.75	72.86	72.81	72.75
	mean	68.72	68.51	68.46	68.43	68.22	68.11	68.10	68.07	67.95	67.92	67.88	67.82	67.76	67.71	67.67	67.62	67.61	67.58	67.55	67.53
	2.5%	65.13	64.55	64.21	63.89	63.50	63.15	62.95	62.73	62.53	62.27	62.23	62.04	62.04	61.62	61.50	61.43	61.18	60.99	60.79	60.70
15-19	97.5%	66.70	66.83	67.04	67.19	67.24	67.20	67.38	67.37	67.38	67.64	67.61	67.65	67.76	67.66	67.93	67.88	67.81	67.90	67.86	67.78
	mean	63.77	63.54	63.50	63.47	63.27	63.16	63.16	63.13	63.01	62.98	62.95	62.89	62.83	62.79	62.76	62.71	62.71	62.68	62.66	62.64
	2.5%	60.21	59.59	59.27	58.98	58.59	58.22	58.04	57.85	57.63	57.37	57.40	57.20	57.12	56.82	56.78	56.59	56.53	56.32	56.09	55.94
20-24	97.5%	61.74	61.86	62.07	62.23	62.28	62.23	62.42	62.40	62.41	62.65	62.63	62.67	62.77	62.67	62.94	62.89	62.82	62.91	62.86	62.79
	mean	58.82	58.60	58.55	58.52	58.31	58.20	58.20	58.17	58.04	58.01	57.99	57.92	57.86	57.82	57.78	57.74	57.73	57.70	57.68	57.66
	2.5%	55.28	54.66	54.34	54.04	53.64	53.28	53.09	52.90	52.68	52.42	52.44	52.24	52.17	51.85	51.81	51.62	51.56	51.33	51.15	50.96
25-29	97.5%	56.85	56.97	57.17	57.31	57.36	57.31	57.48	57.46	57.48	57.73	57.71	57.74	57.84	57.72	58.00	57.94	57.86	57.96	57.91	57.83
	mean	53.97	53.74	53.69	53.65	53.44	53.33	53.32	53.29	53.16	53.12	53.09	53.03	52.96	52.91	52.88	52.83	52.82	52.78	52.76	52.74
	2.5%	50.48	49.85	49.53	49.27	48.85	48.47	48.28	48.11	47.81	47.59	47.62	47.43	47.32	47.03	46.96	46.80	46.67	46.50	46.27	46.14
30-34	97.5%	51.98	52.10	52.30	52.42	52.46	52.40	52.58	52.54	52.59	52.80	52.77	52.82	52.89	52.79	53.07	53.02	52.93	53.03	52.99	52.92
	mean	49.16	48.94	48.88	48.84	48.63	48.52	48.51	48.47	48.35	48.31	48.27	48.21	48.15	48.10	48.06	48.01	48.00	47.96	47.94	47.92
	2.5%	45.75	45.15	44.87	44.57	44.16	43.81	43.62	43.41	43.13	42.94	42.95	42.76	42.69	42.43	42.31	42.21	42.10	42.01	41.75	41.52
35-39	97.5%	47.28	47.40	47.55	47.71	47.73	47.70	47.87	47.81	47.85	48.08	48.08	48.09	48.19	48.11	48.33	48.28	48.23	48.32	48.26	48.25
	mean	44.64	44.45	44.37	44.32	44.13	44.03	44.01	43.98	43.86	43.82	43.79	43.72	43.67	43.63	43.58	43.55	43.54	43.50	43.48	43.46
	2.5%	41.52	40.96	40.68	40.38	39.99	39.66	39.48	39.29	39.05	38.91	38.84	38.64	38.66	38.40	38.23	38.15	38.06	37.98	37.79	37.67
40-44	97.5%	42.45	42.56	42.72	42.84	42.86	42.85	42.98	42.94	43.01	43.20	43.18	43.22	43.33	43.22	43.42	43.40	43.30	43.41	43.33	43.34
	mean	39.89	39.71	39.63	39.58	39.39	39.28	39.27	39.23	39.11	39.08	39.04	38.98	38.92	38.88	38.84	38.80	38.79	38.76	38.74	38.72
	2.5%	36.86	36.35	36.05	35.78	35.39	35.12	34.90	34.77	34.53	34.37	34.34	34.12	34.10	33.89	33.80	33.73	33.59	33.50	33.29	33.14
45-49	97.5%	37.65	37.73	37.88	38.00	38.00	37.99	38.11	38.09	38.14	38.33	38.32	38.34	38.45	38.34	38.52	38.47	38.41	38.53	38.43	38.44
	mean	35.13	34.93	34.85	34.80	34.60	34.49	34.47	34.42	34.30	34.26	34.23	34.15	34.09	34.05	34.01	33.96	33.95	33.92	33.89	33.87
	2.5%	32.19	31.66	31.33	31.06	30.65	30.35	30.21	30.06	29.78	29.65	29.59	29.44	29.38	29.14	29.07	29.02	28.82	28.80	28.49	28.38
50-54	97.5%	33.18	33.32	33.46	33.57	33.60	33.63	33.74	33.71	33.74	33.94	33.94	34.01	33.97	33.97	34.14	34.13	34.06	34.16	34.10	34.09
	mean	30.90	30.76	30.70	30.64	30.47	30.37	30.35	30.30	30.19	30.15	30.11	30.05	29.99	29.95	29.91	29.87	29.85	29.82	29.80	29.78
	2.5%	28.32	27.86	27.55	27.29	26.85	26.63	26.41	26.30	26.10	25.91	25.91	25.65	25.60	25.41	25.30	25.24	25.10	25.11	24.81	24.68
55-59	97.5%	28.72	28.85	28.98	29.05	29.10	29.16	29.27	29.27	29.31	29.43	29.47	29.51	29.50	29.50	29.66	29.68	29.60	29.65	29.68	29.64
	mean	26.58	26.50	26.44	26.37	26.24	26.17	26.15	26.10	26.02	25.99	25.96	25.90	25.85	25.82	25.78	25.75	25.73	25.70	25.69	25.67
	2.5%	24.27	23.95	23.70	23.36	23.04	22.88	22.64	22.60	22.41	22.22	22.14	21.93	21.92	21.77	21.63	21.61	21.45	21.49	21.26	21.21
60-64	97.5%	24.47	24.59	24.66	24.79	24.86	24.90	24.97	24.96	25.01	25.13	25.18	25.23	25.20	25.24	25.34	25.39	25.40	25.39	25.42	25.37
	mean	22.46	22.40	22.35	22.30	22.18	22.13	22.11	22.07	22.00	21.97	21.95	21.90	21.85	21.82	21.80	21.77	21.74	21.73	21.72	21.70
	2.5%	20.41	20.13	19.89	19.66	19.38	19.17	18.98	18.97	18.80	18.60	18.58	18.40	18.26	18.13	18.09	18.04	18.00	17.91	17.85	17.67
65-69	97.5%	20.32	20.44	20.50	20.60	20.68	20.75	20.77	20.83	20.91	20.94	21.06	21.02	21.12	21.16	21.23	21.29	21.30	21.33	21.31	21.36
	mean	18.48	18.45	18.40	18.38	18.33	18.30	18.27	18.25	18.22	18.20	18.17	18.15	18.12	18.11	18.08	18.06	18.06	18.05	18.03	18.02
	2.5%	16.66	16.49	16.28	16.10	15.97	15.81	15.68	15.56	15.53	15.37	15.30	15.22	15.08	15.01	14.86	14.86	14.82	14.75	14.65	14.57
70-74	97.5%	16.48	16.55	16.63	16.71	16.73	16.83	16.85	16.89	16.95	16.99	17.07	17.15	17.16	17.19	17.25	17.29	17.31	17.30	17.34	17.37
	mean	14.77	14.76	14.72	14.71	14.68	14.65	14.64	14.63	14.61	14.60	14.58	14.56	14.55	14.54	14.52	14.51	14.51	14.50	14.49	14.48
	2.5%	13.12	12.98	12.81	12.69	12.59	12.52	12.42	12.34	12.28	12.22	12.15	12.03	11.99	11.93	11.86	11.83	11.74	11.70	11.64	11.57
75-79	97.5%	13.17	13.19	13.24	13.30	13.35	13.44	13.44	13.51	13.53	13.56	13.61	13.66	13.71	13.70	13.78	13.8				

Table A.29. The forecasted Italy female life expectancy by model 3

age 0	97.5%	85.20	85.50	85.77	85.96	86.14	86.24	86.31	86.39	86.49	86.58	86.67	86.76	86.80	86.90	87.00	87.04	87.09	87.13	87.18	87.22
	mean	84.27	84.22	84.19	84.16	84.12	84.07	84.02	83.98	83.94	83.90	83.86	83.81	83.77	83.72	83.67	83.62	83.58	83.53	83.48	83.44
	2.5%	83.28	82.76	82.43	82.14	81.82	81.53	81.25	81.07	80.85	80.60	80.34	80.16	79.92	79.64	79.42	79.23	79.01	78.75	78.58	78.41
1-4	97.5%	84.41	84.70	84.96	85.15	85.31	85.42	85.48	85.55	85.66	85.74	85.82	85.92	85.95	86.05	86.15	86.18	86.23	86.27	86.32	86.36
	mean	83.51	83.47	83.44	83.40	83.36	83.32	83.27	83.24	83.19	83.15	83.11	83.07	83.02	82.98	82.93	82.88	82.84	82.80	82.75	82.71
	2.5%	82.56	82.06	81.74	81.47	81.16	80.88	80.61	80.43	80.24	79.99	79.75	79.55	79.34	79.07	78.83	78.68	78.46	78.21	78.03	77.86
5-9	97.5%	80.44	80.73	80.98	81.17	81.34	81.44	81.50	81.57	81.68	81.76	81.84	81.93	81.97	82.06	82.16	82.20	82.24	82.29	82.33	82.37
	mean	79.55	79.51	79.48	79.45	79.41	79.36	79.32	79.28	79.24	79.20	79.16	79.12	79.07	79.03	78.98	78.93	78.89	78.85	78.80	78.76
	2.5%	78.61	78.12	77.80	77.54	77.23	76.97	76.70	76.52	76.33	76.09	75.85	75.66	75.45	75.17	74.95	74.80	74.58	74.35	74.17	74.00
10-14	97.5%	75.46	75.75	76.00	76.19	76.36	76.45	76.51	76.58	76.69	76.77	76.85	76.95	76.98	77.08	77.17	77.21	77.26	77.30	77.34	77.38
	mean	74.58	74.54	74.51	74.48	74.44	74.39	74.35	74.31	74.27	74.24	74.20	74.15	74.11	74.06	74.02	73.97	73.93	73.89	73.84	73.80
	2.5%	73.65	73.16	72.85	72.60	72.28	72.02	71.76	71.60	71.40	71.16	70.93	70.74	70.54	70.27	70.04	69.91	69.69	69.50	69.30	69.13
15-19	97.5%	70.48	70.77	71.02	71.20	71.37	71.46	71.53	71.60	71.71	71.79	71.86	71.96	71.99	72.09	72.18	72.22	72.26	72.31	72.35	72.39
	mean	69.61	69.57	69.54	69.51	69.47	69.43	69.38	69.35	69.31	69.27	69.23	69.19	69.15	69.10	69.06	69.01	68.97	68.93	68.89	68.85
	2.5%	68.69	68.20	67.90	67.65	67.34	67.09	66.84	66.67	66.49	66.25	66.04	65.83	65.65	65.38	65.18	65.05	64.82	64.60	64.46	64.28
20-24	97.5%	65.51	65.80	66.04	66.23	66.39	66.49	66.55	66.62	66.73	66.81	66.88	66.98	67.01	67.11	67.19	67.23	67.28	67.32	67.37	67.40
	mean	64.66	64.62	64.59	64.56	64.52	64.48	64.44	64.40	64.36	64.33	64.29	64.25	64.21	64.17	64.13	64.08	64.04	64.01	63.96	63.92
	2.5%	63.75	63.28	62.98	62.74	62.44	62.20	61.95	61.79	61.61	61.41	61.17	60.98	60.81	60.57	60.37	60.23	60.02	59.80	59.69	59.52
25-29	97.5%	60.55	60.83	61.07	61.26	61.42	61.51	61.57	61.64	61.74	61.83	61.90	62.00	62.03	62.12	62.21	62.25	62.30	62.34	62.39	62.42
	mean	59.71	59.67	59.64	59.61	59.58	59.54	59.50	59.47	59.43	59.40	59.36	59.32	59.28	59.24	59.20	59.16	59.12	59.09	59.05	59.01
	2.5%	58.82	58.37	58.08	57.85	57.55	57.33	57.09	56.93	56.76	56.56	56.35	56.17	56.00	55.78	55.57	55.45	55.25	55.07	54.96	54.79
30-34	97.5%	55.59	55.87	56.10	56.29	56.45	56.54	56.60	56.67	56.77	56.85	56.92	57.02	57.05	57.15	57.23	57.27	57.31	57.35	57.40	57.44
	mean	54.77	54.73	54.70	54.67	54.64	54.60	54.56	54.53	54.50	54.47	54.43	54.39	54.36	54.32	54.28	54.24	54.21	54.17	54.14	54.10
	2.5%	53.90	53.46	53.18	52.96	52.68	52.45	52.22	52.10	51.91	51.72	51.54	51.35	51.17	50.99	50.80	50.67	50.48	50.33	50.22	50.05
35-39	97.5%	50.64	50.91	51.15	51.32	51.48	51.57	51.63	51.70	51.80	51.88	51.95	52.05	52.07	52.17	52.25	52.29	52.34	52.38	52.42	52.46
	mean	49.84	49.80	49.78	49.75	49.72	49.69	49.65	49.62	49.59	49.56	49.53	49.49	49.46	49.42	49.39	49.35	49.32	49.29	49.26	49.22
	2.5%	48.99	48.57	48.31	48.10	47.83	47.62	47.40	47.28	47.10	46.94	46.76	46.58	46.44	46.27	46.12	45.98	45.82	45.70	45.57	45.43
40-44	97.5%	45.72	45.99	46.22	46.38	46.54	46.62	46.70	46.75	46.85	46.93	47.00	47.10	47.12	47.21	47.30	47.33	47.38	47.43	47.47	47.50
	mean	44.94	44.91	44.88	44.86	44.83	44.80	44.76	44.74	44.71	44.68	44.65	44.62	44.59	44.55	44.52	44.49	44.46	44.43	44.40	44.37
	2.5%	44.13	43.72	43.48	43.27	43.02	42.82	42.62	42.49	42.34	42.19	42.02	41.85	41.71	41.57	41.44	41.28	41.15	41.03	40.92	40.80
45-49	97.5%	40.85	41.10	41.33	41.49	41.63	41.73	41.79	41.86	41.95	42.02	42.10	42.19	42.21	42.30	42.38	42.41	42.45	42.50	42.54	42.58
	mean	40.10	40.07	40.04	40.02	39.99	39.96	39.93	39.91	39.88	39.85	39.83	39.79	39.77	39.73	39.70	39.67	39.64	39.62	39.59	39.56
	2.5%	39.32	38.94	38.70	38.50	38.24	38.08	37.89	37.79	37.63	37.49	37.32	37.17	37.05	36.92	36.79	36.70	36.54	36.43	36.33	36.21
50-54	97.5%	36.06	36.30	36.51	36.67	36.81	36.89	36.95	37.02	37.13	37.18	37.24	37.33	37.36	37.44	37.52	37.56	37.59	37.64	37.68	37.71
	mean	35.33	35.30	35.28	35.26	35.24	35.21	35.18	35.15	35.13	35.10	35.08	35.05	35.02	34.99	34.96	34.93	34.91	34.89	34.86	34.83
	2.5%	34.58	34.23	34.00	33.82	33.58	33.40	33.25	33.16	33.00	32.87	32.71	32.58	32.45	32.34	32.22	32.13	31.99	31.88	31.79	31.67
55-59	97.5%	31.35	31.58	31.79	31.93	32.07	32.16	32.20	32.26	32.33	32.40	32.48	32.56	32.59	32.67	32.73	32.77	32.80	32.86	32.90	32.92
	mean	30.67	30.64	30.62	30.60	30.58	30.55	30.52	30.50	30.48	30.45	30.43	30.40	30.38	30.35	30.33	30.30	30.28	30.25	30.23	30.21
	2.5%	29.96	29.62	29.42	29.25	29.02	28.86	28.70	28.62	28.48	28.37	28.23	28.10	27.98	27.88	27.78	27.68	27.57	27.47	27.38	27.27
60-64	97.5%	26.76	26.97	27.16	27.30	27.42	27.50	27.55	27.61	27.68	27.73	27.81	27.88	27.90	27.99	28.05	28.08	28.10	28.18	28.20	28.23
	mean	26.10	26.08	26.06	26.04	26.02	25.99	25.97	25.95	25.92	25.90	25.88	25.85	25.83	25.80	25.78	25.75	25.73	25.71	25.69	25.67
	2.5%	25.43	25.13	24.94	24.77	24.56	24.40	24.27	24.18	24.05	23.94	23.81	23.69	23.58	23.49	23.40	23.31	23.20	23.11	23.03	22.93
65-69	97.5%	22.28	22.48	22.65	22.78	22.89	22.97	23.01	23.07	23.13	23.18	23.25	23.32	23.34	23.43	23.48	23.51	23.52	23.59	23.62	23.65
	mean	21.67	21.65	21.63	21.61	21.59	21.57	21.54	21.52	21.50	21.48	21.46	21.44	21.42	21.39	21.37	21.34	21.32	21.31	21.28	21.26
	2.5%	21.04	20.77	20.59	20.43	20.24	20.10	19.97	19.88	19.79	19.68	19.55	19.46	19.37	19.26	19.18	19.09	19.00	18.92	18.85	18.78
70-74	97.5%	17.98	18.15	18.31	18.43	18.54	18.61	18.64	18.70	18.76	18.80	18.85	18.92	18.93	19.00	19.05	19.08	19.12	19.17	19.18	19.22
	mean	17.42	17.40	17.38	17.36	17.34	17.32	17.30	17.28	17.26	17.24	17.22	17.20	17.18	17.15	17.13	17.11	17.09	17.07	17.05	17.03
	2.5%	16.85	16.60	16.43	16.29	16.12	16.00	15.88	15.79	15.69	15.59	15.49	15.39	15.30	15.21	15.16	15.07	14.98	14.92	14.85	14.78
75-79	97.5%	13.89	14.05	14.20	14.29	14.39	14.44	14.49	14.52	14.59	14.60	14.67	14.72	14.73	14.79	14.83	14.87	14.90	14.94	14.97	14.9

Table A.30. The forecasted Italy male life expectancy by model 3

age	97.5%	80.58	80.94	81.25	81.47	81.64	81.72	81.80	81.92	82.06	82.09	82.24	82.30	82.35	82.41	82.55	82.55	82.61	82.69	82.68	82.77
	mean	79.45	79.37	79.30	79.23	79.16	79.07	78.98	78.90	78.81	78.73	78.64	78.55	78.45	78.35	78.25	78.14	78.04	77.95	77.84	77.74
	2.5%	78.16	77.46	76.86	76.47	75.94	75.48	75.00	74.61	74.15	73.65	73.23	72.74	72.28	71.83	71.31	70.82	70.43	69.83	69.42	68.89
1-4	97.5%	79.81	80.16	80.46	80.66	80.83	80.91	80.98	81.09	81.23	81.27	81.41	81.48	81.51	81.59	81.70	81.71	81.76	81.83	81.83	81.91
	mean	78.71	78.63	78.56	78.50	78.42	78.33	78.24	78.17	78.08	78.00	77.91	77.82	77.72	77.62	77.52	77.42	77.32	77.23	77.12	77.02
	2.5%	77.46	76.78	76.20	75.80	75.28	74.83	74.35	73.99	73.53	73.05	72.66	72.16	71.69	71.26	70.72	70.25	69.87	69.29	68.88	68.36
5-9	97.5%	75.85	76.18	76.49	76.69	76.86	76.94	77.00	77.11	77.25	77.29	77.44	77.50	77.53	77.61	77.72	77.72	77.77	77.85	77.85	77.93
	mean	74.76	74.68	74.61	74.54	74.46	74.38	74.29	74.22	74.13	74.05	73.96	73.87	73.77	73.67	73.58	73.47	73.37	73.28	73.17	73.07
	2.5%	73.52	72.85	72.27	71.87	71.36	70.92	70.44	70.07	69.61	69.14	68.74	68.24	67.78	67.36	66.85	66.37	65.97	65.39	65.02	64.48
10-14	97.5%	70.87	71.20	71.51	71.71	71.87	71.95	72.02	72.13	72.26	72.30	72.45	72.51	72.54	72.62	72.73	72.74	72.78	72.86	72.86	72.93
	mean	69.79	69.71	69.64	69.57	69.50	69.41	69.32	69.25	69.16	69.08	69.00	68.90	68.81	68.71	68.61	68.51	68.41	68.32	68.21	68.11
	2.5%	68.55	67.89	67.32	66.92	66.41	65.98	65.51	65.14	64.68	64.22	63.82	63.32	62.88	62.46	61.93	61.47	61.07	60.51	60.12	59.62
15-19	97.5%	65.90	66.23	66.53	66.73	66.89	66.97	67.04	67.15	67.28	67.32	67.47	67.53	67.56	67.63	67.75	67.75	67.80	67.87	67.87	67.95
	mean	64.82	64.74	64.68	64.61	64.54	64.45	64.36	64.29	64.21	64.13	64.04	63.95	63.86	63.76	63.66	63.56	63.46	63.37	63.27	63.17
	2.5%	63.60	62.94	62.38	62.00	61.49	61.06	60.60	60.23	59.77	59.32	58.93	58.44	58.00	57.59	57.05	56.62	56.24	55.64	55.28	54.76
20-24	97.5%	60.96	61.29	61.58	61.78	61.94	62.02	62.09	62.19	62.32	62.35	62.50	62.56	62.59	62.68	62.77	62.78	62.83	62.91	62.92	62.98
	mean	59.92	59.85	59.79	59.73	59.65	59.58	59.49	59.42	59.35	59.27	59.19	59.10	59.02	58.92	58.83	58.74	58.65	58.56	58.46	58.37
	2.5%	58.75	58.13	57.58	57.23	56.76	56.33	55.88	55.54	55.13	54.75	54.33	53.88	53.45	53.07	52.55	52.16	51.76	51.23	50.86	50.46
25-29	97.5%	56.05	56.36	56.64	56.83	56.99	57.07	57.13	57.24	57.36	57.40	57.54	57.60	57.64	57.71	57.81	57.82	57.86	57.95	57.96	58.01
	mean	55.07	55.01	54.96	54.91	54.85	54.79	54.72	54.67	54.60	54.54	54.48	54.41	54.35	54.27	54.21	54.13	54.06	54.00	53.92	53.85
	2.5%	53.98	53.44	52.98	52.66	52.28	51.94	51.61	51.30	50.99	50.68	50.40	50.01	49.77	49.40	49.03	48.77	48.47	48.03	47.82	47.52
30-34	97.5%	51.14	51.45	51.71	51.89	52.05	52.13	52.18	52.29	52.40	52.45	52.59	52.65	52.68	52.75	52.87	52.86	52.91	52.99	53.00	53.04
	mean	50.21	50.16	50.12	50.07	50.02	49.97	49.91	49.86	49.81	49.76	49.71	49.65	49.59	49.53	49.47	49.41	49.35	49.30	49.24	49.18
	2.5%	49.19	48.69	48.29	48.02	47.67	47.37	47.08	46.84	46.53	46.28	46.06	45.74	45.52	45.27	44.95	44.75	44.47	44.14	43.97	43.79
35-39	97.5%	46.24	46.53	46.79	46.96	47.12	47.20	47.26	47.35	47.47	47.51	47.64	47.69	47.73	47.80	47.91	47.91	47.96	48.05	48.05	48.11
	mean	45.36	45.31	45.28	45.24	45.20	45.15	45.10	45.06	45.01	44.97	44.93	44.88	44.83	44.78	44.73	44.67	44.63	44.58	44.53	44.48
	2.5%	44.40	43.96	43.58	43.34	43.04	42.78	42.53	42.32	42.11	41.88	41.67	41.42	41.20	40.96	40.75	40.59	40.35	40.13	39.91	39.80
40-44	97.5%	41.38	41.64	41.90	42.07	42.20	42.29	42.34	42.44	42.55	42.60	42.71	42.77	42.81	42.88	42.98	42.99	43.04	43.12	43.12	43.18
	mean	40.54	40.50	40.47	40.44	40.40	40.36	40.31	40.28	40.24	40.20	40.17	40.12	40.08	40.04	40.00	39.95	39.91	39.87	39.83	39.79
	2.5%	39.65	39.25	38.90	38.69	38.42	38.20	37.99	37.82	37.61	37.41	37.22	37.02	36.82	36.66	36.47	36.37	36.16	35.94	35.77	35.70
45-49	97.5%	36.56	36.81	37.05	37.21	37.35	37.42	37.47	37.58	37.68	37.72	37.83	37.88	37.93	37.99	38.09	38.11	38.14	38.21	38.22	38.28
	mean	35.78	35.74	35.72	35.69	35.66	35.62	35.58	35.55	35.52	35.49	35.46	35.42	35.39	35.35	35.32	35.28	35.25	35.21	35.18	35.15
	2.5%	34.96	34.59	34.28	34.10	33.86	33.69	33.49	33.32	33.15	33.00	32.84	32.67	32.50	32.38	32.20	32.10	31.95	31.82	31.65	31.54
50-54	97.5%	31.86	32.08	32.31	32.45	32.58	32.64	32.71	32.79	32.88	32.93	33.03	33.09	33.12	33.17	33.28	33.30	33.34	33.40	33.42	33.45
	mean	31.12	31.09	31.07	31.05	31.02	30.99	30.96	30.93	30.90	30.88	30.85	30.82	30.79	30.76	30.73	30.70	30.67	30.65	30.62	30.59
	2.5%	30.37	30.04	29.78	29.62	29.41	29.23	29.10	28.96	28.79	28.66	28.54	28.40	28.24	28.14	28.02	27.92	27.80	27.64	27.56	27.45
55-59	97.5%	27.30	27.50	27.70	27.83	27.95	28.02	28.07	28.14	28.23	28.28	28.37	28.43	28.46	28.52	28.62	28.62	28.66	28.72	28.74	28.77
	mean	26.61	26.59	26.57	26.55	26.52	26.50	26.47	26.45	26.42	26.40	26.38	26.35	26.33	26.30	26.27	26.25	26.22	26.20	26.18	26.15
	2.5%	25.93	25.64	25.39	25.25	25.06	24.94	24.81	24.68	24.53	24.42	24.30	24.19	24.08	23.95	23.85	23.78	23.65	23.55	23.47	23.41
60-64	97.5%	22.93	23.10	23.29	23.41	23.51	23.58	23.63	23.69	23.78	23.82	23.90	23.95	23.98	24.04	24.12	24.13	24.17	24.23	24.24	24.28
	mean	22.29	22.27	22.25	22.23	22.21	22.19	22.16	22.14	22.12	22.10	22.08	22.06	22.03	22.01	21.99	21.96	21.94	21.92	21.90	21.88
	2.5%	21.65	21.39	21.19	21.05	20.90	20.77	20.65	20.55	20.41	20.31	20.21	20.11	20.03	19.92	19.81	19.74	19.64	19.53	19.43	19.41
65-69	97.5%	18.81	18.98	19.13	19.25	19.34	19.39	19.44	19.49	19.56	19.62	19.71	19.72	19.75	19.81	19.86	19.91	19.92	19.98	19.99	20.02
	mean	18.24	18.21	18.20	18.18	18.16	18.14	18.12	18.10	18.08	18.06	18.04	18.02	18.00	17.98	17.95	17.93	17.91	17.89	17.87	17.85
	2.5%	17.66	17.44	17.23	17.11	16.97	16.86	16.74	16.66	16.54	16.44	16.36	16.26	16.19	16.07	16.02	15.94	15.85	15.76	15.69	15.62
70-74	97.5%	14.99	15.13	15.27	15.38	15.46	15.50	15.53	15.59	15.63	15.70	15.75	15.78	15.81	15.86	15.90	15.93	15.96	16.02	16.02	16.04
	mean	14.48	14.46	14.44	14.43	14.41	14.39	14.37	14.35	14.33	14.32	14.30	14.28	14.26	14.24	14.22	14.20	14.18	14.17	14.15	14.13
	2.5%	13.97	13.77	13.60	13.50	13.37	13.27	13.16	13.09	12.98	12.90	12.82	12.74	12.68	12.58	12.53	12.45	12.38	12.29	12.23	12.21
75-79	97.5%	11.46	11.58	11.69	11.78	11.85	11.88	11.92	11.96	12.00	12.04	12.11	12.12	12.13	12.17	12.23	12.25				

Table A.31. The forecasted Italy female life expectancy by model 8

age	97.5%	85.21	85.62	85.96	86.26	86.53	86.80	87.07	87.31	87.58	87.81	88.04	88.25	88.53	88.73	88.94	89.13	89.32	89.51	89.69	89.87
0	mean	84.36	84.52	84.69	84.87	85.04	85.20	85.37	85.54	85.70	85.87	86.02	86.18	86.33	86.48	86.62	86.76	86.91	87.05	87.19	87.33
	2.5%	83.44	83.40	83.38	83.39	83.42	83.44	83.46	83.58	83.61	83.66	83.72	83.81	83.87	83.99	84.09	84.15	84.17	84.31	84.34	84.38
1-4	97.5%	84.43	84.80	85.13	85.41	85.67	85.92	86.19	86.41	86.67	86.90	87.11	87.32	87.59	87.78	88.00	88.18	88.35	88.55	88.73	88.90
	mean	83.59	83.74	83.90	84.06	84.22	84.37	84.53	84.68	84.83	84.99	85.14	85.28	85.43	85.57	85.70	85.84	85.98	86.12	86.25	86.38
	2.5%	82.69	82.65	82.64	82.62	82.63	82.66	82.69	82.78	82.79	82.83	82.87	82.98	83.02	83.13	83.21	83.27	83.30	83.41	83.45	83.46
5-9	97.5%	80.45	80.82	81.15	81.43	81.69	81.93	82.20	82.42	82.68	82.91	83.12	83.32	83.60	83.79	84.00	84.18	84.36	84.55	84.73	84.90
	mean	79.62	79.77	79.92	80.08	80.24	80.39	80.54	80.70	80.85	81.01	81.15	81.30	81.44	81.58	81.71	81.85	81.99	82.12	82.26	82.39
	2.5%	78.72	78.69	78.67	78.65	78.66	78.68	78.72	78.81	78.82	78.85	78.89	79.00	79.05	79.15	79.23	79.28	79.32	79.42	79.46	79.48
10-14	97.5%	75.47	75.84	76.16	76.44	76.70	76.94	77.21	77.43	77.69	77.91	78.12	78.33	78.60	78.79	79.00	79.18	79.36	79.56	79.73	79.90
	mean	74.64	74.79	74.95	75.11	75.26	75.41	75.56	75.72	75.87	76.02	76.17	76.31	76.45	76.59	76.73	76.86	77.00	77.13	77.27	77.40
	2.5%	73.75	73.72	73.70	73.68	73.69	73.71	73.76	73.84	73.86	73.88	73.93	74.03	74.07	74.17	74.26	74.32	74.35	74.45	74.47	74.50
15-19	97.5%	70.49	70.85	71.18	71.45	71.71	71.95	72.22	72.43	72.69	72.92	73.13	73.34	73.61	73.80	74.01	74.19	74.37	74.56	74.74	74.90
	mean	69.66	69.81	69.97	70.12	70.28	70.43	70.58	70.73	70.88	71.03	71.18	71.32	71.47	71.60	71.74	71.87	72.01	72.14	72.28	72.41
	2.5%	68.78	68.75	68.73	68.71	68.72	68.75	68.79	68.87	68.89	68.91	68.96	69.06	69.10	69.21	69.29	69.34	69.37	69.47	69.51	69.52
20-24	97.5%	65.52	65.88	66.20	66.47	66.73	66.97	67.23	67.45	67.71	67.94	68.14	68.35	68.62	68.81	69.01	69.20	69.38	69.57	69.75	69.91
	mean	64.70	64.85	65.00	65.16	65.31	65.46	65.61	65.76	65.91	66.06	66.21	66.35	66.49	66.63	66.76	66.90	67.03	67.17	67.30	67.43
	2.5%	63.83	63.79	63.78	63.77	63.78	63.80	63.84	63.92	63.94	63.96	64.02	64.11	64.16	64.27	64.34	64.39	64.42	64.53	64.57	64.58
25-29	97.5%	60.57	60.92	61.23	61.51	61.76	62.00	62.26	62.48	62.73	62.96	63.16	63.37	63.63	63.82	64.02	64.21	64.39	64.58	64.76	64.92
	mean	59.76	59.90	60.05	60.21	60.36	60.51	60.66	60.81	60.95	61.11	61.25	61.39	61.53	61.67	61.80	61.93	62.07	62.20	62.33	62.46
	2.5%	58.91	58.87	58.86	58.83	58.85	58.88	58.91	58.98	59.01	59.04	59.10	59.17	59.24	59.34	59.42	59.48	59.48	59.61	59.64	59.66
30-34	97.5%	55.61	55.96	56.26	56.54	56.78	57.03	57.28	57.50	57.75	57.97	58.18	58.38	58.64	58.84	59.03	59.22	59.40	59.59	59.77	59.92
	mean	54.80	54.95	55.09	55.25	55.40	55.54	55.69	55.84	55.99	56.14	56.28	56.42	56.56	56.69	56.82	56.96	57.09	57.22	57.35	57.48
	2.5%	53.96	53.92	53.91	53.90	53.91	53.93	53.96	54.04	54.06	54.10	54.16	54.24	54.29	54.40	54.48	54.51	54.54	54.66	54.69	54.70
35-39	97.5%	50.65	51.00	51.30	51.57	51.82	52.06	52.31	52.52	52.78	53.00	53.20	53.40	53.67	53.85	54.06	54.23	54.41	54.60	54.77	54.93
	mean	49.86	50.00	50.15	50.30	50.45	50.59	50.74	50.89	51.03	51.18	51.32	51.46	51.60	51.73	51.86	51.99	52.12	52.25	52.38	52.51
	2.5%	49.04	48.99	48.98	48.96	48.98	49.00	49.02	49.10	49.13	49.17	49.22	49.29	49.35	49.47	49.54	49.60	49.61	49.72	49.76	49.78
40-44	97.5%	45.73	46.06	46.36	46.63	46.88	47.11	47.36	47.57	47.82	48.05	48.25	48.44	48.70	48.88	49.09	49.27	49.44	49.63	49.81	49.95
	mean	44.95	45.09	45.23	45.38	45.53	45.67	45.81	45.96	46.10	46.25	46.38	46.52	46.66	46.79	46.91	47.05	47.17	47.30	47.43	47.56
	2.5%	44.15	44.10	44.09	44.07	44.10	44.12	44.12	44.20	44.24	44.27	44.33	44.39	44.46	44.57	44.61	44.70	44.71	44.81	44.85	44.87
45-49	97.5%	40.86	41.20	41.47	41.74	41.98	42.21	42.46	42.66	42.90	43.12	43.32	43.51	43.77	43.95	44.15	44.33	44.50	44.68	44.85	44.99
	mean	40.10	40.24	40.38	40.52	40.67	40.80	40.94	41.09	41.22	41.37	41.50	41.63	41.77	41.90	42.02	42.15	42.27	42.40	42.53	42.65
	2.5%	39.32	39.28	39.27	39.25	39.29	39.30	39.30	39.39	39.41	39.45	39.51	39.57	39.63	39.73	39.80	39.83	39.86	39.95	40.00	40.02
50-54	97.5%	36.08	36.40	36.66	36.92	37.16	37.37	37.62	37.80	38.05	38.27	38.47	38.63	38.89	39.08	39.26	39.44	39.60	39.80	39.95	40.08
	mean	35.33	35.47	35.60	35.74	35.88	36.01	36.15	36.29	36.42	36.56	36.69	36.82	36.95	37.07	37.19	37.32	37.44	37.56	37.69	37.80
	2.5%	34.57	34.54	34.53	34.51	34.55	34.55	34.59	34.65	34.68	34.71	34.76	34.83	34.89	34.99	35.03	35.08	35.13	35.19	35.23	35.24
55-59	97.5%	31.38	31.68	31.94	32.18	32.41	32.64	32.86	33.03	33.26	33.47	33.66	33.84	34.07	34.25	34.44	34.59	34.77	34.94	35.10	35.24
	mean	30.67	30.79	30.92	31.06	31.19	31.32	31.45	31.58	31.71	31.84	31.97	32.09	32.22	32.34	32.45	32.57	32.69	32.81	32.93	33.04
	2.5%	29.93	29.91	29.91	29.89	29.92	29.94	29.96	30.03	30.07	30.08	30.16	30.21	30.27	30.35	30.41	30.45	30.49	30.55	30.57	30.60
60-64	97.5%	26.80	27.07	27.30	27.53	27.75	27.97	28.16	28.34	28.56	28.76	28.94	29.10	29.34	29.49	29.68	29.83	29.99	30.14	30.30	30.45
	mean	26.10	26.22	26.34	26.47	26.59	26.71	26.84	26.96	27.08	27.21	27.33	27.44	27.56	27.67	27.78	27.89	28.00	28.11	28.22	28.33
	2.5%	25.41	25.38	25.37	25.36	25.39	25.42	25.43	25.49	25.53	25.54	25.57	25.67	25.71	25.76	25.81	25.84	25.90	25.96	26.03	26.02
65-69	97.5%	22.33	22.58	22.79	23.00	23.20	23.39	23.58	23.74	23.95	24.13	24.31	24.46	24.66	24.81	24.98	25.12	25.28	25.42	25.56	25.70
	mean	21.68	21.78	21.89	22.01	22.12	22.23	22.34	22.46	22.57	22.68	22.79	22.89	23.00	23.10	23.20	23.30	23.40	23.50	23.61	23.70
	2.5%	21.00	20.98	20.98	20.97	21.00	21.02	21.04	21.08	21.12	21.14	21.18	21.24	21.28	21.33	21.36	21.39	21.42	21.51	21.54	21.57
70-74	97.5%	18.01	18.24	18.43	18.61	18.78	18.97	19.13	19.28	19.46	19.63	19.79	19.91	20.09	20.23	20.37	20.50	20.66	20.79	20.92	21.02
	mean	17.38	17.48	17.57	17.66	17.76	17.85	17.95	18.04	18.14	18.24	18.33	18.42	18.51	18.59	18.67	18.76	18.85	18.93	19.02	19.10
	2.5%	16.74	16.71	16.69	16.66	16.68	16.69	16.72	16.74	16.77	16.79	16.83	16.87	16.90	16.92	16.92	16.96	17.01	17.05	17.05	
75-79	97.5%	13.91	14.09	14.26	14.41	14.57	14.73	14.85	14.98	15.14	15.26	15.40	15.50	15.65	15.76	15.87	15.99	16.12	16.21	16.31	16.43
	mean	13.32	13.39	13.45	13.52	13.59	13.66	13.73	13.80	13.87	13.94	14.01	14.07	14.14	14.20	14.26	14.32	14.38	14.45	14.51	14.57
	2.5%	12.72	12.66	12.62	12.60	12.57	12.56	12.54	12.56	12.56	12.55	12.54	12.57	12.58	12.57	12.58	12.56	12.58	12.60	12.63	12.64
80-84	97.5%	10.16	10.28	10.42	10.53	10.63	10.75	10.85	10.94	11.04	11.15	11.22	11.30	11.39	11.49	11.59	11.63	11.73	11.79	11.87	11.93
	mean	9.63	9.66	9.69	9.72	9.75	9.79	9.82	9.86	9.89	9.93	9.96	9.99	10.03	10.06	10.08	10.12	10.15	10.18	10.22	10.25
	2.5%	9.09	9.01	8.93	8.87	8.84	8.80	8.77	8.74	8.71	8.69	8.66	8.64	8.62	8.57	8.59	8.55	8.52	8.50	8.49	8.46
85-89	97.5%	7.09	7.16	7.23	7.31	7.36	7.43	7.46	7.49	7.57	7.62	7.65	7.67	7.73	7.78	7.81	7.82	7.86	7.91	7.96	7.97
	mean	6.60	6.59	6.58	6.58	6.58	6.58	6.58	6.58	6.58	6.59	6.58	6.58	6.58	6.58	6.57	6.57	6.56	6.56	6.57	6.57
	2.5%																				

Table A.32. The forecasted Italy male life expectancy by model 8

age 0	97.5%	80.32	80.75	81.15	81.55	81.83	82.18	82.48	82.77	83.07	83.39	83.66	83.93	84.24	84.46	84.76	85.00	85.27	85.57	85.76	85.99
	mean	79.17	79.36	79.51	79.68	79.84	80.00	80.17	80.35	80.53	80.72	80.90	81.08	81.26	81.42	81.56	81.72	81.87	82.02	82.17	82.31
	2.5%	77.84	77.70	77.48	77.42	77.17	77.18	76.96	77.07	77.00	76.96	76.96	77.04	77.02	77.01	76.97	76.86	76.84	76.83	76.83	76.54
1-4	97.5%	79.55	79.95	80.33	80.71	80.98	81.31	81.60	81.87	82.16	82.47	82.74	83.00	83.30	83.52	83.80	84.04	84.31	84.61	84.80	85.01
	mean	78.42	78.59	78.72	78.88	79.02	79.17	79.33	79.50	79.67	79.85	80.02	80.19	80.36	80.51	80.65	80.80	80.94	81.09	81.23	81.37
	2.5%	77.10	76.95	76.73	76.65	76.40	76.39	76.17	76.25	76.17	76.14	76.10	76.18	76.16	76.14	76.09	75.95	75.95	75.92	75.90	75.62
5-9	97.5%	75.58	75.98	76.36	76.73	77.00	77.33	77.62	77.88	78.17	78.49	78.75	79.01	79.31	79.52	79.81	80.04	80.32	80.61	80.80	81.02
	mean	74.45	74.62	74.75	74.91	75.05	75.19	75.35	75.52	75.68	75.86	76.03	76.20	76.37	76.52	76.66	76.81	76.95	77.09	77.24	77.37
	2.5%	73.14	72.98	72.76	72.68	72.44	72.43	72.20	72.28	72.19	72.17	72.12	72.20	72.17	72.15	72.11	71.96	71.97	71.94	71.92	71.63
10-14	97.5%	70.60	71.00	71.37	71.75	72.01	72.34	72.63	72.89	73.18	73.50	73.76	74.02	74.31	74.53	74.82	75.05	75.32	75.61	75.81	76.02
	mean	69.48	69.64	69.78	69.93	70.07	70.21	70.37	70.54	70.70	70.88	71.05	71.22	71.39	71.54	71.67	71.82	71.96	72.11	72.25	72.38
	2.5%	68.17	68.02	67.80	67.71	67.46	67.47	67.23	67.32	67.22	67.20	67.14	67.24	67.21	67.18	67.13	66.99	66.98	66.96	66.92	66.69
15-19	97.5%	65.62	66.02	66.39	66.76	67.03	67.36	67.64	67.90	68.20	68.51	68.77	69.02	69.32	69.54	69.83	70.05	70.33	70.62	70.81	71.03
	mean	64.51	64.67	64.81	64.96	65.10	65.24	65.40	65.56	65.73	65.90	66.07	66.24	66.41	66.56	66.69	66.84	66.98	67.12	67.26	67.40
	2.5%	63.22	63.06	62.84	62.76	62.51	62.51	62.26	62.35	62.26	62.25	62.19	62.28	62.26	62.23	62.18	62.01	62.02	62.03	61.95	61.73
20-24	97.5%	60.69	61.08	61.45	61.81	62.08	62.40	62.68	62.95	63.22	63.54	63.80	64.06	64.36	64.57	64.86	65.09	65.35	65.64	65.85	66.06
	mean	59.62	59.78	59.92	60.08	60.23	60.37	60.53	60.70	60.87	61.05	61.22	61.39	61.56	61.71	61.85	62.00	62.15	62.30	62.44	62.58
	2.5%	58.38	58.24	58.07	58.04	57.79	57.84	57.67	57.72	57.66	57.66	57.71	57.70	57.70	57.66	57.62	57.63	57.59	57.54	57.55	57.33
25-29	97.5%	55.80	56.18	56.53	56.89	57.16	57.48	57.76	58.03	58.30	58.63	58.87	59.12	59.44	59.63	59.91	60.17	60.39	60.68	60.93	61.12
	mean	54.79	54.96	55.11	55.28	55.44	55.59	55.77	55.94	56.11	56.29	56.47	56.64	56.82	56.98	57.13	57.29	57.45	57.60	57.76	57.90
	2.5%	53.63	53.53	53.43	53.41	53.29	53.32	53.29	53.34	53.33	53.37	53.43	53.40	53.46	53.51	53.57	53.56	53.62	53.58	53.63	53.54
30-34	97.5%	50.91	51.28	51.64	52.00	52.26	52.56	52.85	53.11	53.39	53.70	53.94	54.21	54.51	54.69	54.98	55.24	55.44	55.74	55.98	56.18
	mean	49.96	50.14	50.29	50.47	50.64	50.80	50.98	51.15	51.32	51.51	51.68	51.85	52.03	52.19	52.34	52.51	52.67	52.84	52.99	53.14
	2.5%	48.89	48.82	48.75	48.77	48.67	48.78	48.73	48.79	48.82	48.83	48.85	48.95	49.00	49.06	49.07	49.14	49.22	49.27	49.24	49.24
35-39	97.5%	46.09	46.45	46.78	47.14	47.40	47.69	47.98	48.23	48.50	48.80	49.04	49.31	49.60	49.79	50.07	50.31	50.54	50.83	51.05	51.25
	mean	45.20	45.37	45.53	45.71	45.88	46.05	46.23	46.41	46.58	46.76	46.93	47.10	47.28	47.44	47.60	47.77	47.94	48.10	48.26	48.41
	2.5%	44.18	44.16	44.12	44.14	44.11	44.19	44.22	44.28	44.35	44.33	44.42	44.46	44.54	44.67	44.67	44.77	44.81	44.90	44.93	44.99
40-44	97.5%	41.28	41.62	41.94	42.29	42.54	42.82	43.11	43.37	43.63	43.92	44.15	44.43	44.70	44.89	45.18	45.43	45.64	45.92	46.15	46.34
	mean	40.44	40.61	40.78	40.96	41.13	41.30	41.48	41.66	41.83	42.01	42.19	42.36	42.54	42.70	42.86	43.03	43.20	43.37	43.52	43.68
	2.5%	39.49	39.51	39.47	39.53	39.53	39.62	39.67	39.76	39.82	39.88	39.95	39.99	40.08	40.18	40.24	40.34	40.43	40.50	40.58	40.63
45-49	97.5%	36.49	36.80	37.11	37.46	37.71	37.98	38.25	38.51	38.77	39.06	39.27	39.55	39.81	40.02	40.31	40.53	40.75	41.03	41.27	41.45
	mean	35.68	35.86	36.03	36.21	36.38	36.55	36.73	36.91	37.09	37.27	37.44	37.62	37.79	37.96	38.12	38.29	38.46	38.62	38.78	38.94
	2.5%	34.79	34.82	34.84	34.90	34.91	34.99	35.05	35.20	35.25	35.33	35.40	35.46	35.55	35.67	35.74	35.85	35.94	36.05	36.10	36.14
50-54	97.5%	31.81	32.11	32.40	32.72	32.97	33.23	33.50	33.74	33.98	34.28	34.49	34.77	35.02	35.23	35.49	35.72	35.93	36.20	36.43	36.61
	mean	31.04	31.22	31.38	31.56	31.74	31.91	32.08	32.26	32.43	32.61	32.78	32.96	33.13	33.30	33.46	33.63	33.80	33.96	34.12	34.28
	2.5%	30.17	30.22	30.24	30.32	30.37	30.46	30.55	30.67	30.73	30.83	30.92	31.01	31.11	31.22	31.30	31.41	31.50	31.60	31.70	31.81
55-59	97.5%	27.26	27.53	27.81	28.12	28.37	28.59	28.85	29.11	29.33	29.60	29.81	30.07	30.30	30.50	30.76	30.96	31.17	31.42	31.64	31.84
	mean	26.54	26.71	26.87	27.04	27.21	27.37	27.54	27.71	27.88	28.05	28.21	28.38	28.55	28.70	28.86	29.03	29.18	29.34	29.49	29.64
	2.5%	25.71	25.75	25.81	25.88	25.93	26.03	26.14	26.24	26.30	26.38	26.48	26.57	26.67	26.78	26.86	26.97	27.05	27.14	27.23	27.32
60-64	97.5%	22.93	23.20	23.43	23.71	23.92	24.17	24.40	24.61	24.84	25.07	25.28	25.52	25.75	25.92	26.15	26.35	26.55	26.77	26.98	27.17
	mean	22.26	22.42	22.56	22.72	22.87	23.02	23.18	23.34	23.49	23.65	23.80	23.96	24.11	24.26	24.40	24.55	24.70	24.84	24.98	25.12
	2.5%	21.47	21.54	21.56	21.63	21.69	21.76	21.88	21.95	22.02	22.10	22.19	22.28	22.38	22.46	22.55	22.66	22.70	22.81	22.89	23.00
65-69	97.5%	18.85	19.08	19.30	19.54	19.72	19.92	20.12	20.32	20.52	20.71	20.90	21.10	21.30	21.44	21.65	21.82	22.00	22.18	22.36	22.52
	mean	18.20	18.33	18.45	18.58	18.70	18.82	18.95	19.08	19.21	19.33	19.46	19.59	19.71	19.83	19.95	20.07	20.19	20.31	20.43	20.54
	2.5%	17.45	17.47	17.49	17.51	17.53	17.59	17.68	17.73	17.76	17.81	17.90	17.95	18.03	18.09	18.16	18.23	18.26	18.33	18.39	18.50
70-74	97.5%	14.99	15.19	15.37	15.57	15.73	15.88	16.05	16.21	16.37	16.57	16.69	16.87	17.02	17.15	17.31	17.46	17.61	17.77	17.92	18.02
	mean	14.41	14.50	14.59	14.69	14.79	14.88	14.98	15.08	15.18	15.27	15.37	15.47	15.56	15.65	15.74	15.84	15.92	16.02	16.11	16.19
	2.5%	13.68	13.70	13.71	13.70	13.70	13.72	13.77	13.79	13.80	13.87	13.90	13.96	14.00	14.04	14.07	14.13	14.15	14.19	14.24	14.29
75-79	97.5%	11.42	11.59	11.72	11.87	11.97	12.09	12.22	12.32	12.43	12.57	12.67	12.77	12.89	12.98	13.09	13.21				

Table A.33. The forecasted Japan female life expectancy by model 3

age 0	97.5%	87.19	87.46	87.71	87.88	88.05	88.12	88.20	88.30	88.38	88.42	88.55	88.60	88.68	88.76	88.85	88.88	88.91	89.02	89.00	89.09
	mean	86.34	86.30	86.28	86.25	86.22	86.18	86.14	86.12	86.08	86.05	86.02	85.98	85.95	85.91	85.87	85.83	85.80	85.77	85.73	85.69
	2.5%	85.44	85.04	84.70	84.48	84.19	83.98	83.69	83.52	83.35	83.18	82.91	82.76	82.68	82.42	82.22	82.10	81.91	81.72	81.61	81.50
1-4	97.5%	86.34	86.60	86.84	87.00	87.17	87.24	87.31	87.41	87.48	87.53	87.66	87.70	87.79	87.86	87.93	87.99	88.00	88.10	88.09	88.18
	mean	85.51	85.47	85.45	85.43	85.39	85.36	85.32	85.29	85.26	85.23	85.20	85.17	85.13	85.10	85.06	85.02	84.99	84.96	84.92	84.89
	2.5%	84.64	84.26	83.92	83.72	83.44	83.24	82.97	82.80	82.65	82.48	82.21	82.09	81.97	81.75	81.57	81.41	81.25	81.10	80.96	80.87
5-9	97.5%	82.38	82.65	82.88	83.04	83.21	83.27	83.34	83.44	83.52	83.56	83.69	83.73	83.82	83.89	83.96	84.01	84.03	84.13	84.12	84.20
	mean	81.57	81.54	81.51	81.49	81.46	81.43	81.39	81.36	81.33	81.30	81.27	81.24	81.21	81.17	81.14	81.10	81.07	81.04	81.00	80.97
	2.5%	80.72	80.35	80.02	79.82	79.55	79.36	79.09	78.93	78.77	78.61	78.37	78.24	78.13	77.90	77.72	77.58	77.43	77.28	77.15	77.05
10-14	97.5%	77.41	77.67	77.91	78.06	78.22	78.29	78.36	78.45	78.54	78.58	78.71	78.75	78.84	78.91	78.97	79.03	79.05	79.14	79.13	79.22
	mean	76.60	76.57	76.54	76.52	76.49	76.46	76.42	76.39	76.36	76.34	76.31	76.27	76.24	76.20	76.17	76.13	76.10	76.07	76.04	76.01
	2.5%	75.76	75.39	75.06	74.86	74.60	74.40	74.15	73.98	73.83	73.67	73.43	73.31	73.19	72.97	72.78	72.66	72.49	72.35	72.23	72.13
15-19	97.5%	72.43	72.69	72.92	73.08	73.24	73.31	73.38	73.47	73.55	73.59	73.72	73.76	73.85	73.92	73.98	74.04	74.06	74.15	74.14	74.23
	mean	71.63	71.59	71.57	71.55	71.52	71.48	71.45	71.42	71.39	71.36	71.33	71.30	71.27	71.23	71.20	71.16	71.13	71.10	71.07	71.04
	2.5%	70.79	70.42	70.10	69.90	69.64	69.44	69.19	69.02	68.87	68.72	68.47	68.36	68.24	68.02	67.83	67.71	67.54	67.42	67.29	67.20
20-24	97.5%	67.47	67.72	67.95	68.11	68.27	68.34	68.40	68.50	68.58	68.62	68.74	68.79	68.88	68.95	69.01	69.07	69.08	69.17	69.17	69.26
	mean	66.67	66.64	66.62	66.59	66.56	66.53	66.50	66.47	66.44	66.41	66.39	66.35	66.32	66.29	66.25	66.22	66.19	66.16	66.13	66.09
	2.5%	65.84	65.48	65.16	64.97	64.71	64.52	64.27	64.11	63.96	63.81	63.56	63.46	63.34	63.12	62.94	62.83	62.65	62.54	62.41	62.32
25-29	97.5%	62.53	62.78	63.01	63.16	63.32	63.39	63.45	63.54	63.62	63.67	63.79	63.83	63.91	63.98	64.05	64.10	64.12	64.21	64.21	64.29
	mean	61.75	61.71	61.69	61.67	61.64	61.61	61.58	61.55	61.52	61.49	61.47	61.43	61.40	61.37	61.34	61.30	61.27	61.25	61.21	61.18
	2.5%	60.93	60.58	60.27	60.08	59.82	59.64	59.40	59.23	59.08	58.95	58.70	58.61	58.48	58.27	58.11	57.99	57.80	57.71	57.59	57.49
30-34	97.5%	57.60	57.85	58.07	58.22	58.38	58.45	58.51	58.60	58.68	58.72	58.84	58.89	58.96	59.03	59.10	59.15	59.17	59.26	59.26	59.34
	mean	56.83	56.80	56.78	56.76	56.73	56.70	56.66	56.64	56.61	56.58	56.56	56.53	56.50	56.46	56.43	56.40	56.37	56.34	56.31	56.28
	2.5%	56.03	55.69	55.39	55.19	54.94	54.77	54.53	54.37	54.22	54.10	53.86	53.75	53.65	53.43	53.27	53.15	52.97	52.89	52.76	52.67
35-39	97.5%	52.69	52.93	53.14	53.30	53.45	53.51	53.57	53.67	53.74	53.79	53.90	53.95	54.02	54.09	54.15	54.20	54.22	54.31	54.31	54.39
	mean	51.93	51.90	51.88	51.86	51.84	51.81	51.77	51.75	51.72	51.70	51.67	51.64	51.61	51.58	51.55	51.51	51.49	51.46	51.43	51.40
	2.5%	51.15	50.82	50.52	50.33	50.09	49.92	49.70	49.54	49.41	49.27	49.04	48.95	48.83	48.63	48.46	48.34	48.20	48.09	47.98	47.89
40-44	97.5%	47.80	48.03	48.25	48.40	48.54	48.61	48.67	48.76	48.84	48.87	48.99	49.05	49.11	49.18	49.24	49.28	49.31	49.39	49.39	49.47
	mean	47.07	47.04	47.02	47.00	46.97	46.94	46.91	46.89	46.86	46.83	46.81	46.78	46.75	46.72	46.69	46.66	46.63	46.61	46.58	46.55
	2.5%	46.30	45.98	45.69	45.51	45.28	45.11	44.89	44.74	44.61	44.48	44.26	44.16	44.07	43.87	43.71	43.59	43.44	43.33	43.22	43.15
45-49	97.5%	42.96	43.18	43.40	43.54	43.68	43.75	43.80	43.89	43.96	44.00	44.11	44.17	44.24	44.30	44.35	44.41	44.42	44.51	44.50	44.58
	mean	42.24	42.21	42.19	42.17	42.15	42.12	42.09	42.07	42.04	42.02	41.99	41.96	41.94	41.91	41.88	41.85	41.82	41.80	41.77	41.74
	2.5%	41.50	41.19	40.90	40.73	40.50	40.34	40.13	39.99	39.86	39.74	39.53	39.44	39.33	39.15	39.00	38.89	38.72	38.63	38.52	38.45
50-54	97.5%	38.18	38.40	38.61	38.74	38.88	38.94	39.00	39.08	39.15	39.19	39.30	39.35	39.41	39.47	39.52	39.58	39.60	39.69	39.67	39.76
	mean	37.49	37.46	37.44	37.42	37.40	37.37	37.34	37.32	37.29	37.27	37.24	37.22	37.19	37.16	37.13	37.10	37.08	37.05	37.03	37.00
	2.5%	36.77	36.46	36.20	36.02	35.80	35.65	35.44	35.31	35.18	35.06	34.85	34.78	34.66	34.50	34.34	34.24	34.10	33.99	33.90	33.82
55-59	97.5%	33.48	33.69	33.89	34.02	34.15	34.21	34.26	34.34	34.41	34.45	34.55	34.60	34.67	34.72	34.76	34.82	34.86	34.93	34.91	35.00
	mean	32.81	32.78	32.76	32.75	32.72	32.70	32.67	32.65	32.62	32.60	32.58	32.55	32.52	32.50	32.47	32.44	32.42	32.39	32.37	32.34
	2.5%	32.11	31.83	31.57	31.41	31.19	31.04	30.85	30.72	30.59	30.48	30.29	30.21	30.09	29.94	29.78	29.71	29.55	29.47	29.37	29.29
60-64	97.5%	28.85	29.04	29.23	29.34	29.48	29.53	29.58	29.65	29.71	29.76	29.84	29.89	29.96	30.01	30.05	30.10	30.14	30.19	30.20	30.26
	mean	28.22	28.19	28.18	28.16	28.14	28.11	28.09	28.07	28.05	28.03	28.00	27.98	27.96	27.93	27.91	27.88	27.86	27.84	27.81	27.79
	2.5%	27.57	27.31	27.07	26.91	26.72	26.57	26.40	26.28	26.17	26.07	25.89	25.79	25.71	25.56	25.43	25.33	25.20	25.13	25.05	24.98
65-69	97.5%	24.29	24.47	24.64	24.75	24.87	24.92	24.97	25.03	25.10	25.14	25.22	25.26	25.31	25.36	25.41	25.45	25.48	25.53	25.54	25.60
	mean	23.72	23.70	23.69	23.67	23.65	23.63	23.61	23.59	23.57	23.55	23.53	23.51	23.49	23.47	23.45	23.42	23.40	23.38	23.36	23.34
	2.5%	23.13	22.90	22.68	22.53	22.36	22.23	22.09	21.96	21.85	21.77	21.62	21.51	21.45	21.31	21.20	21.13	21.01	20.92	20.84	20.80
70-74	97.5%	19.85	20.02	20.17	20.27	20.37	20.42	20.47	20.51	20.57	20.61	20.68	20.73	20.77	20.82	20.86	20.90	20.92	20.95	20.98	21.04
	mean	19.34	19.32	19.31	19.30	19.28	19.26	19.24	19.23	19.21	19.19	19.17	19.15	19.14	19.12	19.10	19.07	19.06	19.04	19.02	19.00
	2.5%	18.81	18.60	18.42	18.27	18.13	18.01	17.87	17.78	17.70	17.60	17.49	17.38	17.33	17.19	17.11	17.06	16.94	16.88	16.82	16.75
75-79	97.5%	15.59	15.73	15.85	15.95	16.03	16.07	16.11	16.16	16.20	16.25	16.30	16.34	16.39	16.43	16.48	16.50	16.51	16.54	16.56	16.6

Table A.34. The forecasted Japan male life expectancy by model 3

age 0	97.5%	81.13	81.44	81.71	81.88	82.10	82.18	82.28	82.38	82.50	82.57	82.69	82.78	82.84	82.93	83.04	83.10	83.14	83.23	83.27	83.29
	mean	80.16	80.12	80.09	80.06	80.03	79.99	79.94	79.91	79.87	79.84	79.80	79.76	79.72	79.68	79.64	79.59	79.56	79.52	79.48	79.44
	2.5%	79.15	78.70	78.31	78.07	77.78	77.52	77.26	77.06	76.83	76.67	76.40	76.24	76.07	75.85	75.62	75.45	75.25	75.09	74.84	74.78
1-4	97.5%	80.28	80.59	80.84	81.01	81.22	81.31	81.39	81.49	81.61	81.68	81.78	81.89	81.94	82.02	82.14	82.19	82.23	82.32	82.36	82.38
	mean	79.34	79.30	79.27	79.25	79.21	79.17	79.13	79.10	79.06	79.03	79.00	78.96	78.92	78.88	78.84	78.80	78.76	78.73	78.69	78.65
	2.5%	78.37	77.93	77.56	77.33	77.04	76.80	76.54	76.36	76.16	75.98	75.74	75.56	75.42	75.20	74.99	74.84	74.64	74.49	74.26	74.18
5-9	97.5%	76.33	76.63	76.88	77.05	77.26	77.34	77.43	77.52	77.65	77.71	77.81	77.92	77.97	78.05	78.17	78.22	78.25	78.35	78.38	78.41
	mean	75.40	75.36	75.34	75.31	75.28	75.24	75.20	75.17	75.13	75.10	75.07	75.03	74.99	74.95	74.91	74.87	74.84	74.80	74.77	74.73
	2.5%	74.44	74.01	73.65	73.42	73.15	72.91	72.66	72.47	72.28	72.11	71.87	71.69	71.57	71.35	71.14	70.98	70.80	70.65	70.43	70.37
10-14	97.5%	71.36	71.66	71.91	72.08	72.28	72.37	72.45	72.55	72.67	72.74	72.84	72.94	72.99	73.07	73.19	73.24	73.27	73.37	73.41	73.43
	mean	70.44	70.40	70.38	70.35	70.32	70.28	70.24	70.21	70.17	70.14	70.11	70.07	70.03	69.99	69.96	69.91	69.88	69.85	69.81	69.77
	2.5%	69.49	69.06	68.70	68.48	68.21	67.97	67.72	67.54	67.36	67.18	66.95	66.76	66.64	66.42	66.22	66.06	65.88	65.73	65.53	65.46
15-19	97.5%	66.39	66.69	66.94	67.11	67.31	67.39	67.48	67.57	67.70	67.76	67.86	67.96	68.01	68.10	68.21	68.27	68.29	68.40	68.42	68.45
	mean	65.48	65.44	65.41	65.39	65.35	65.32	65.28	65.25	65.21	65.18	65.15	65.11	65.07	65.03	65.00	64.95	64.92	64.89	64.85	64.82
	2.5%	64.53	64.11	63.75	63.53	63.27	63.02	62.77	62.59	62.43	62.24	62.01	61.82	61.71	61.49	61.29	61.15	60.96	60.80	60.60	60.53
20-24	97.5%	61.47	61.76	62.00	62.17	62.37	62.45	62.54	62.62	62.76	62.82	62.92	63.01	63.07	63.15	63.27	63.31	63.35	63.44	63.47	63.49
	mean	60.57	60.53	60.50	60.48	60.45	60.41	60.37	60.34	60.31	60.27	60.24	60.20	60.17	60.13	60.09	60.05	60.02	59.99	59.95	59.92
	2.5%	59.64	59.22	58.87	58.64	58.39	58.15	57.90	57.74	57.57	57.38	57.16	56.98	56.86	56.67	56.45	56.31	56.12	55.99	55.80	55.71
25-29	97.5%	56.61	56.90	57.13	57.30	57.49	57.57	57.64	57.74	57.86	57.93	58.02	58.10	58.16	58.23	58.35	58.41	58.44	58.54	58.57	58.58
	mean	55.73	55.70	55.67	55.65	55.62	55.58	55.54	55.51	55.48	55.45	55.42	55.39	55.35	55.31	55.28	55.24	55.21	55.18	55.14	55.11
	2.5%	54.83	54.43	54.09	53.88	53.63	53.40	53.16	53.01	52.83	52.67	52.46	52.30	52.18	52.00	51.79	51.67	51.49	51.34	51.15	51.05
30-34	97.5%	51.75	52.03	52.25	52.42	52.60	52.67	52.75	52.84	52.97	53.03	53.12	53.19	53.25	53.33	53.45	53.50	53.53	53.62	53.65	53.66
	mean	50.89	50.86	50.84	50.81	50.78	50.75	50.71	50.69	50.65	50.63	50.60	50.56	50.53	50.49	50.46	50.42	50.39	50.37	50.33	50.30
	2.5%	50.02	49.64	49.31	49.11	48.85	48.65	48.41	48.27	48.10	47.95	47.75	47.58	47.49	47.29	47.10	46.97	46.82	46.67	46.51	46.42
35-39	97.5%	46.89	47.16	47.38	47.54	47.72	47.79	47.87	47.96	48.08	48.14	48.23	48.31	48.36	48.43	48.56	48.61	48.63	48.72	48.75	48.76
	mean	46.06	46.03	46.01	45.98	45.95	45.92	45.89	45.86	45.83	45.80	45.78	45.74	45.71	45.68	45.65	45.61	45.58	45.55	45.52	45.49
	2.5%	45.21	44.84	44.53	44.34	44.08	43.89	43.68	43.52	43.37	43.22	43.03	42.87	42.77	42.59	42.39	42.27	42.13	42.00	41.83	41.74
40-44	97.5%	42.06	42.33	42.54	42.69	42.87	42.94	43.01	43.10	43.23	43.28	43.37	43.43	43.49	43.57	43.68	43.71	43.75	43.84	43.88	43.89
	mean	41.26	41.22	41.20	41.18	41.15	41.12	41.09	41.06	41.03	41.01	40.98	40.95	40.92	40.88	40.85	40.82	40.79	40.77	40.73	40.71
	2.5%	40.43	40.07	39.78	39.59	39.35	39.17	38.93	38.80	38.66	38.52	38.31	38.18	38.07	37.90	37.73	37.60	37.46	37.34	37.18	37.10
45-49	97.5%	37.31	37.56	37.76	37.91	38.08	38.15	38.23	38.31	38.43	38.48	38.56	38.63	38.69	38.78	38.88	38.91	38.94	39.02	39.07	39.09
	mean	36.52	36.49	36.47	36.45	36.42	36.39	36.36	36.33	36.31	36.28	36.25	36.22	36.20	36.16	36.13	36.10	36.07	36.05	36.02	35.99
	2.5%	35.73	35.38	35.09	34.91	34.68	34.50	34.29	34.17	34.00	33.89	33.70	33.55	33.44	33.28	33.11	33.00	32.86	32.77	32.60	32.52
50-54	97.5%	32.65	32.89	33.09	33.24	33.40	33.46	33.54	33.62	33.73	33.78	33.85	33.91	33.97	34.06	34.15	34.19	34.22	34.31	34.33	34.35
	mean	31.89	31.86	31.84	31.82	31.79	31.76	31.73	31.71	31.68	31.66	31.63	31.60	31.57	31.54	31.51	31.48	31.46	31.43	31.40	31.37
	2.5%	31.12	30.79	30.52	30.35	30.11	29.94	29.75	29.62	29.46	29.36	29.17	29.03	28.93	28.76	28.62	28.52	28.37	28.28	28.12	28.04
55-59	97.5%	28.14	28.36	28.56	28.70	28.85	28.92	28.98	29.05	29.16	29.23	29.28	29.34	29.39	29.46	29.56	29.60	29.62	29.72	29.72	29.74
	mean	27.43	27.41	27.39	27.37	27.35	27.32	27.29	27.27	27.24	27.22	27.20	27.17	27.15	27.12	27.09	27.06	27.04	27.01	26.99	26.96
	2.5%	26.72	26.41	26.16	26.01	25.81	25.61	25.45	25.34	25.18	25.05	24.91	24.78	24.70	24.55	24.40	24.33	24.18	24.08	23.94	23.87
60-64	97.5%	23.80	24.01	24.19	24.32	24.46	24.51	24.58	24.65	24.74	24.79	24.86	24.90	24.96	25.03	25.12	25.14	25.17	25.24	25.27	25.29
	mean	23.15	23.12	23.11	23.09	23.07	23.05	23.02	23.00	22.98	22.96	22.93	22.91	22.89	22.86	22.84	22.81	22.79	22.72	22.75	22.72
	2.5%	22.49	22.22	21.98	21.84	21.65	21.49	21.32	21.22	21.09	20.98	20.85	20.72	20.66	20.52	20.37	20.31	20.18	20.07	19.98	19.90
65-69	97.5%	19.71	19.90	20.05	20.18	20.29	20.34	20.41	20.47	20.54	20.60	20.66	20.69	20.75	20.82	20.88	20.91	20.93	21.00	21.05	21.04
	mean	19.12	19.10	19.09	19.07	19.05	19.03	19.01	18.99	18.97	18.95	18.94	18.91	18.89	18.87	18.85	18.83	18.81	18.79	18.77	18.75
	2.5%	18.53	18.30	18.09	17.95	17.81	17.66	17.53	17.43	17.31	17.22	17.07	16.97	16.93	16.79	16.67	16.62	16.50	16.42	16.32	16.22
70-74	97.5%	15.84	16.00	16.14	16.23	16.33	16.37	16.44	16.47	16.55	16.60	16.65	16.69	16.73	16.78	16.84	16.88	16.89	16.95	16.97	16.97
	mean	15.33	15.31	15.30	15.29	15.27	15.25	15.23	15.22	15.20	15.18	15.17	15.15	15.13	15.11	15.09	15.07	15.06	15.04	15.02	15.01
	2.5%	14.81	14.61	14.43	14.32	14.19	14.07	13.95	13.86	13.78	13.70	13.58	13.49	13.41	13.33	13.23	13.16	13.08	13.00	12.95	12.86
75-79	97.5%	12.23	12.36	12.47	12.56	12.63	12.67	12.72	12.75	12.81	12.86	12.90	12.95	12.96	13.01	13.06	13.10				

Table A.35. The forecasted Japan female life expectancy by model 8

age 0	97.5%	87.46	87.88	88.25	88.62	88.95	89.28	89.60	89.91	90.23	90.52	90.86	91.15	91.43	91.68	91.98	92.23	92.51	92.75	92.99	93.25
	mean	86.64	86.89	87.13	87.39	87.64	87.89	88.14	88.39	88.63	88.88	89.12	89.37	89.61	89.84	90.07	90.30	90.53	90.75	90.97	91.19
	2.5%	85.71	85.80	85.84	85.99	86.11	86.27	86.38	86.54	86.67	86.76	86.90	87.04	87.22	87.37	87.53	87.62	87.85	87.96	87.98	88.07
1-4	97.5%	86.61	87.02	87.37	87.72	88.05	88.38	88.69	88.98	89.30	89.59	89.92	90.21	90.48	90.72	91.02	91.26	91.54	91.78	92.02	92.28
	mean	85.81	86.05	86.29	86.53	86.78	87.02	87.26	87.50	87.74	87.98	88.22	88.46	88.69	88.92	89.15	89.37	89.60	89.82	90.03	90.25
	2.5%	84.90	85.00	85.03	85.18	85.28	85.44	85.56	85.71	85.82	85.93	86.04	86.16	86.36	86.51	86.65	86.75	86.96	87.07	87.09	87.18
5-9	97.5%	82.66	83.06	83.41	83.75	84.08	84.41	84.71	85.00	85.32	85.60	85.94	86.22	86.49	86.74	87.03	87.28	87.55	87.78	88.03	88.29
	mean	81.87	82.11	82.34	82.58	82.82	83.06	83.30	83.54	83.78	84.02	84.25	84.49	84.73	84.95	85.18	85.40	85.63	85.84	86.06	86.27
	2.5%	80.97	81.07	81.12	81.26	81.35	81.52	81.63	81.80	81.90	81.99	82.11	82.24	82.43	82.58	82.70	82.83	83.01	83.15	83.15	83.21
10-14	97.5%	77.71	78.10	78.45	78.78	79.11	79.43	79.73	80.03	80.33	80.62	80.95	81.24	81.50	81.75	82.04	82.29	82.56	82.79	83.03	83.29
	mean	76.92	77.16	77.39	77.63	77.86	78.10	78.34	78.58	78.81	79.05	79.28	79.52	79.75	79.98	80.21	80.42	80.65	80.86	81.08	81.29
	2.5%	76.02	76.13	76.17	76.32	76.41	76.57	76.70	76.86	76.95	77.05	77.16	77.31	77.48	77.63	77.77	77.89	78.05	78.19	78.21	78.27
15-19	97.5%	72.73	73.11	73.46	73.79	74.12	74.45	74.74	75.04	75.34	75.63	75.96	76.25	76.51	76.76	77.04	77.29	77.56	77.80	78.04	78.30
	mean	71.94	72.18	72.41	72.65	72.89	73.12	73.36	73.60	73.83	74.07	74.30	74.54	74.77	74.99	75.22	75.44	75.66	75.88	76.09	76.30
	2.5%	71.05	71.16	71.21	71.35	71.45	71.60	71.73	71.88	71.98	72.08	72.20	72.34	72.51	72.66	72.78	72.92	73.09	73.22	73.24	73.29
20-24	97.5%	67.77	68.15	68.50	68.83	69.15	69.47	69.76	70.06	70.36	70.65	70.97	71.26	71.52	71.77	72.06	72.30	72.57	72.81	73.05	73.30
	mean	66.99	67.23	67.45	67.69	67.93	68.16	68.40	68.63	68.87	69.10	69.33	69.57	69.80	70.02	70.25	70.46	70.69	70.90	71.11	71.32
	2.5%	66.11	66.22	66.26	66.40	66.51	66.66	66.79	66.94	67.04	67.12	67.25	67.39	67.56	67.71	67.84	67.98	68.14	68.27	68.26	68.34
25-29	97.5%	62.83	63.21	63.55	63.87	64.20	64.51	64.81	65.10	65.40	65.69	66.01	66.28	66.55	66.80	67.08	67.33	67.60	67.82	68.07	68.32
	mean	62.06	62.30	62.52	62.76	62.99	63.23	63.46	63.69	63.92	64.16	64.39	64.62	64.85	65.07	65.30	65.51	65.73	65.94	66.16	66.36
	2.5%	61.18	61.30	61.36	61.49	61.60	61.76	61.88	62.03	62.12	62.23	62.34	62.49	62.63	62.80	62.91	63.06	63.24	63.35	63.35	63.42
30-34	97.5%	57.91	58.28	58.62	58.93	59.25	59.57	59.85	60.14	60.44	60.73	61.05	61.32	61.58	61.82	62.11	62.36	62.62	62.84	63.09	63.34
	mean	57.15	57.38	57.61	57.84	58.07	58.30	58.53	58.76	58.99	59.23	59.45	59.68	59.91	60.13	60.35	60.56	60.79	60.99	61.20	61.41
	2.5%	56.29	56.40	56.46	56.59	56.70	56.87	56.97	57.14	57.22	57.34	57.44	57.57	57.74	57.88	58.01	58.14	58.34	58.42	58.45	58.54
35-39	97.5%	53.01	53.36	53.70	54.00	54.32	54.63	54.91	55.20	55.49	55.78	56.09	56.36	56.63	56.87	57.14	57.40	57.65	57.89	58.13	58.37
	mean	52.26	52.49	52.71	52.94	53.17	53.39	53.62	53.85	54.08	54.31	54.53	54.76	54.99	55.20	55.42	55.64	55.86	56.06	56.27	56.47
	2.5%	51.41	51.52	51.59	51.71	51.83	51.98	52.09	52.26	52.35	52.46	52.58	52.71	52.88	53.01	53.13	53.25	53.47	53.55	53.58	53.70
40-44	97.5%	48.13	48.47	48.80	49.10	49.41	49.71	49.99	50.27	50.57	50.85	51.16	51.42	51.69	51.93	52.20	52.44	52.69	52.94	53.16	53.41
	mean	47.39	47.61	47.83	48.06	48.28	48.51	48.74	48.96	49.18	49.41	49.63	49.86	50.08	50.30	50.51	50.72	50.94	51.14	51.35	51.55
	2.5%	46.55	46.65	46.72	46.87	46.97	47.13	47.24	47.41	47.50	47.62	47.75	47.87	48.03	48.15	48.27	48.40	48.60	48.68	48.74	48.83
45-49	97.5%	43.29	43.63	43.94	44.23	44.54	44.85	45.11	45.39	45.68	45.95	46.26	46.52	46.78	47.01	47.28	47.53	47.78	48.00	48.24	48.48
	mean	42.57	42.79	43.00	43.23	43.45	43.67	43.89	44.11	44.33	44.56	44.77	45.00	45.21	45.43	45.64	45.85	46.06	46.26	46.46	46.66
	2.5%	41.74	41.85	41.92	42.06	42.17	42.33	42.44	42.60	42.70	42.82	42.93	43.06	43.21	43.32	43.46	43.59	43.76	43.84	43.91	43.95
50-54	97.5%	38.52	38.85	39.16	39.44	39.73	40.03	40.29	40.56	40.84	41.11	41.41	41.66	41.91	42.16	42.40	42.66	42.90	43.12	43.35	43.59
	mean	37.82	38.04	38.24	38.46	38.68	38.89	39.11	39.33	39.54	39.76	39.97	40.19	40.40	40.61	40.82	41.02	41.23	41.42	41.62	41.82
	2.5%	37.01	37.11	37.20	37.33	37.43	37.59	37.70	37.85	37.97	38.07	38.18	38.32	38.44	38.57	38.71	38.83	38.97	39.04	39.11	39.16
55-59	97.5%	33.54	33.86	34.17	34.45	34.74	35.04	35.30	35.57	35.85	36.13	36.43	36.67	36.94	37.18	37.43	37.68	37.93	38.16	38.38	38.61
	mean	32.83	33.05	33.25	33.47	33.69	33.90	34.12	34.34	34.56	34.78	34.99	35.21	35.42	35.63	35.84	36.04	36.25	36.45	36.65	36.85
	2.5%	32.02	32.12	32.21	32.35	32.45	32.60	32.72	32.86	32.98	33.10	33.20	33.34	33.47	33.60	33.72	33.86	33.99	34.07	34.12	34.19
60-64	97.5%	28.58	28.89	29.21	29.51	29.79	30.09	30.37	30.63	30.92	31.20	31.48	31.74	32.00	32.26	32.52	32.76	32.99	33.24	33.46	33.69
	mean	27.89	28.11	28.32	28.55	28.77	28.99	29.21	29.44	29.66	29.89	30.11	30.34	30.57	30.79	31.01	31.23	31.45	31.66	31.88	32.09
	2.5%	27.14	27.24	27.37	27.51	27.66	27.81	27.93	28.12	28.27	28.37	28.54	28.70	28.88	29.07	29.28	29.43	29.64	29.79	30.00	30.13
65-69	97.5%	24.02	24.31	24.62	24.89	25.16	25.45	25.71	25.96	26.24	26.49	26.77	27.03	27.26	27.50	27.75	27.99	28.22	28.44	28.66	28.91
	mean	23.39	23.60	23.80	24.02	24.23	24.44	24.65	24.87	25.08	25.30	25.51	25.73	25.94	26.15	26.36	26.57	26.79	26.99	27.20	27.40
	2.5%	22.71	22.82	22.94	23.08	23.22	23.37	23.50	23.66	23.79	23.90	24.08	24.23	24.42	24.58	24.77	24.93	25.11	25.30	25.44	25.61
70-74	97.5%	19.68	19.95	20.22	20.46	20.70	20.97	21.20	21.43	21.69	21.93	22.17	22.42	22.63	22.85	23.08	23.29	23.51	23.72	23.93	24.16
	mean	19.12	19.31	19.49	19.68	19.88	20.07	20.26	20.46	20.65	20.85	21.04	21.24	21.44	21.63	21.82	22.02	22.21	22.40	22.59	22.78
	2.5%	18.52	18.62	18.74	18.86	18.98	19.11	19.23	19.38	19.49	19.62	19.77	19.91	20.08	20.21	20.39	20.54	20.68	20.86	20.99	21.14
75-79	97.5%	15.47	15.71	15.92	16.15	16.35	16.58	16.78	16.99	17.22	17.44	17.65	17.86	18.07	18.27	18.47	18.65	18.85	19.05	19.24	19.

Table A.36. The forecasted Japan male life expectancy by model 8

age 0	97.5%	80.91	81.41	81.81	82.24	82.64	83.04	83.39	83.75	84.13	84.54	84.91	85.31	85.68	86.05	86.42	86.77	87.16	87.51	87.89	88.21
	mean	79.97	80.23	80.49	80.76	81.03	81.31	81.59	81.87	82.15	82.43	82.71	83.00	83.29	83.58	83.87	84.16	84.46	84.75	85.04	85.34
	2.5%	78.98	79.00	79.09	79.16	79.29	79.42	79.53	79.71	79.81	80.01	80.17	80.33	80.54	80.75	80.89	81.11	81.33	81.56	81.79	81.99
1-4	97.5%	80.07	80.55	80.94	81.35	81.74	82.12	82.46	82.83	83.21	83.60	83.97	84.36	84.73	85.09	85.47	85.81	86.18	86.54	86.92	87.23
	mean	79.15	79.40	79.65	79.91	80.17	80.44	80.71	80.99	81.26	81.53	81.81	82.09	82.38	82.66	82.94	83.23	83.52	83.81	84.10	84.40
	2.5%	78.18	78.20	78.30	78.37	78.46	78.61	78.72	78.89	78.97	79.19	79.32	79.49	79.71	79.88	80.02	80.26	80.44	80.66	80.92	81.10
5-9	97.5%	76.12	76.59	76.99	77.39	77.77	78.15	78.49	78.86	79.24	79.61	79.98	80.37	80.74	81.10	81.48	81.82	82.19	82.55	82.92	83.24
	mean	75.21	75.46	75.70	75.96	76.22	76.49	76.76	77.03	77.30	77.57	77.84	78.13	78.41	78.69	78.97	79.26	79.55	79.83	80.12	80.42
	2.5%	74.25	74.27	74.37	74.44	74.53	74.69	74.79	74.96	75.04	75.26	75.39	75.55	75.78	75.94	76.08	76.32	76.51	76.73	76.95	77.14
10-14	97.5%	71.15	71.62	72.01	72.41	72.80	73.17	73.52	73.87	74.25	74.63	75.00	75.38	75.75	76.11	76.48	76.83	77.20	77.55	77.93	78.25
	mean	70.24	70.49	70.73	71.00	71.25	71.51	71.78	72.05	72.32	72.59	72.86	73.15	73.43	73.71	73.99	74.28	74.56	74.85	75.14	75.43
	2.5%	69.28	69.31	69.40	69.48	69.58	69.73	69.83	70.00	70.07	70.29	70.43	70.58	70.80	70.97	71.11	71.34	71.55	71.75	71.98	72.16
15-19	97.5%	66.18	66.65	67.04	67.44	67.82	68.19	68.54	68.89	69.26	69.65	70.02	70.40	70.76	71.13	71.49	71.84	72.21	72.56	72.94	73.26
	mean	65.28	65.53	65.77	66.03	66.28	66.54	66.81	67.08	67.35	67.62	67.89	68.17	68.45	68.73	69.01	69.30	69.58	69.87	70.16	70.45
	2.5%	64.32	64.35	64.45	64.53	64.61	64.76	64.86	65.04	65.11	65.33	65.46	65.62	65.84	66.01	66.15	66.39	66.59	66.77	67.00	67.19
20-24	97.5%	61.27	61.72	62.10	62.50	62.87	63.25	63.58	63.94	64.31	64.69	65.05	65.43	65.80	66.15	66.52	66.86	67.24	67.59	67.96	68.28
	mean	60.37	60.61	60.85	61.11	61.36	61.62	61.88	62.15	62.41	62.68	62.95	63.23	63.51	63.78	64.06	64.34	64.63	64.91	65.20	65.49
	2.5%	59.42	59.45	59.55	59.63	59.71	59.86	59.97	60.13	60.21	60.42	60.54	60.72	60.92	61.11	61.23	61.46	61.67	61.85	62.07	62.27
25-29	97.5%	56.41	56.86	57.21	57.60	57.96	58.34	58.67	59.02	59.38	59.77	60.12	60.49	60.85	61.21	61.57	61.91	62.29	62.64	63.00	63.32
	mean	55.53	55.77	56.00	56.25	56.50	56.76	57.02	57.28	57.54	57.81	58.07	58.35	58.63	58.90	59.17	59.45	59.74	60.02	60.30	60.59
	2.5%	54.60	54.63	54.73	54.82	54.91	55.05	55.18	55.34	55.43	55.63	55.76	55.92	56.13	56.31	56.43	56.65	56.86	57.06	57.25	57.47
30-34	97.5%	51.55	51.96	52.33	52.71	53.07	53.44	53.75	54.09	54.47	54.84	55.18	55.56	55.92	56.27	56.63	56.96	57.34	57.68	58.05	58.36
	mean	50.68	50.92	51.15	51.40	51.64	51.89	52.15	52.42	52.67	52.93	53.19	53.47	53.74	54.01	54.28	54.56	54.84	55.12	55.40	55.69
	2.5%	49.76	49.81	49.93	50.00	50.10	50.24	50.36	50.53	50.65	50.83	50.94	51.12	51.32	51.52	51.63	51.84	52.07	52.25	52.42	52.66
35-39	97.5%	46.70	47.10	47.46	47.82	48.17	48.53	48.85	49.20	49.57	49.92	50.27	50.63	50.99	51.34	51.69	52.03	52.40	52.74	53.10	53.41
	mean	45.84	46.08	46.30	46.55	46.79	47.04	47.29	47.55	47.81	48.07	48.32	48.59	48.87	49.13	49.40	49.68	49.96	50.23	50.51	50.79
	2.5%	44.93	44.99	45.11	45.19	45.30	45.43	45.59	45.72	45.84	46.01	46.15	46.33	46.53	46.71	46.82	47.06	47.25	47.44	47.65	47.86
40-44	97.5%	41.88	42.28	42.62	42.97	43.31	43.67	43.98	44.32	44.68	45.02	45.37	45.75	46.09	46.42	46.78	47.10	47.48	47.81	48.17	48.47
	mean	41.04	41.27	41.50	41.74	41.98	42.22	42.47	42.72	42.97	43.23	43.48	43.75	44.02	44.28	44.55	44.82	45.09	45.36	45.64	45.92
	2.5%	40.14	40.21	40.32	40.42	40.54	40.67	40.81	40.95	41.06	41.24	41.38	41.56	41.73	41.92	42.04	42.28	42.46	42.66	42.86	43.07
45-49	97.5%	37.15	37.52	37.85	38.19	38.53	38.88	39.18	39.50	39.85	40.19	40.52	40.91	41.23	41.57	41.91	42.25	42.59	42.94	43.28	43.58
	mean	36.32	36.55	36.76	37.00	37.23	37.47	37.71	37.96	38.20	38.46	38.70	38.96	39.23	39.49	39.75	40.02	40.29	40.55	40.82	41.10
	2.5%	35.44	35.53	35.64	35.73	35.85	35.96	36.11	36.27	36.37	36.54	36.68	36.87	37.04	37.21	37.34	37.56	37.75	37.93	38.13	38.33
50-54	97.5%	32.52	32.87	33.19	33.53	33.85	34.18	34.49	34.78	35.14	35.47	35.79	36.14	36.46	36.80	37.11	37.44	37.80	38.13	38.47	38.76
	mean	31.72	31.94	32.14	32.37	32.59	32.82	33.06	33.30	33.53	33.78	34.02	34.27	34.53	34.78	35.03	35.29	35.56	35.81	36.08	36.34
	2.5%	30.85	30.94	31.04	31.13	31.26	31.38	31.51	31.65	31.76	31.93	32.07	32.24	32.41	32.59	32.70	32.91	33.07	33.26	33.46	33.64
55-59	97.5%	28.05	28.39	28.69	29.01	29.32	29.64	29.91	30.24	30.56	30.88	31.16	31.52	31.83	32.16	32.47	32.78	33.11	33.42	33.78	34.05
	mean	27.30	27.50	27.70	27.92	28.14	28.35	28.58	28.81	29.04	29.28	29.51	29.75	30.00	30.24	30.49	30.74	30.99	31.24	31.49	31.75
	2.5%	26.47	26.57	26.66	26.78	26.87	26.98	27.12	27.24	27.37	27.55	27.68	27.84	28.01	28.17	28.30	28.52	28.69	28.82	29.01	29.22
60-64	97.5%	23.68	24.00	24.28	24.58	24.85	25.16	25.43	25.72	26.04	26.33	26.61	26.94	27.25	27.54	27.85	28.16	28.46	28.76	29.09	29.36
	mean	22.96	23.16	23.34	23.54	23.75	23.95	24.17	24.38	24.60	24.82	25.04	25.27	25.51	25.73	25.97	26.20	26.45	26.68	26.92	27.17
	2.5%	22.18	22.26	22.36	22.46	22.56	22.67	22.79	22.91	23.03	23.17	23.32	23.47	23.64	23.79	23.90	24.10	24.25	24.40	24.56	24.73
65-69	97.5%	19.58	19.85	20.10	20.38	20.64	20.91	21.17	21.44	21.71	21.99	22.25	22.56	22.87	23.14	23.43	23.70	24.00	24.26	24.58	24.85
	mean	18.89	19.07	19.25	19.43	19.62	19.81	20.01	20.22	20.41	20.62	20.83	21.04	21.26	21.48	21.69	21.92	22.14	22.37	22.59	22.82
	2.5%	18.16	18.26	18.37	18.45	18.58	18.67	18.79	18.89	19.00	19.12	19.28	19.41	19.56	19.70	19.86	20.02	20.15	20.29	20.47	20.60
70-74	97.5%	15.71	15.95	16.17	16.40	16.64	16.88	17.11	17.32	17.59	17.84	18.07	18.36	18.63	18.88	19.13	19.38	19.65	19.91	20.20	20.46
	mean	15.10	15.26	15.42	15.58	15.75	15.92	16.10	16.28	16.46	16.65	16.83	17.03	17.23	17.42	17.62	17.82	18.03	18.23	18.43	18.64
	2.5%	14.44	14.53	14.63	14.71	14.82	14.92	15.03	15.14	15.25	15.38	15.48	15.62	15.75	15.87	15.98	16.13	16.26	16.40	16.53	16.65
75-79	97.5%	12.14	12.32	12.51	12.73	12.92	13.10	13.30	13.49	13.70	13.93	14.13	14.36	14.59	14.80	15.02	15.24				

Table A.37. The forecasted Netherlands female life expectancy by model 3

age 0	97.5%	83.47	83.73	83.97	84.14	84.29	84.39	84.46	84.55	84.66	84.71	84.83	84.89	84.95	85.00	85.10	85.11	85.15	85.24	85.28	85.35
	mean	82.65	82.62	82.59	82.57	82.53	82.50	82.46	82.43	82.40	82.37	82.34	82.30	82.26	82.22	82.19	82.14	82.11	82.08	82.04	82.00
	2.5%	81.78	81.38	81.03	80.81	80.51	80.27	80.04	79.88	79.69	79.46	79.26	79.03	78.86	78.66	78.43	78.24	78.08	77.88	77.68	77.55
1-4	97.5%	82.69	82.93	83.15	83.32	83.46	83.56	83.62	83.70	83.80	83.87	83.98	84.04	84.10	84.14	84.23	84.24	84.29	84.37	84.42	84.47
	mean	81.93	81.89	81.87	81.85	81.82	81.79	81.75	81.73	81.70	81.67	81.64	81.61	81.58	81.54	81.51	81.47	81.44	81.41	81.38	81.34
	2.5%	81.12	80.75	80.43	80.24	79.97	79.75	79.53	79.39	79.22	79.05	78.84	78.64	78.50	78.31	78.11	77.93	77.79	77.60	77.46	77.32
5-9	97.5%	78.72	78.96	79.18	79.34	79.48	79.58	79.65	79.72	79.82	79.89	80.00	80.05	80.12	80.16	80.24	80.26	80.30	80.38	80.43	80.49
	mean	77.97	77.94	77.92	77.90	77.87	77.84	77.80	77.78	77.75	77.72	77.70	77.66	77.63	77.60	77.57	77.53	77.50	77.47	77.44	77.41
	2.5%	77.17	76.82	76.51	76.32	76.05	75.85	75.63	75.49	75.34	75.16	74.97	74.79	74.64	74.45	74.26	74.10	73.95	73.78	73.64	73.50
10-14	97.5%	73.74	73.98	74.19	74.36	74.50	74.59	74.66	74.74	74.83	74.90	75.01	75.06	75.13	75.16	75.25	75.27	75.31	75.39	75.44	75.50
	mean	73.00	72.97	72.95	72.93	72.90	72.87	72.84	72.81	72.78	72.76	72.73	72.70	72.67	72.64	72.61	72.57	72.54	72.52	72.48	72.45
	2.5%	72.21	71.87	71.56	71.37	71.11	70.92	70.70	70.58	70.41	70.23	70.05	69.87	69.73	69.56	69.38	69.23	69.08	68.91	68.77	68.65
15-19	97.5%	68.76	69.00	69.21	69.37	69.51	69.61	69.68	69.75	69.84	69.91	70.02	70.07	70.14	70.17	70.26	70.28	70.32	70.40	70.45	70.51
	mean	68.03	68.00	67.98	67.96	67.93	67.90	67.87	67.85	67.82	67.79	67.77	67.74	67.71	67.68	67.65	67.61	67.58	67.56	67.53	67.50
	2.5%	67.26	66.91	66.61	66.43	66.18	65.99	65.77	65.65	65.48	65.32	65.14	64.97	64.83	64.67	64.48	64.34	64.19	64.03	63.90	63.78
20-24	97.5%	63.80	64.03	64.24	64.40	64.53	64.63	64.70	64.77	64.87	64.94	65.04	65.09	65.16	65.20	65.28	65.30	65.34	65.41	65.46	65.53
	mean	63.08	63.05	63.03	63.01	62.99	62.96	62.93	62.91	62.88	62.86	62.83	62.80	62.77	62.74	62.72	62.68	62.66	62.63	62.60	62.58
	2.5%	62.32	61.99	61.70	61.53	61.28	61.09	60.89	60.77	60.61	60.46	60.27	60.13	59.99	59.84	59.68	59.52	59.38	59.26	59.10	59.00
25-29	97.5%	58.84	59.07	59.28	59.43	59.57	59.66	59.73	59.80	59.89	59.96	60.07	60.11	60.18	60.21	60.30	60.32	60.36	60.43	60.49	60.55
	mean	58.13	58.11	58.09	58.07	58.05	58.02	57.99	57.97	57.95	57.92	57.90	57.87	57.85	57.82	57.79	57.76	57.74	57.72	57.69	57.66
	2.5%	57.39	57.08	56.79	56.63	56.40	56.21	56.03	55.91	55.76	55.63	55.47	55.31	55.18	55.03	54.89	54.74	54.62	54.51	54.34	54.24
30-34	97.5%	53.88	54.10	54.31	54.46	54.60	54.69	54.76	54.83	54.91	54.99	55.09	55.14	55.19	55.24	55.32	55.35	55.38	55.45	55.50	55.57
	mean	53.19	53.16	53.15	53.13	53.11	53.08	53.06	53.04	53.01	52.99	52.97	52.94	52.92	52.89	52.87	52.84	52.82	52.80	52.77	52.75
	2.5%	52.47	52.17	51.89	51.73	51.51	51.33	51.15	51.04	50.90	50.77	50.61	50.48	50.35	50.21	50.08	49.93	49.83	49.70	49.58	49.47
35-39	97.5%	48.95	49.17	49.37	49.52	49.66	49.74	49.81	49.89	49.96	50.03	50.13	50.18	50.24	50.29	50.36	50.39	50.42	50.49	50.54	50.61
	mean	48.28	48.26	48.24	48.23	48.20	48.18	48.15	48.14	48.12	48.10	48.08	48.05	48.03	48.00	47.98	47.95	47.93	47.91	47.89	47.86
	2.5%	47.58	47.29	47.04	46.87	46.67	46.49	46.33	46.22	46.09	45.97	45.82	45.69	45.58	45.43	45.32	45.17	45.08	44.97	44.86	44.77
40-44	97.5%	44.06	44.27	44.47	44.61	44.73	44.82	44.89	44.96	45.04	45.10	45.21	45.25	45.31	45.35	45.43	45.45	45.49	45.55	45.60	45.67
	mean	43.41	43.39	43.37	43.36	43.34	43.31	43.29	43.27	43.25	43.23	43.22	43.19	43.17	43.15	43.13	43.10	43.08	43.06	43.04	43.02
	2.5%	42.73	42.46	42.22	42.06	41.87	41.70	41.54	41.44	41.33	41.20	41.07	40.96	40.85	40.72	40.59	40.45	40.39	40.28	40.18	40.10
45-49	97.5%	39.22	39.42	39.62	39.75	39.88	39.96	40.03	40.09	40.17	40.23	40.33	40.36	40.43	40.47	40.54	40.56	40.60	40.67	40.71	40.78
	mean	38.60	38.58	38.56	38.55	38.53	38.51	38.49	38.47	38.46	38.44	38.42	38.40	38.38	38.36	38.34	38.31	38.30	38.28	38.26	38.24
	2.5%	37.96	37.69	37.46	37.32	37.14	36.99	36.84	36.74	36.63	36.51	36.38	36.28	36.18	36.07	35.96	35.83	35.76	35.68	35.56	35.50
50-54	97.5%	34.50	34.69	34.88	35.01	35.12	35.20	35.26	35.32	35.40	35.47	35.55	35.59	35.65	35.68	35.76	35.78	35.81	35.87	35.92	35.97
	mean	33.91	33.89	33.88	33.86	33.85	33.83	33.81	33.79	33.77	33.76	33.74	33.72	33.70	33.68	33.66	33.64	33.63	33.61	33.59	33.57
	2.5%	33.29	33.04	32.83	32.70	32.53	32.39	32.25	32.15	32.06	31.95	31.84	31.75	31.64	31.53	31.44	31.32	31.24	31.17	31.08	31.01
55-59	97.5%	29.93	30.11	30.28	30.40	30.52	30.59	30.64	30.70	30.77	30.84	30.92	30.97	31.00	31.06	31.12	31.13	31.17	31.21	31.27	31.31
	mean	29.36	29.34	29.33	29.32	29.30	29.29	29.27	29.25	29.24	29.22	29.21	29.19	29.17	29.15	29.13	29.11	29.10	29.08	29.07	29.05
	2.5%	28.78	28.55	28.36	28.23	28.08	27.94	27.81	27.72	27.62	27.53	27.42	27.32	27.24	27.14	27.03	26.94	26.86	26.79	26.71	26.67
60-64	97.5%	25.49	25.67	25.82	25.93	26.03	26.11	26.16	26.21	26.28	26.33	26.41	26.46	26.50	26.53	26.61	26.63	26.65	26.68	26.73	26.79
	mean	24.96	24.94	24.93	24.92	24.90	24.89	24.87	24.86	24.84	24.83	24.81	24.79	24.78	24.76	24.74	24.72	24.71	24.70	24.68	24.66
	2.5%	24.41	24.20	24.02	23.90	23.76	23.62	23.53	23.44	23.34	23.27	23.15	23.07	22.99	22.89	22.82	22.70	22.63	22.57	22.50	22.46
65-69	97.5%	21.19	21.35	21.49	21.59	21.70	21.76	21.81	21.85	21.93	21.96	22.03	22.09	22.12	22.14	22.20	22.23	22.26	22.29	22.33	22.39
	mean	20.70	20.69	20.68	20.67	20.65	20.64	20.62	20.61	20.59	20.58	20.57	20.55	20.54	20.52	20.50	20.48	20.47	20.46	20.44	20.43
	2.5%	20.20	19.99	19.83	19.74	19.60	19.48	19.38	19.31	19.22	19.14	19.04	18.97	18.90	18.81	18.72	18.65	18.58	18.51	18.45	18.41
70-74	97.5%	17.06	17.20	17.32	17.41	17.51	17.56	17.60	17.65	17.71	17.76	17.81	17.85	17.89	17.90	17.97	17.99	18.01	18.02	18.07	18.13
	mean	16.61	16.60	16.59	16.58	16.57	16.55	16.53	16.52	16.51	16.50	16.49	16.47	16.46	16.44	16.43	16.41	16.40	16.39	16.37	16.36
	2.5%	16.16	15.98	15.84	15.74	15.62	15.52	15.44	15.36	15.28	15.23	15.13	15.05	14.99	14.90	14.83	14.76	14.70	14.65	14.59	14.55
75-79	97.5%	13.16	13.27	13.39	13.48	13.55	13.59	13.62	13.66	13.71	13.76	13.81	13.85	13.87	13.89	13.94	13.96	13.98	14.00	14.04	14.

Table A.38. The forecasted Netherlands male life expectancy by model 3

age	97.5%	80.04	80.35	80.67	80.86	81.03	81.13	81.23	81.31	81.43	81.49	81.60	81.65	81.71	81.82	81.88	81.92	81.98	82.04	82.08	82.17
	mean	78.95	78.88	78.82	78.77	78.70	78.63	78.55	78.49	78.42	78.35	78.28	78.20	78.12	78.04	77.96	77.87	77.80	77.72	77.64	77.56
	2.5%	77.75	77.10	76.61	76.25	75.80	75.38	75.02	74.67	74.28	73.93	73.55	73.13	72.84	72.46	72.00	71.64	71.26	70.90	70.57	70.25
1-4	97.5%	79.31	79.58	79.89	80.06	80.23	80.34	80.42	80.51	80.61	80.68	80.77	80.82	80.87	80.98	81.03	81.07	81.14	81.17	81.23	81.31
	mean	78.27	78.20	78.14	78.09	78.03	77.96	77.88	77.83	77.76	77.69	77.63	77.55	77.48	77.40	77.32	77.24	77.17	77.10	77.02	76.94
	2.5%	77.14	76.52	76.06	75.74	75.30	74.92	74.58	74.26	73.87	73.55	73.17	72.75	72.50	72.15	71.73	71.36	71.00	70.70	70.40	70.09
5-9	97.5%	75.35	75.63	75.93	76.10	76.26	76.37	76.45	76.54	76.65	76.70	76.80	76.84	76.90	77.00	77.05	77.11	77.16	77.19	77.25	77.33
	mean	74.33	74.26	74.21	74.16	74.09	74.03	73.95	73.90	73.83	73.77	73.70	73.63	73.55	73.48	73.40	73.32	73.25	73.18	73.10	73.03
	2.5%	73.22	72.61	72.16	71.85	71.42	71.04	70.70	70.40	70.01	69.72	69.33	68.92	68.70	68.31	67.91	67.55	67.23	66.92	66.62	66.34
10-14	97.5%	70.39	70.65	70.95	71.12	71.29	71.39	71.46	71.55	71.67	71.72	71.81	71.86	71.92	72.02	72.07	72.12	72.18	72.21	72.26	72.34
	mean	69.37	69.31	69.25	69.20	69.14	69.07	69.00	68.94	68.88	68.82	68.75	68.68	68.61	68.53	68.46	68.38	68.31	68.24	68.16	68.09
	2.5%	68.27	67.67	67.24	66.92	66.51	66.13	65.79	65.50	65.12	64.81	64.44	64.05	63.83	63.46	63.08	62.69	62.36	62.06	61.80	61.48
15-19	97.5%	65.41	65.68	65.97	66.15	66.31	66.41	66.48	66.57	66.68	66.74	66.83	66.87	66.93	67.03	67.08	67.14	67.20	67.22	67.28	67.36
	mean	64.41	64.35	64.29	64.24	64.18	64.12	64.05	63.99	63.93	63.86	63.80	63.73	63.66	63.58	63.51	63.43	63.36	63.30	63.22	63.15
	2.5%	63.33	62.74	62.31	61.99	61.59	61.21	60.88	60.60	60.21	59.92	59.55	59.18	58.99	58.60	58.21	57.84	57.51	57.25	56.96	56.69
20-24	97.5%	60.47	60.73	61.02	61.19	61.34	61.45	61.52	61.61	61.72	61.77	61.86	61.91	61.97	62.06	62.11	62.16	62.23	62.25	62.31	62.39
	mean	59.49	59.43	59.38	59.33	59.27	59.21	59.14	59.09	59.02	58.97	58.90	58.84	58.77	58.70	58.63	58.55	58.49	58.42	58.35	58.28
	2.5%	58.44	57.87	57.46	57.15	56.76	56.38	56.07	55.82	55.42	55.16	54.80	54.46	54.24	53.90	53.51	53.17	52.87	52.57	52.33	52.04
25-29	97.5%	55.56	55.81	56.09	56.25	56.40	56.51	56.58	56.66	56.77	56.82	56.90	56.95	57.01	57.12	57.16	57.21	57.26	57.30	57.36	57.43
	mean	54.63	54.57	54.53	54.48	54.43	54.38	54.32	54.27	54.22	54.17	54.11	54.05	54.00	53.93	53.87	53.81	53.75	53.70	53.64	53.58
	2.5%	53.64	53.11	52.74	52.48	52.13	51.79	51.54	51.31	50.96	50.69	50.44	50.11	49.94	49.62	49.31	49.06	48.80	48.51	48.32	48.12
30-34	97.5%	50.64	50.88	51.15	51.31	51.46	51.55	51.62	51.71	51.82	51.87	51.95	51.99	52.05	52.15	52.20	52.24	52.30	52.34	52.40	52.47
	mean	49.75	49.70	49.66	49.63	49.58	49.53	49.48	49.44	49.39	49.35	49.30	49.25	49.20	49.15	49.10	49.04	48.99	48.95	48.89	48.84
	2.5%	48.83	48.35	48.01	47.76	47.46	47.15	46.93	46.72	46.45	46.21	46.00	45.70	45.57	45.31	45.06	44.83	44.64	44.39	44.26	44.08
35-39	97.5%	45.74	45.97	46.23	46.39	46.52	46.62	46.69	46.77	46.87	46.92	47.01	47.05	47.11	47.20	47.24	47.29	47.35	47.38	47.46	47.51
	mean	44.90	44.86	44.82	44.79	44.75	44.71	44.67	44.63	44.59	44.55	44.51	44.47	44.43	44.38	44.34	44.29	44.26	44.22	44.17	44.13
	2.5%	44.03	43.61	43.30	43.08	42.81	42.55	42.35	42.17	41.93	41.76	41.56	41.33	41.22	40.98	40.80	40.65	40.46	40.28	40.13	39.99
40-44	97.5%	40.85	41.08	41.32	41.48	41.60	41.70	41.77	41.84	41.94	41.99	42.06	42.13	42.18	42.26	42.31	42.35	42.42	42.46	42.52	42.57
	mean	40.06	40.02	39.99	39.96	39.93	39.89	39.85	39.82	39.78	39.75	39.72	39.68	39.64	39.60	39.57	39.52	39.49	39.46	39.42	39.39
	2.5%	39.24	38.86	38.57	38.37	38.14	37.90	37.72	37.56	37.34	37.20	37.02	36.82	36.76	36.50	36.38	36.24	36.10	35.95	35.81	35.69
45-49	97.5%	36.03	36.24	36.47	36.61	36.73	36.83	36.90	36.98	37.06	37.10	37.18	37.24	37.30	37.37	37.43	37.46	37.52	37.56	37.63	37.67
	mean	35.28	35.25	35.23	35.20	35.17	35.14	35.11	35.08	35.05	35.02	34.99	34.96	34.93	34.90	34.87	34.83	34.80	34.78	34.75	34.72
	2.5%	34.52	34.18	33.93	33.75	33.56	33.35	33.18	33.06	32.87	32.73	32.58	32.42	32.37	32.18	32.07	31.94	31.81	31.70	31.62	31.50
50-54	97.5%	31.31	31.52	31.73	31.86	31.97	32.07	32.13	32.19	32.30	32.33	32.41	32.46	32.51	32.58	32.63	32.68	32.74	32.75	32.82	32.87
	mean	30.62	30.59	30.57	30.54	30.52	30.49	30.46	30.43	30.41	30.38	30.35	30.32	30.30	30.27	30.24	30.21	30.18	30.16	30.13	30.10
	2.5%	29.91	29.60	29.36	29.20	29.03	28.84	28.67	28.57	28.39	28.29	28.14	28.00	27.96	27.79	27.67	27.56	27.43	27.36	27.27	27.18
55-59	97.5%	26.76	26.96	27.15	27.28	27.38	27.46	27.53	27.58	27.67	27.72	27.79	27.83	27.87	27.94	27.99	28.03	28.08	28.12	28.16	28.20
	mean	26.11	26.09	26.06	26.04	26.02	25.99	25.96	25.94	25.91	25.89	25.87	25.84	25.81	25.79	25.76	25.73	25.71	25.69	25.66	25.63
	2.5%	25.46	25.17	24.96	24.81	24.63	24.47	24.32	24.22	24.07	23.97	23.82	23.71	23.63	23.51	23.39	23.30	23.19	23.08	23.02	22.96
60-64	97.5%	22.42	22.60	22.78	22.91	23.01	23.07	23.13	23.18	23.25	23.29	23.35	23.39	23.43	23.50	23.54	23.58	23.64	23.65	23.71	23.75
	mean	21.82	21.80	21.78	21.76	21.74	21.71	21.68	21.66	21.64	21.62	21.60	21.57	21.55	21.52	21.50	21.47	21.45	21.43	21.41	21.38
	2.5%	21.22	20.96	20.77	20.64	20.47	20.33	20.18	20.09	19.95	19.87	19.73	19.62	19.56	19.45	19.34	19.27	19.15	19.08	19.02	18.94
65-69	97.5%	18.31	18.47	18.62	18.74	18.83	18.88	18.94	18.98	19.03	19.08	19.13	19.17	19.21	19.27	19.31	19.34	19.39	19.41	19.44	19.50
	mean	17.76	17.74	17.72	17.70	17.68	17.66	17.63	17.61	17.59	17.57	17.55	17.53	17.51	17.48	17.46	17.44	17.42	17.40	17.38	17.36
	2.5%	17.21	16.98	16.80	16.69	16.56	16.43	16.30	16.21	16.09	16.01	15.89	15.79	15.74	15.66	15.57	15.49	15.41	15.32	15.27	15.17
70-74	97.5%	14.47	14.61	14.75	14.84	14.93	14.97	15.01	15.06	15.11	15.14	15.20	15.22	15.25	15.31	15.34	15.37	15.38	15.43	15.45	15.49
	mean	14.00	13.97	13.96	13.94	13.92	13.90	13.88	13.86	13.84	13.83	13.81	13.79	13.77	13.75	13.73	13.71	13.69	13.67	13.65	13.64
	2.5%	13.52	13.32	13.17	13.07	12.95	12.84	12.73	12.64	12.55	12.47	12.38	12.30	12.24	12.17	12.10	12.01	11.96	11.90	11.84	11.77
75-79	97.5%	11.00	11.12	11.22	11.30	11.38	11.42	11.44	11.48	11.52	11.56	11.60	11.63	11.65	11.70	11.73	11.74				

Table A.39. The forecasted Netherlands female life expectancy by model 8

age 0	97.5%	83.73	84.02	84.30	84.56	84.78	85.01	85.21	85.42	85.62	85.81	86.02	86.23	86.43	86.60	86.80	86.99	87.15	87.36	87.53	87.69
	mean	82.95	83.07	83.19	83.32	83.44	83.56	83.69	83.81	83.94	84.07	84.19	84.32	84.44	84.56	84.68	84.80	84.93	85.05	85.18	85.30
	2.5%	82.12	82.05	81.97	81.97	81.94	81.94	81.96	81.99	81.99	82.04	82.03	82.08	82.15	82.20	82.25	82.28	82.39	82.44	82.45	82.50
1-4	97.5%	82.95	83.22	83.48	83.72	83.95	84.15	84.36	84.55	84.75	84.93	85.14	85.33	85.53	85.69	85.89	86.08	86.23	86.43	86.59	86.76
	mean	82.21	82.33	82.44	82.57	82.68	82.80	82.91	83.03	83.15	83.27	83.39	83.51	83.63	83.75	83.86	83.98	84.10	84.22	84.34	84.45
	2.5%	81.44	81.37	81.30	81.30	81.28	81.28	81.28	81.32	81.33	81.37	81.37	81.45	81.49	81.53	81.55	81.60	81.71	81.75	81.77	81.82
5-9	97.5%	78.98	79.25	79.50	79.75	79.97	80.17	80.38	80.56	80.76	80.94	81.15	81.34	81.54	81.70	81.90	82.08	82.24	82.44	82.59	82.76
	mean	78.25	78.36	78.48	78.60	78.71	78.82	78.94	79.06	79.18	79.30	79.41	79.53	79.65	79.77	79.88	80.00	80.12	80.23	80.35	80.47
	2.5%	77.48	77.41	77.35	77.35	77.34	77.34	77.35	77.38	77.37	77.41	77.42	77.49	77.54	77.58	77.60	77.68	77.77	77.80	77.81	77.86
10-14	97.5%	73.99	74.26	74.51	74.76	74.98	75.18	75.39	75.57	75.77	75.95	76.15	76.35	76.54	76.71	76.90	77.09	77.24	77.44	77.59	77.76
	mean	73.27	73.38	73.50	73.62	73.73	73.84	73.96	74.08	74.19	74.31	74.43	74.55	74.67	74.78	74.89	75.01	75.13	75.25	75.37	75.48
	2.5%	72.51	72.44	72.37	72.38	72.37	72.37	72.38	72.40	72.42	72.45	72.45	72.53	72.57	72.62	72.64	72.70	72.80	72.83	72.85	72.89
15-19	97.5%	69.01	69.28	69.53	69.77	69.99	70.20	70.40	70.58	70.77	70.95	71.16	71.36	71.55	71.72	71.91	72.10	72.24	72.45	72.60	72.76
	mean	68.30	68.40	68.52	68.64	68.75	68.87	68.98	69.10	69.21	69.33	69.45	69.57	69.68	69.80	69.91	70.03	70.15	70.26	70.38	70.50
	2.5%	67.54	67.48	67.41	67.42	67.41	67.41	67.41	67.45	67.46	67.48	67.49	67.56	67.61	67.65	67.67	67.74	67.84	67.87	67.89	67.93
20-24	97.5%	64.06	64.32	64.56	64.80	65.02	65.22	65.42	65.60	65.79	65.97	66.17	66.38	66.56	66.73	66.92	67.11	67.26	67.45	67.61	67.78
	mean	63.35	63.46	63.57	63.69	63.80	63.92	64.03	64.15	64.26	64.38	64.50	64.62	64.73	64.85	64.96	65.08	65.20	65.31	65.43	65.54
	2.5%	62.60	62.55	62.50	62.50	62.49	62.49	62.50	62.54	62.58	62.61	62.60	62.66	62.72	62.77	62.82	62.88	62.98	63.02	63.03	63.07
25-29	97.5%	59.11	59.36	59.60	59.84	60.05	60.25	60.46	60.63	60.82	61.00	61.19	61.40	61.58	61.75	61.94	62.12	62.28	62.47	62.63	62.79
	mean	58.41	58.52	58.64	58.76	58.86	58.98	59.09	59.21	59.32	59.44	59.56	59.68	59.79	59.90	60.01	60.13	60.25	60.37	60.48	60.60
	2.5%	57.69	57.64	57.59	57.60	57.59	57.62	57.62	57.66	57.70	57.74	57.75	57.81	57.84	57.90	57.97	58.04	58.14	58.19	58.18	58.24
30-34	97.5%	54.15	54.41	54.64	54.88	55.09	55.28	55.49	55.66	55.85	56.02	56.22	56.42	56.60	56.77	56.96	57.14	57.29	57.48	57.65	57.80
	mean	53.47	53.58	53.69	53.81	53.91	54.03	54.14	54.26	54.37	54.49	54.60	54.72	54.83	54.95	55.05	55.17	55.29	55.41	55.52	55.64
	2.5%	52.75	52.72	52.67	52.68	52.67	52.68	52.71	52.75	52.78	52.81	52.82	52.88	52.92	52.99	53.05	53.12	53.20	53.26	53.26	53.33
35-39	97.5%	49.23	49.48	49.71	49.93	50.14	50.33	50.53	50.70	50.88	51.06	51.26	51.45	51.63	51.80	51.98	52.17	52.32	52.51	52.67	52.82
	mean	48.55	48.66	48.77	48.89	48.99	49.10	49.22	49.33	49.44	49.56	49.67	49.78	49.90	50.01	50.11	50.23	50.35	50.46	50.58	50.69
	2.5%	47.85	47.82	47.78	47.78	47.77	47.79	47.82	47.85	47.88	47.94	47.95	48.00	48.04	48.11	48.16	48.22	48.32	48.37	48.38	48.44
40-44	97.5%	44.33	44.58	44.79	45.01	45.23	45.40	45.61	45.76	45.96	46.12	46.31	46.51	46.68	46.86	47.03	47.22	47.36	47.55	47.71	47.87
	mean	43.67	43.78	43.89	44.00	44.10	44.21	44.32	44.43	44.54	44.66	44.77	44.88	44.99	45.10	45.21	45.32	45.44	45.55	45.66	45.77
	2.5%	42.98	42.97	42.92	42.92	42.92	42.94	42.97	43.01	43.01	43.10	43.11	43.16	43.21	43.26	43.33	43.38	43.47	43.50	43.51	43.60
45-49	97.5%	39.49	39.73	39.94	40.16	40.35	40.53	40.73	40.89	41.07	41.23	41.42	41.60	41.78	41.94	42.11	42.30	42.45	42.62	42.79	42.96
	mean	38.85	38.96	39.06	39.17	39.27	39.38	39.49	39.59	39.70	39.81	39.92	40.03	40.14	40.25	40.35	40.46	40.58	40.68	40.79	40.90
	2.5%	38.18	38.17	38.13	38.13	38.13	38.15	38.19	38.23	38.23	38.29	38.33	38.38	38.43	38.49	38.53	38.59	38.68	38.72	38.74	38.82
50-54	97.5%	34.77	35.00	35.19	35.40	35.59	35.76	35.94	36.10	36.26	36.42	36.61	36.79	36.95	37.09	37.27	37.44	37.61	37.76	37.94	38.07
	mean	34.16	34.25	34.35	34.46	34.55	34.66	34.76	34.86	34.96	35.07	35.17	35.28	35.38	35.49	35.59	35.69	35.80	35.91	36.01	36.12
	2.5%	33.50	33.50	33.47	33.47	33.48	33.50	33.54	33.56	33.58	33.64	33.67	33.73	33.78	33.82	33.88	33.94	34.00	34.06	34.07	34.16
55-59	97.5%	30.32	30.54	30.73	30.94	31.14	31.30	31.49	31.66	31.85	31.99	32.19	32.35	32.54	32.69	32.87	33.04	33.19	33.37	33.54	33.67
	mean	29.73	29.83	29.94	30.04	30.14	30.25	30.36	30.46	30.57	30.68	30.78	30.89	31.00	31.11	31.21	31.32	31.43	31.53	31.64	31.75
	2.5%	29.13	29.11	29.10	29.12	29.11	29.14	29.18	29.21	29.23	29.29	29.33	29.37	29.43	29.49	29.53	29.59	29.67	29.70	29.71	29.81
60-64	97.5%	25.85	26.07	26.26	26.47	26.65	26.80	27.00	27.15	27.32	27.46	27.66	27.82	27.99	28.13	28.30	28.45	28.61	28.79	28.92	29.05
	mean	25.29	25.39	25.49	25.59	25.68	25.78	25.88	25.99	26.09	26.19	26.29	26.39	26.49	26.59	26.69	26.80	26.90	27.00	27.10	27.20
	2.5%	24.70	24.69	24.68	24.69	24.68	24.70	24.72	24.75	24.79	24.83	24.88	24.89	24.97	25.00	25.05	25.12	25.19	25.21	25.24	25.32
65-69	97.5%	21.52	21.71	21.89	22.09	22.26	22.42	22.57	22.73	22.88	23.02	23.20	23.35	23.52	23.64	23.80	23.95	24.08	24.24	24.36	24.49
	mean	20.96	21.05	21.14	21.24	21.32	21.41	21.51	21.60	21.69	21.79	21.88	21.97	22.07	22.16	22.25	22.34	22.44	22.53	22.62	22.71
	2.5%	20.41	20.38	20.36	20.36	20.35	20.37	20.38	20.41	20.46	20.48	20.50	20.54	20.59	20.62	20.67	20.72	20.77	20.79	20.82	20.88
70-74	97.5%	17.30	17.48	17.64	17.81	17.96	18.11	18.25	18.41	18.53	18.67	18.81	18.95	19.08	19.21	19.37	19.51	19.60	19.73	19.86	19.98
	mean	16.76	16.84	16.91	16.99	17.06	17.14	17.22	17.30	17.37	17.45	17.53	17.61	17.69	17.77	17.84	17.93	18.01	18.08	18.16	18.24
	2.5%	16.23	16.19	16.16	16.14	16.13	16.13	16.12	16.15	16.17	16.18	16.20	16.21	16.26	16.26	16.30	16.34	16.37	16.41	16.42	16.46
75-79	97.5%	13.34	13.49	13.62	13.79	13.89	14.04	14.15	14.26	14.38	14.49	14.62	14.72	14.85	14.96	15.09	15.18	15.28	15.41	15.49	15.

Table A.40. The forecasted Netherlands male life expectancy by model 8

age 0	97.5%	79.75	80.05	80.31	80.59	80.81	80.96	81.10	81.23	81.42	81.57	81.74	81.92	82.12	82.14	82.35	82.47	82.57	82.71	82.80	82.87
	mean	78.48	78.51	78.52	78.57	78.59	78.63	78.67	78.72	78.75	78.80	78.84	78.88	78.94	78.98	79.02	79.06	79.12	79.16	79.21	79.27
	2.5%	76.96	76.73	76.47	76.24	76.12	75.97	75.83	75.81	75.66	75.69	75.53	75.49	75.43	75.31	75.24	75.20	75.20	75.13	75.12	75.05
1-4	97.5%	79.02	79.28	79.53	79.79	80.00	80.15	80.25	80.40	80.57	80.70	80.88	81.02	81.21	81.24	81.44	81.57	81.66	81.79	81.88	81.94
	mean	77.78	77.81	77.80	77.84	77.86	77.89	77.91	77.96	77.98	78.03	78.06	78.10	78.14	78.18	78.21	78.25	78.30	78.34	78.38	78.43
	2.5%	76.31	76.06	75.81	75.62	75.49	75.36	75.22	75.20	75.05	75.01	74.88	74.85	74.77	74.72	74.58	74.57	74.54	74.49	74.42	74.36
5-9	97.5%	75.05	75.32	75.56	75.81	76.02	76.17	76.27	76.42	76.59	76.72	76.89	77.03	77.23	77.26	77.46	77.58	77.67	77.80	77.89	77.95
	mean	73.84	73.85	73.85	73.89	73.90	73.93	73.95	74.00	74.02	74.06	74.10	74.13	74.17	74.21	74.24	74.28	74.33	74.37	74.41	74.46
	2.5%	72.37	72.13	71.89	71.69	71.57	71.43	71.27	71.28	71.10	71.07	70.96	70.92	70.85	70.76	70.63	70.64	70.62	70.56	70.48	70.44
10-14	97.5%	70.08	70.34	70.59	70.83	71.04	71.18	71.28	71.44	71.60	71.73	71.91	72.04	72.24	72.27	72.46	72.59	72.67	72.80	72.89	72.95
	mean	68.87	68.89	68.88	68.92	68.93	68.95	68.98	69.02	69.04	69.09	69.12	69.15	69.20	69.23	69.26	69.30	69.35	69.38	69.43	69.47
	2.5%	67.41	67.17	66.93	66.73	66.61	66.47	66.32	66.34	66.15	66.12	66.00	65.97	65.91	65.82	65.69	65.69	65.65	65.60	65.54	65.47
15-19	97.5%	65.12	65.36	65.61	65.85	66.06	66.20	66.30	66.45	66.61	66.74	66.92	67.05	67.25	67.27	67.48	67.59	67.68	67.81	67.90	67.96
	mean	63.90	63.92	63.92	63.95	63.96	63.99	64.01	64.05	64.07	64.12	64.15	64.18	64.22	64.25	64.29	64.32	64.37	64.41	64.45	64.50
	2.5%	62.45	62.22	61.98	61.78	61.67	61.53	61.37	61.39	61.21	61.17	61.05	61.02	60.98	60.86	60.74	60.76	60.70	60.64	60.59	60.54
20-24	97.5%	60.18	60.42	60.65	60.90	61.09	61.24	61.33	61.47	61.64	61.77	61.95	62.07	62.28	62.31	62.50	62.62	62.70	62.83	62.92	62.97
	mean	58.98	59.00	58.99	59.02	59.04	59.06	59.09	59.12	59.14	59.19	59.22	59.25	59.29	59.32	59.36	59.39	59.44	59.48	59.52	59.57
	2.5%	57.54	57.31	57.09	56.89	56.79	56.65	56.52	56.51	56.36	56.30	56.23	56.18	56.13	56.03	55.93	55.90	55.87	55.84	55.81	55.75
25-29	97.5%	55.32	55.59	55.80	56.05	56.25	56.40	56.51	56.65	56.80	56.97	57.13	57.26	57.45	57.52	57.68	57.81	57.91	58.04	58.13	58.19
	mean	54.18	54.21	54.22	54.26	54.28	54.31	54.34	54.38	54.41	54.46	54.49	54.52	54.57	54.60	54.64	54.67	54.72	54.76	54.80	54.85
	2.5%	52.77	52.60	52.39	52.23	52.14	51.98	51.88	51.87	51.75	51.68	51.60	51.56	51.50	51.40	51.32	51.29	51.27	51.23	51.17	51.10
30-34	97.5%	50.41	50.67	50.88	51.12	51.31	51.47	51.57	51.71	51.86	52.02	52.18	52.31	52.51	52.56	52.74	52.85	52.95	53.08	53.18	53.24
	mean	49.29	49.32	49.32	49.36	49.38	49.42	49.44	49.49	49.51	49.56	49.59	49.62	49.67	49.70	49.74	49.77	49.82	49.86	49.90	49.94
	2.5%	47.90	47.72	47.53	47.37	47.27	47.17	47.07	47.06	46.95	46.86	46.78	46.74	46.69	46.58	46.52	46.49	46.48	46.48	46.39	46.34
35-39	97.5%	45.58	45.83	46.03	46.27	46.47	46.63	46.76	46.90	47.05	47.20	47.38	47.52	47.69	47.74	47.96	48.05	48.15	48.30	48.38	48.47
	mean	44.48	44.52	44.54	44.58	44.61	44.65	44.68	44.73	44.76	44.81	44.84	44.88	44.92	44.96	45.00	45.04	45.09	45.13	45.17	45.22
	2.5%	43.16	42.99	42.83	42.71	42.60	42.49	42.39	42.37	42.25	42.20	42.13	42.09	42.04	41.96	41.87	41.84	41.82	41.79	41.75	41.70
40-44	97.5%	40.71	40.97	41.16	41.39	41.58	41.74	41.86	41.99	42.16	42.29	42.46	42.60	42.80	42.83	43.04	43.13	43.23	43.39	43.46	43.55
	mean	39.64	39.68	39.70	39.75	39.77	39.81	39.84	39.89	39.92	39.97	40.00	40.04	40.08	40.12	40.16	40.19	40.24	40.28	40.32	40.37
	2.5%	38.35	38.20	38.03	37.91	37.82	37.71	37.64	37.60	37.51	37.47	37.40	37.35	37.27	37.24	37.18	37.12	37.08	37.08	37.06	37.00
45-49	97.5%	35.90	36.14	36.33	36.55	36.72	36.89	37.00	37.14	37.30	37.42	37.59	37.71	37.90	37.96	38.14	38.26	38.33	38.47	38.56	38.65
	mean	34.86	34.90	34.92	34.96	34.99	35.02	35.05	35.10	35.12	35.17	35.21	35.24	35.28	35.32	35.36	35.39	35.44	35.48	35.52	35.56
	2.5%	33.59	33.45	33.31	33.17	33.09	33.00	32.93	32.92	32.84	32.77	32.70	32.67	32.61	32.55	32.50	32.44	32.42	32.38	32.38	32.35
50-54	97.5%	31.21	31.44	31.63	31.84	32.00	32.14	32.28	32.42	32.57	32.67	32.83	32.95	33.14	33.21	33.39	33.49	33.53	33.69	33.77	33.88
	mean	30.22	30.27	30.28	30.33	30.36	30.40	30.43	30.47	30.50	30.55	30.58	30.61	30.66	30.69	30.73	30.76	30.81	30.85	30.89	30.94
	2.5%	28.98	28.89	28.76	28.65	28.56	28.49	28.44	28.40	28.33	28.25	28.21	28.19	28.14	28.09	28.05	28.00	27.96	27.92	27.92	27.87
55-59	97.5%	26.72	26.93	27.10	27.30	27.46	27.60	27.73	27.88	27.99	28.08	28.24	28.34	28.50	28.59	28.77	28.86	28.92	29.06	29.14	29.21
	mean	25.77	25.81	25.83	25.87	25.90	25.93	25.96	26.00	26.03	26.07	26.11	26.13	26.18	26.21	26.24	26.27	26.32	26.35	26.39	26.43
	2.5%	24.54	24.50	24.39	24.26	24.20	24.11	24.06	24.04	23.98	23.90	23.87	23.85	23.79	23.75	23.68	23.65	23.63	23.63	23.60	23.54
60-64	97.5%	22.46	22.64	22.78	22.98	23.13	23.26	23.35	23.49	23.62	23.68	23.83	23.95	24.07	24.18	24.33	24.42	24.49	24.61	24.66	24.72
	mean	21.55	21.59	21.60	21.64	21.66	21.69	21.72	21.75	21.78	21.82	21.85	21.88	21.91	21.95	21.97	22.01	22.04	22.08	22.11	22.15
	2.5%	20.37	20.33	20.24	20.11	20.08	20.02	19.96	19.91	19.88	19.82	19.77	19.76	19.73	19.69	19.60	19.58	19.55	19.57	19.51	19.49
65-69	97.5%	18.39	18.55	18.70	18.84	18.97	19.06	19.17	19.28	19.36	19.44	19.56	19.67	19.75	19.85	19.94	20.04	20.09	20.21	20.25	20.33
	mean	17.53	17.55	17.56	17.58	17.60	17.62	17.64	17.66	17.68	17.70	17.72	17.74	17.77	17.79	17.81	17.83	17.86	17.88	17.90	17.93
	2.5%	16.34	16.36	16.25	16.15	16.14	16.06	16.00	15.95	15.92	15.85	15.81	15.78	15.74	15.71	15.71	15.62	15.62	15.58	15.55	15.48
70-74	97.5%	14.63	14.78	14.87	15.01	15.12	15.21	15.27	15.39	15.48	15.54	15.63	15.71	15.81	15.86	15.97	16.07	16.09	16.18	16.23	16.28
	mean	13.86	13.88	13.88	13.90	13.91	13.93	13.94	13.96	13.97	14.00	14.01	14.03	14.05	14.07	14.08	14.10	14.12	14.14	14.16	14.18
	2.5%	12.96	12.88	12.77	12.72	12.64	12.59	12.53	12.49	12.43	12.40	12.34	12.31	12.29	12.24	12.21	12.17	12.13	12.10	12.11	12.03
75-79	97.5%	11.18	11.27	11.34	11.43	11.46	11.53	11.57	11.61	11.65	11.67	11.74	11.80	11.83	11.88	11.93	11.				

Table A.41. The forecasted Norway female life expectancy by model 3

age 0	97.5%	84.28	84.48	84.67	84.80	84.92	84.98	85.05	85.12	85.19	85.25	85.32	85.37	85.42	85.48	85.57	85.59	85.60	85.69	85.71	85.76
	mean	83.67	83.64	83.62	83.61	83.59	83.56	83.54	83.52	83.49	83.47	83.45	83.43	83.40	83.38	83.35	83.32	83.30	83.28	83.25	83.23
	2.5%	83.02	82.70	82.47	82.30	82.09	81.93	81.73	81.65	81.47	81.32	81.18	81.05	80.91	80.77	80.63	80.52	80.37	80.22	80.16	80.00
1-4	97.5%	83.44	83.64	83.82	83.94	84.05	84.11	84.18	84.25	84.32	84.37	84.44	84.49	84.54	84.60	84.68	84.70	84.71	84.79	84.81	84.86
	mean	82.86	82.84	82.82	82.81	82.79	82.76	82.74	82.72	82.70	82.68	82.66	82.63	82.61	82.59	82.56	82.54	82.52	82.49	82.47	82.45
	2.5%	82.25	81.94	81.72	81.57	81.37	81.22	81.04	80.96	80.79	80.66	80.52	80.39	80.27	80.13	80.01	79.91	79.78	79.63	79.55	79.45
5-9	97.5%	79.47	79.66	79.84	79.96	80.07	80.13	80.20	80.27	80.34	80.38	80.45	80.50	80.55	80.61	80.69	80.72	80.72	80.80	80.82	80.88
	mean	78.89	78.87	78.86	78.84	78.82	78.80	78.78	78.76	78.74	78.72	78.70	78.68	78.65	78.63	78.61	78.58	78.56	78.54	78.52	78.49
	2.5%	78.29	77.99	77.78	77.63	77.44	77.29	77.12	77.03	76.87	76.73	76.61	76.48	76.37	76.25	76.12	76.02	75.88	75.74	75.67	75.57
10-14	97.5%	74.49	74.68	74.85	74.98	75.09	75.14	75.21	75.28	75.35	75.40	75.46	75.51	75.56	75.62	75.70	75.72	75.73	75.81	75.83	75.88
	mean	73.92	73.90	73.88	73.87	73.85	73.83	73.80	73.79	73.77	73.75	73.73	73.71	73.69	73.66	73.64	73.62	73.60	73.58	73.55	73.53
	2.5%	73.33	73.03	72.82	72.67	72.49	72.34	72.17	72.09	71.94	71.80	71.69	71.56	71.45	71.34	71.21	71.10	70.97	70.83	70.77	70.67
15-19	97.5%	69.51	69.69	69.87	70.00	70.10	70.15	70.23	70.29	70.36	70.41	70.48	70.53	70.58	70.63	70.72	70.73	70.74	70.82	70.84	70.90
	mean	68.95	68.92	68.91	68.90	68.88	68.86	68.83	68.82	68.80	68.78	68.76	68.74	68.72	68.69	68.67	68.65	68.63	68.61	68.59	68.57
	2.5%	68.36	68.07	67.86	67.71	67.53	67.39	67.23	67.15	67.00	66.86	66.74	66.62	66.51	66.40	66.29	66.18	66.05	65.92	65.84	65.75
20-24	97.5%	64.55	64.74	64.91	65.03	65.14	65.19	65.26	65.32	65.39	65.44	65.50	65.55	65.60	65.66	65.74	65.76	65.77	65.85	65.87	65.92
	mean	64.00	63.98	63.97	63.96	63.94	63.92	63.90	63.88	63.86	63.85	63.83	63.81	63.79	63.76	63.74	63.72	63.70	63.69	63.66	63.64
	2.5%	63.43	63.15	62.95	62.81	62.63	62.50	62.35	62.27	62.12	61.99	61.88	61.77	61.66	61.55	61.44	61.34	61.24	61.11	61.03	60.95
25-29	97.5%	59.60	59.78	59.95	60.07	60.18	60.22	60.30	60.36	60.42	60.46	60.53	60.59	60.64	60.69	60.76	60.79	60.79	60.87	60.89	60.95
	mean	59.07	59.05	59.04	59.02	59.01	58.99	58.97	58.95	58.94	58.92	58.91	58.89	58.87	58.85	58.83	58.81	58.79	58.77	58.75	58.74
	2.5%	58.51	58.25	58.06	57.92	57.75	57.63	57.49	57.42	57.28	57.15	57.04	56.95	56.84	56.73	56.64	56.55	56.45	56.33	56.26	56.19
30-34	97.5%	54.65	54.82	54.99	55.10	55.21	55.26	55.33	55.39	55.45	55.49	55.56	55.61	55.66	55.71	55.79	55.81	55.82	55.90	55.92	55.97
	mean	54.13	54.11	54.10	54.09	54.07	54.05	54.03	54.02	54.00	53.99	53.97	53.96	53.94	53.92	53.90	53.88	53.87	53.85	53.83	53.82
	2.5%	53.59	53.33	53.15	53.02	52.86	52.73	52.61	52.54	52.40	52.29	52.18	52.09	51.99	51.89	51.82	51.72	51.63	51.53	51.46	51.39
35-39	97.5%	49.72	49.89	50.06	50.16	50.26	50.31	50.38	50.44	50.50	50.54	50.62	50.66	50.71	50.76	50.83	50.86	50.86	50.94	50.97	51.01
	mean	49.22	49.20	49.19	49.18	49.16	49.15	49.13	49.12	49.10	49.09	49.07	49.06	49.04	49.02	49.01	48.99	48.98	48.96	48.95	48.93
	2.5%	48.69	48.45	48.28	48.16	48.01	47.88	47.77	47.70	47.58	47.46	47.37	47.29	47.20	47.09	47.02	46.94	46.85	46.78	46.71	46.64
40-44	97.5%	44.81	44.98	45.14	45.24	45.34	45.38	45.46	45.51	45.57	45.61	45.68	45.73	45.77	45.83	45.90	45.92	45.92	46.00	46.03	46.08
	mean	44.32	44.31	44.30	44.29	44.27	44.26	44.24	44.23	44.22	44.20	44.19	44.17	44.16	44.14	44.13	44.11	44.10	44.08	44.07	44.05
	2.5%	43.82	43.58	43.42	43.30	43.17	43.06	42.94	42.88	42.76	42.66	42.55	42.48	42.40	42.31	42.23	42.16	42.08	42.01	41.95	41.88
45-49	97.5%	39.96	40.11	40.28	40.37	40.46	40.51	40.58	40.63	40.69	40.73	40.79	40.84	40.89	40.94	41.00	41.02	41.03	41.11	41.13	41.19
	mean	39.48	39.46	39.46	39.45	39.43	39.42	39.40	39.39	39.38	39.37	39.35	39.34	39.32	39.31	39.29	39.28	39.26	39.25	39.24	39.22
	2.5%	38.99	38.77	38.62	38.50	38.37	38.25	38.14	38.09	37.96	37.87	37.79	37.72	37.63	37.54	37.47	37.39	37.32	37.25	37.20	37.12
50-54	97.5%	35.18	35.33	35.49	35.58	35.67	35.72	35.78	35.82	35.89	35.92	35.99	36.03	36.08	36.13	36.19	36.21	36.22	36.29	36.32	36.36
	mean	34.73	34.71	34.70	34.69	34.68	34.67	34.65	34.64	34.63	34.62	34.61	34.59	34.58	34.56	34.55	34.53	34.52	34.51	34.49	34.48
	2.5%	34.26	34.05	33.90	33.78	33.66	33.55	33.45	33.39	33.28	33.19	33.11	33.04	32.96	32.89	32.81	32.74	32.66	32.60	32.56	32.49
55-59	97.5%	30.53	30.67	30.82	30.91	30.99	31.04	31.10	31.14	31.20	31.23	31.30	31.34	31.38	31.43	31.48	31.50	31.51	31.57	31.61	31.66
	mean	30.10	30.09	30.08	30.07	30.06	30.05	30.03	30.02	30.01	30.00	29.99	29.97	29.96	29.94	29.93	29.92	29.90	29.89	29.88	29.87
	2.5%	29.66	29.46	29.32	29.21	29.10	28.99	28.90	28.84	28.73	28.66	28.58	28.51	28.44	28.36	28.29	28.24	28.15	28.10	28.05	28.01
60-64	97.5%	25.99	26.13	26.26	26.35	26.43	26.47	26.53	26.57	26.61	26.66	26.71	26.74	26.79	26.84	26.88	26.90	26.91	26.96	27.01	27.04
	mean	25.59	25.58	25.57	25.56	25.55	25.54	25.52	25.51	25.50	25.49	25.48	25.47	25.46	25.44	25.43	25.42	25.41	25.39	25.38	25.37
	2.5%	25.17	24.99	24.86	24.76	24.65	24.55	24.46	24.41	24.31	24.24	24.17	24.10	24.04	23.96	23.91	23.84	23.78	23.72	23.67	23.64
65-69	97.5%	21.61	21.75	21.87	21.93	22.02	22.06	22.11	22.14	22.19	22.22	22.27	22.30	22.35	22.39	22.43	22.45	22.46	22.51	22.54	22.58
	mean	21.24	21.23	21.22	21.21	21.20	21.19	21.18	21.17	21.16	21.15	21.14	21.13	21.11	21.10	21.09	21.08	21.07	21.06	21.04	21.03
	2.5%	20.86	20.69	20.56	20.48	20.37	20.29	20.19	20.15	20.06	19.99	19.93	19.86	19.81	19.74	19.70	19.65	19.59	19.53	19.47	19.44
70-74	97.5%	17.44	17.56	17.67	17.73	17.80	17.85	17.88	17.92	17.96	17.99	18.03	18.06	18.10	18.14	18.17	18.19	18.20	18.24	18.29	18.31
	mean	17.11	17.10	17.09	17.08	17.07	17.06	17.05	17.04	17.03	17.02	17.01	17.00	16.99	16.98	16.97	16.95	16.94	16.93	16.92	16.91
	2.5%	16.76	16.61	16.49	16.42	16.32	16.25	16.16	16.12	16.04	15.97	15.93	15.87	15.82	15.76	15.71	15.68	15.61	15.56	15.52	15.48
75-79	97.5%	13.47	13.57	13.65	13.71	13.77	13.80	13.83	13.86	13.89	13.92	13.95	13.99	14.01	14.04	14.08	14.09	14.08	14.14	14.16	14

Table A.42. The forecasted Norway male life expectancy by model 3

age 0	97.5%	80.44	80.69	80.92	81.04	81.20	81.28	81.33	81.43	81.53	81.60	81.69	81.73	81.77	81.85	81.93	81.96	82.01	82.05	82.09	82.11
	mean	79.67	79.64	79.61	79.58	79.55	79.51	79.47	79.44	79.40	79.37	79.34	79.30	79.26	79.22	79.18	79.14	79.10	79.07	79.03	78.99
	2.5%	78.85	78.44	78.14	77.93	77.66	77.41	77.17	76.97	76.76	76.57	76.39	76.19	76.01	75.80	75.65	75.50	75.32	75.10	74.92	74.78
1-4	97.5%	79.62	79.86	80.08	80.19	80.34	80.43	80.47	80.56	80.66	80.74	80.82	80.86	80.90	80.97	81.06	81.07	81.12	81.18	81.21	81.23
	mean	78.88	78.84	78.82	78.79	78.76	78.72	78.68	78.65	78.62	78.58	78.55	78.51	78.48	78.44	78.40	78.36	78.33	78.29	78.25	78.22
	2.5%	78.08	77.69	77.41	77.20	76.94	76.71	76.47	76.27	76.09	75.89	75.71	75.54	75.35	75.17	75.01	74.87	74.67	74.49	74.32	74.19
5-9	97.5%	75.65	75.89	76.11	76.22	76.37	76.45	76.50	76.59	76.68	76.76	76.84	76.89	76.92	76.99	77.07	77.09	77.14	77.20	77.22	77.24
	mean	74.92	74.89	74.86	74.83	74.80	74.77	74.73	74.70	74.66	74.63	74.60	74.56	74.53	74.49	74.45	74.41	74.38	74.34	74.31	74.27
	2.5%	74.14	73.75	73.48	73.26	73.01	72.79	72.56	72.37	72.18	71.99	71.83	71.64	71.47	71.28	71.13	70.99	70.80	70.60	70.45	70.32
10-14	97.5%	70.68	70.91	71.13	71.24	71.38	71.47	71.51	71.61	71.70	71.77	71.85	71.90	71.94	72.01	72.09	72.11	72.15	72.21	72.24	72.25
	mean	69.96	69.92	69.89	69.87	69.84	69.80	69.76	69.74	69.70	69.67	69.64	69.60	69.57	69.53	69.50	69.46	69.42	69.39	69.35	69.32
	2.5%	69.19	68.80	68.53	68.33	68.07	67.86	67.63	67.44	67.25	67.07	66.91	66.73	66.57	66.38	66.24	66.10	65.91	65.74	65.57	65.45
15-19	97.5%	65.70	65.93	66.15	66.26	66.41	66.49	66.53	66.63	66.71	66.79	66.87	66.92	66.95	67.02	67.10	67.12	67.17	67.23	67.25	67.27
	mean	64.99	64.95	64.93	64.90	64.87	64.84	64.80	64.77	64.74	64.71	64.68	64.64	64.61	64.57	64.53	64.50	64.46	64.43	64.40	64.36
	2.5%	64.22	63.84	63.58	63.38	63.13	62.92	62.70	62.50	62.32	62.14	61.98	61.79	61.64	61.47	61.32	61.18	61.00	60.84	60.66	60.54
20-24	97.5%	60.78	61.00	61.21	61.33	61.46	61.54	61.59	61.68	61.77	61.84	61.91	61.97	62.01	62.07	62.15	62.17	62.21	62.27	62.28	62.32
	mean	60.08	60.05	60.03	60.00	59.97	59.94	59.90	59.88	59.85	59.82	59.79	59.75	59.72	59.69	59.65	59.62	59.59	59.56	59.52	59.49
	2.5%	59.34	58.97	58.72	58.54	58.28	58.09	57.87	57.69	57.52	57.36	57.21	57.02	56.87	56.71	56.57	56.44	56.28	56.10	55.97	55.83
25-29	97.5%	55.92	56.14	56.34	56.44	56.57	56.65	56.69	56.78	56.86	56.94	57.00	57.06	57.10	57.16	57.24	57.26	57.29	57.35	57.37	57.41
	mean	55.26	55.23	55.21	55.19	55.16	55.14	55.10	55.08	55.05	55.03	55.00	54.97	54.94	54.91	54.88	54.85	54.82	54.79	54.76	54.74
	2.5%	54.57	54.23	54.00	53.83	53.60	53.43	53.22	53.08	52.89	52.75	52.61	52.46	52.34	52.18	52.05	51.96	51.79	51.67	51.49	51.38
30-34	97.5%	51.07	51.27	51.46	51.56	51.69	51.76	51.81	51.89	51.97	52.04	52.11	52.15	52.19	52.24	52.32	52.34	52.39	52.43	52.46	52.48
	mean	50.46	50.43	50.41	50.40	50.37	50.35	50.32	50.30	50.28	50.26	50.24	50.21	50.19	50.16	50.14	50.11	50.09	50.06	50.04	50.02
	2.5%	49.82	49.51	49.31	49.16	48.96	48.80	48.62	48.51	48.36	48.20	48.13	48.00	47.88	47.75	47.65	47.53	47.44	47.30	47.17	47.08
35-39	97.5%	46.23	46.42	46.60	46.70	46.82	46.87	46.92	47.01	47.08	47.16	47.21	47.25	47.29	47.36	47.42	47.44	47.49	47.52	47.55	47.58
	mean	45.65	45.63	45.62	45.60	45.58	45.56	45.54	45.52	45.50	45.48	45.46	45.44	45.42	45.40	45.38	45.35	45.33	45.31	45.29	45.27
	2.5%	45.06	44.78	44.59	44.46	44.27	44.14	43.98	43.89	43.76	43.62	43.53	43.43	43.34	43.21	43.11	43.03	42.93	42.84	42.73	42.64
40-44	97.5%	41.42	41.60	41.76	41.86	41.97	42.03	42.07	42.15	42.21	42.29	42.35	42.39	42.42	42.48	42.53	42.56	42.61	42.65	42.67	42.71
	mean	40.87	40.85	40.84	40.82	40.81	40.79	40.77	40.75	40.73	40.72	40.70	40.68	40.66	40.64	40.62	40.60	40.58	40.57	40.55	40.53
	2.5%	40.31	40.07	39.88	39.75	39.60	39.47	39.32	39.25	39.13	39.01	38.92	38.84	38.73	38.64	38.55	38.49	38.38	38.31	38.21	38.14
45-49	97.5%	36.63	36.80	36.96	37.05	37.16	37.23	37.26	37.33	37.39	37.46	37.52	37.56	37.59	37.66	37.70	37.73	37.78	37.81	37.83	37.86
	mean	36.11	36.09	36.08	36.07	36.05	36.03	36.01	36.00	35.98	35.97	35.95	35.93	35.91	35.90	35.88	35.86	35.84	35.83	35.81	35.79
	2.5%	35.59	35.35	35.18	35.05	34.91	34.79	34.66	34.58	34.46	34.37	34.26	34.21	34.10	34.00	33.93	33.86	33.78	33.71	33.62	33.56
50-54	97.5%	31.93	32.09	32.24	32.33	32.44	32.48	32.53	32.60	32.66	32.71	32.77	32.82	32.84	32.90	32.95	32.98	33.02	33.05	33.07	33.10
	mean	31.44	31.42	31.41	31.40	31.38	31.36	31.34	31.33	31.32	31.30	31.29	31.27	31.25	31.23	31.22	31.20	31.18	31.17	31.15	31.14
	2.5%	30.95	30.71	30.56	30.44	30.30	30.19	30.06	29.99	29.88	29.79	29.71	29.63	29.55	29.45	29.39	29.33	29.22	29.18	29.10	29.03
55-59	97.5%	27.37	27.53	27.67	27.75	27.84	27.90	27.93	28.00	28.07	28.11	28.16	28.20	28.23	28.29	28.33	28.37	28.39	28.42	28.43	28.46
	mean	26.92	26.90	26.89	26.88	26.86	26.84	26.83	26.81	26.80	26.79	26.77	26.75	26.74	26.72	26.71	26.69	26.67	26.66	26.65	26.63
	2.5%	26.46	26.24	26.09	25.98	25.86	25.74	25.63	25.57	25.46	25.38	25.30	25.23	25.16	25.07	25.02	24.94	24.87	24.81	24.74	24.67
60-64	97.5%	23.00	23.14	23.27	23.33	23.42	23.47	23.50	23.56	23.62	23.66	23.70	23.75	23.77	23.83	23.86	23.89	23.92	23.96	23.97	24.00
	mean	22.57	22.55	22.54	22.53	22.52	22.50	22.49	22.48	22.46	22.45	22.44	22.42	22.41	22.39	22.38	22.36	22.35	22.33	22.32	22.31
	2.5%	22.15	21.95	21.82	21.72	21.60	21.51	21.40	21.35	21.26	21.18	21.10	21.03	20.96	20.89	20.84	20.78	20.71	20.66	20.59	20.54
65-69	97.5%	18.84	18.97	19.08	19.15	19.22	19.27	19.30	19.34	19.40	19.44	19.46	19.50	19.54	19.58	19.61	19.64	19.66	19.69	19.70	19.73
	mean	18.45	18.44	18.43	18.42	18.41	18.39	18.38	18.37	18.35	18.34	18.33	18.31	18.30	18.29	18.27	18.26	18.25	18.23	18.22	18.21
	2.5%	18.07	17.89	17.77	17.69	17.58	17.50	17.40	17.36	17.28	17.20	17.15	17.08	17.01	16.95	16.90	16.83	16.80	16.73	16.68	16.63
70-74	97.5%	14.96	15.06	15.15	15.22	15.28	15.32	15.35	15.39	15.43	15.47	15.50	15.52	15.54	15.58	15.62	15.64	15.66	15.69	15.69	15.70
	mean	14.62	14.60	14.59	14.58	14.57	14.56	14.54	14.53	14.52	14.51	14.50	14.49	14.48	14.46	14.45	14.44	14.43	14.42	14.40	14.39
	2.5%	14.28	14.13	14.02	13.94	13.85	13.78	13.71	13.66	13.59	13.53	13.47	13.43	13.37	13.30	13.26	13.21	13.18	13.13	13.07	13.03
75-79	97.5%	11.38	11.47	11.54	11.59	11.64	11.66	11.69	11.73	11.75	11.79	11.81	11.83	11.86	11.88	11.91	11.91				

Table A.43. The forecasted Norway female life expectancy by model 8

age	97.5%	84.44	84.72	84.97	85.23	85.46	85.66	85.90	86.11	86.33	86.53	86.73	86.94	87.13	87.33	87.50	87.70	87.85	88.05	88.23	88.39
0	mean	83.85	84.00	84.15	84.31	84.46	84.60	84.75	84.90	85.05	85.20	85.34	85.49	85.64	85.78	85.93	86.07	86.21	86.36	86.51	86.65
	2.5%	83.24	83.24	83.26	83.29	83.32	83.39	83.45	83.54	83.60	83.66	83.75	83.85	83.95	84.03	84.12	84.20	84.30	84.44	84.51	84.58
1-4	97.5%	83.59	83.86	84.09	84.34	84.56	84.76	84.99	85.20	85.40	85.60	85.80	86.00	86.18	86.38	86.55	86.74	86.90	87.10	87.26	87.43
	mean	83.02	83.17	83.30	83.45	83.59	83.73	83.88	84.02	84.16	84.31	84.45	84.59	84.73	84.87	85.01	85.15	85.29	85.44	85.58	85.71
	2.5%	82.43	82.42	82.44	82.47	82.51	82.56	82.61	82.71	82.77	82.83	82.91	82.99	83.10	83.16	83.24	83.34	83.42	83.54	83.61	83.69
5-9	97.5%	79.63	79.89	80.12	80.37	80.58	80.77	81.01	81.21	81.41	81.62	81.81	82.01	82.20	82.39	82.56	82.75	82.90	83.11	83.27	83.43
	mean	79.07	79.21	79.34	79.49	79.63	79.77	79.91	80.05	80.19	80.34	80.48	80.62	80.76	80.90	81.04	81.18	81.31	81.46	81.60	81.73
	2.5%	78.48	78.49	78.50	78.54	78.57	78.62	78.69	78.76	78.84	78.89	78.96	79.05	79.15	79.22	79.28	79.39	79.47	79.60	79.67	79.75
10-14	97.5%	74.65	74.91	75.14	75.39	75.59	75.79	76.02	76.22	76.43	76.63	76.82	77.02	77.20	77.40	77.56	77.76	77.91	78.11	78.27	78.44
	mean	74.10	74.24	74.37	74.52	74.66	74.79	74.93	75.08	75.22	75.36	75.50	75.64	75.78	75.92	76.06	76.19	76.33	76.47	76.61	76.75
	2.5%	73.51	73.52	73.54	73.58	73.61	73.67	73.73	73.81	73.88	73.93	74.02	74.11	74.19	74.27	74.33	74.44	74.52	74.62	74.71	74.79
15-19	97.5%	69.67	69.92	70.15	70.40	70.60	70.80	71.03	71.23	71.43	71.63	71.83	72.02	72.21	72.41	72.57	72.77	72.92	73.12	73.28	73.44
	mean	69.12	69.26	69.39	69.53	69.67	69.81	69.95	70.09	70.23	70.38	70.51	70.65	70.79	70.93	71.07	71.21	71.34	71.49	71.62	71.76
	2.5%	68.54	68.55	68.56	68.60	68.64	68.69	68.76	68.84	68.90	68.95	69.04	69.12	69.21	69.29	69.34	69.46	69.53	69.64	69.73	69.80
20-24	97.5%	64.72	64.97	65.19	65.44	65.64	65.83	66.07	66.26	66.46	66.66	66.85	67.05	67.24	67.43	67.59	67.79	67.93	68.13	68.29	68.45
	mean	64.18	64.32	64.45	64.59	64.73	64.87	65.01	65.15	65.29	65.43	65.57	65.70	65.84	65.98	66.12	66.25	66.39	66.53	66.67	66.80
	2.5%	63.61	63.63	63.64	63.70	63.73	63.78	63.86	63.93	64.00	64.06	64.13	64.22	64.32	64.38	64.46	64.57	64.62	64.73	64.84	64.92
25-29	97.5%	59.77	60.02	60.24	60.48	60.69	60.87	61.10	61.29	61.49	61.69	61.88	62.08	62.26	62.45	62.62	62.80	62.95	63.15	63.31	63.47
	mean	59.25	59.39	59.52	59.66	59.80	59.93	60.07	60.21	60.35	60.49	60.63	60.76	60.90	61.04	61.17	61.31	61.45	61.59	61.72	61.86
	2.5%	58.70	58.72	58.74	58.80	58.84	58.89	58.95	59.05	59.12	59.18	59.25	59.35	59.43	59.49	59.57	59.68	59.75	59.88	59.97	60.04
30-34	97.5%	54.83	55.07	55.29	55.53	55.72	55.90	56.13	56.32	56.52	56.72	56.90	57.10	57.28	57.47	57.64	57.82	57.97	58.16	58.34	58.49
	mean	54.31	54.45	54.58	54.72	54.86	54.99	55.13	55.26	55.40	55.54	55.68	55.81	55.95	56.09	56.22	56.36	56.49	56.63	56.77	56.90
	2.5%	53.77	53.81	53.83	53.88	53.92	53.98	54.05	54.14	54.19	54.26	54.34	54.43	54.52	54.60	54.67	54.78	54.86	55.00	55.08	55.16
35-39	97.5%	49.91	50.14	50.36	50.59	50.78	50.96	51.18	51.37	51.56	51.76	51.95	52.14	52.31	52.51	52.67	52.86	53.00	53.19	53.38	53.52
	mean	49.40	49.54	49.67	49.80	49.94	50.07	50.21	50.34	50.48	50.62	50.75	50.88	51.02	51.16	51.29	51.42	51.56	51.70	51.83	51.96
	2.5%	48.87	48.91	48.93	48.99	49.04	49.10	49.17	49.24	49.32	49.38	49.46	49.55	49.63	49.71	49.80	49.89	49.96	50.10	50.20	50.28
40-44	97.5%	45.01	45.23	45.44	45.66	45.86	46.03	46.25	46.44	46.62	46.82	47.01	47.20	47.37	47.55	47.72	47.91	48.04	48.23	48.41	48.55
	mean	44.51	44.64	44.77	44.90	45.04	45.17	45.30	45.43	45.57	45.70	45.83	45.97	46.10	46.23	46.37	46.50	46.63	46.77	46.90	47.03
	2.5%	43.99	44.03	44.06	44.12	44.17	44.23	44.28	44.37	44.44	44.51	44.58	44.67	44.75	44.84	44.91	45.02	45.08	45.21	45.31	45.39
45-49	97.5%	40.13	40.35	40.55	40.77	40.96	41.13	41.35	41.53	41.71	41.91	42.09	42.28	42.44	42.62	42.79	42.97	43.11	43.29	43.47	43.61
	mean	39.64	39.77	39.90	40.03	40.16	40.29	40.41	40.54	40.68	40.81	40.94	41.07	41.20	41.33	41.46	41.59	41.72	41.85	41.98	42.11
	2.5%	39.13	39.17	39.20	39.26	39.30	39.37	39.42	39.50	39.58	39.63	39.70	39.79	39.88	39.96	40.04	40.13	40.20	40.32	40.42	40.50
50-54	97.5%	35.32	35.54	35.74	35.95	36.13	36.30	36.51	36.68	36.86	37.05	37.23	37.40	37.57	37.74	37.90	38.08	38.22	38.40	38.56	38.70
	mean	34.85	34.98	35.10	35.23	35.35	35.47	35.60	35.72	35.85	35.98	36.10	36.23	36.36	36.48	36.61	36.73	36.86	36.99	37.12	37.24
	2.5%	34.35	34.39	34.42	34.47	34.52	34.58	34.64	34.71	34.78	34.83	34.90	34.99	35.08	35.15	35.22	35.33	35.39	35.49	35.60	35.68
55-59	97.5%	30.65	30.85	31.04	31.25	31.42	31.59	31.78	31.95	32.13	32.31	32.48	32.65	32.81	32.98	33.13	33.30	33.43	33.61	33.75	33.89
	mean	30.19	30.31	30.43	30.55	30.67	30.79	30.91	31.03	31.15	31.27	31.39	31.51	31.63	31.75	31.87	31.99	32.11	32.24	32.36	32.48
	2.5%	29.71	29.75	29.78	29.83	29.86	29.92	29.98	30.05	30.11	30.17	30.24	30.31	30.40	30.47	30.53	30.63	30.69	30.78	30.90	30.95
60-64	97.5%	26.07	26.27	26.44	26.64	26.80	26.96	27.14	27.31	27.48	27.64	27.81	27.97	28.12	28.30	28.42	28.59	28.72	28.89	29.02	29.17
	mean	25.63	25.74	25.85	25.97	26.08	26.19	26.30	26.42	26.53	26.65	26.76	26.87	26.99	27.10	27.22	27.33	27.45	27.56	27.68	27.79
	2.5%	25.16	25.20	25.23	25.27	25.30	25.36	25.41	25.47	25.54	25.59	25.66	25.72	25.81	25.86	25.93	26.02	26.06	26.17	26.26	26.30
65-69	97.5%	21.66	21.85	22.01	22.19	22.35	22.49	22.65	22.82	22.97	23.12	23.28	23.43	23.58	23.72	23.86	24.01	24.13	24.29	24.42	24.55
	mean	21.24	21.35	21.45	21.55	21.66	21.76	21.86	21.97	22.07	22.18	22.29	22.39	22.50	22.60	22.71	22.81	22.92	23.03	23.14	23.24
	2.5%	20.80	20.83	20.84	20.89	20.92	20.97	21.00	21.09	21.14	21.16	21.23	21.30	21.37	21.40	21.48	21.57	21.60	21.69	21.76	21.82
70-74	97.5%	17.44	17.60	17.76	17.91	18.06	18.19	18.33	18.48	18.63	18.76	18.90	19.03	19.17	19.30	19.44	19.57	19.69	19.82	19.93	20.06
	mean	17.04	17.13	17.22	17.31	17.40	17.49	17.58	17.68	17.77	17.87	17.96	18.05	18.15	18.24	18.33	18.43	18.52	18.62	18.71	18.80
	2.5%	16.61	16.63	16.65	16.67	16.70	16.73	16.78	16.83	16.88	16.90	16.95	17.01	17.07	17.09	17.15	17.23	17.26	17.34	17.39	17.43
75-79	97.5%	13.47	13.61	13.72	13.85	13.96	14.07	14.19	14.30	14.42	14.53	14.65	14.78	14.88	14.99	15.09	15.21	15.31	15.42	15.52	15.61
	mean	13.11	13.17	13.24	13.32	13.39	13.46	13.53	13.60	13.67	13.75	13.82	13.89	13.97	14.04	14.11	14.19	14.26	14.34	14.41	14.48
	2.5%	12.71	12.72	12.72	12.74	12.76	12.78	12.81	12.84	12.86	12.89	12.91	12.95	13.01	13.02	13.05	13.10	13.13	13.18	13.21	13.23
80-84	97.5%	9.87	9.97	10.05	10.14	10.23	10.32	10.40	10.48	10.57	10.66	10.74	10.83	10.92	10.99	11.08	11.16	11.24	11.32	11.40	11.47
	mean	9.56	9.61	9.65	9.71	9.75	9.80	9.85	9.90	9.95	10.00	10.05	10.10	10.15	10.21	10.26	10.31	10.36	10.41	10.47	10.52
	2.5%	9.23	9.23	9.21	9.22	9.22	9.23	9.24	9.24	9.26	9.26	9.27	9.29	9.32	9.33	9.34	9.35	9.36	9.40	9.42	9.43
85-89	97.5%	6.81	6.85	6.91	6.96	7.00	7.05	7.09	7.13	7.17	7.22	7.27	7.30	7.35	7.38	7.42	7.46	7.51	7.55	7.59	7.64
	mean	6.55	6.56	6.58	6.59	6.61	6.62	6.63	6.65	6.66	6.68	6.69	6.70	6.72	6.74	6.75	6.77	6.78	6.80	6.82	6.83

Table A.44. The forecasted Norway male life expectancy by model 8

age 0	97.5%	80.59	80.94	81.25	81.55	81.80	82.09	82.32	82.58	82.86	83.09	83.32	83.55	83.78	83.98	84.21	84.45	84.64	84.88	85.06	85.24
	mean	79.87	80.03	80.19	80.36	80.52	80.68	80.85	81.02	81.19	81.36	81.52	81.70	81.86	82.02	82.18	82.35	82.52	82.69	82.86	83.02
	2.5%	79.10	79.09	79.05	79.05	79.09	79.14	79.21	79.27	79.34	79.40	79.46	79.54	79.62	79.77	79.83	79.92	80.03	80.11	80.24	80.32
1-4	97.5%	79.76	80.10	80.40	80.69	80.92	81.20	81.42	81.67	81.95	82.17	82.39	82.61	82.84	83.04	83.26	83.50	83.69	83.91	84.09	84.28
	mean	79.06	79.22	79.37	79.53	79.68	79.84	80.00	80.16	80.32	80.48	80.63	80.80	80.96	81.12	81.27	81.43	81.60	81.76	81.92	82.08
	2.5%	78.32	78.31	78.27	78.26	78.31	78.35	78.41	78.46	78.51	78.57	78.63	78.69	78.78	78.91	78.97	79.05	79.17	79.24	79.34	79.44
5-9	97.5%	75.79	76.12	76.42	76.70	76.93	77.21	77.43	77.68	77.96	78.18	78.40	78.62	78.85	79.05	79.26	79.50	79.70	79.92	80.09	80.28
	mean	75.09	75.25	75.40	75.56	75.71	75.86	76.02	76.18	76.33	76.49	76.65	76.81	76.97	77.13	77.28	77.44	77.60	77.77	77.93	78.09
	2.5%	74.36	74.35	74.31	74.31	74.34	74.39	74.45	74.49	74.55	74.61	74.67	74.72	74.81	74.94	75.00	75.07	75.19	75.26	75.36	75.45
10-14	97.5%	70.81	71.14	71.44	71.72	71.95	72.23	72.45	72.69	72.97	73.19	73.41	73.63	73.86	74.06	74.27	74.51	74.70	74.92	75.10	75.28
	mean	70.12	70.28	70.42	70.58	70.73	70.88	71.04	71.20	71.36	71.52	71.67	71.83	71.99	72.15	72.30	72.46	72.62	72.78	72.95	73.10
	2.5%	69.39	69.38	69.34	69.34	69.38	69.43	69.48	69.53	69.59	69.65	69.70	69.76	69.84	69.97	70.04	70.11	70.25	70.29	70.41	70.49
15-19	97.5%	65.83	66.16	66.46	66.74	66.97	67.24	67.47	67.71	67.98	68.20	68.42	68.64	68.87	69.07	69.28	69.52	69.70	69.93	70.11	70.29
	mean	65.15	65.30	65.45	65.61	65.76	65.91	66.07	66.23	66.38	66.54	66.69	66.85	67.01	67.16	67.32	67.48	67.64	67.80	67.96	68.12
	2.5%	64.42	64.42	64.38	64.37	64.42	64.46	64.51	64.56	64.62	64.67	64.73	64.80	64.87	65.01	65.07	65.14	65.27	65.31	65.44	65.52
20-24	97.5%	60.91	61.23	61.52	61.79	62.02	62.29	62.52	62.75	63.02	63.24	63.45	63.67	63.90	64.10	64.31	64.54	64.73	64.95	65.13	65.31
	mean	60.24	60.39	60.54	60.69	60.84	60.99	61.14	61.30	61.45	61.60	61.76	61.92	62.07	62.22	62.37	62.53	62.69	62.85	63.01	63.17
	2.5%	59.54	59.53	59.49	59.49	59.53	59.58	59.64	59.68	59.74	59.79	59.83	59.90	59.99	60.11	60.18	60.25	60.37	60.44	60.54	60.64
25-29	97.5%	56.05	56.36	56.64	56.90	57.13	57.40	57.61	57.84	58.11	58.32	58.53	58.74	58.98	59.17	59.37	59.60	59.79	60.01	60.19	60.38
	mean	55.41	55.56	55.70	55.85	56.00	56.15	56.30	56.46	56.61	56.76	56.91	57.07	57.23	57.38	57.53	57.68	57.84	58.00	58.16	58.31
	2.5%	54.73	54.74	54.72	54.73	54.77	54.83	54.88	54.92	55.00	55.07	55.11	55.18	55.26	55.39	55.45	55.55	55.64	55.74	55.82	55.93
30-34	97.5%	51.21	51.50	51.77	52.04	52.27	52.52	52.73	52.96	53.22	53.43	53.62	53.85	54.07	54.25	54.45	54.68	54.86	55.08	55.26	55.45
	mean	50.59	50.74	50.88	51.03	51.18	51.33	51.48	51.63	51.78	51.93	52.08	52.24	52.39	52.55	52.69	52.85	53.01	53.16	53.32	53.48
	2.5%	49.95	49.95	49.94	49.97	50.01	50.07	50.13	50.17	50.28	50.32	50.39	50.46	50.57	50.67	50.75	50.84	50.93	51.03	51.13	51.22
35-39	97.5%	46.36	46.65	46.91	47.16	47.39	47.64	47.84	48.06	48.31	48.53	48.73	48.95	49.15	49.35	49.54	49.76	49.94	50.15	50.34	50.52
	mean	45.76	45.91	46.05	46.20	46.34	46.49	46.64	46.79	46.94	47.09	47.24	47.39	47.55	47.70	47.84	48.00	48.15	48.31	48.47	48.62
	2.5%	45.15	45.16	45.15	45.18	45.23	45.28	45.35	45.41	45.51	45.57	45.63	45.72	45.83	45.93	46.01	46.10	46.18	46.28	46.41	46.48
40-44	97.5%	41.54	41.82	42.07	42.31	42.53	42.77	42.98	43.18	43.43	43.63	43.84	44.06	44.26	44.46	44.63	44.86	45.04	45.24	45.44	45.63
	mean	40.97	41.11	41.25	41.40	41.54	41.68	41.83	41.98	42.12	42.27	42.42	42.58	42.73	42.87	43.02	43.17	43.33	43.48	43.63	43.78
	2.5%	40.37	40.39	40.40	40.43	40.49	40.54	40.62	40.69	40.78	40.85	40.90	41.01	41.12	41.20	41.30	41.38	41.47	41.57	41.71	41.79
45-49	97.5%	36.75	37.02	37.26	37.49	37.70	37.93	38.13	38.34	38.56	38.78	38.98	39.19	39.39	39.57	39.74	39.97	40.15	40.34	40.54	40.72
	mean	36.19	36.33	36.46	36.60	36.74	36.88	37.02	37.17	37.31	37.45	37.59	37.75	37.89	38.03	38.18	38.32	38.47	38.62	38.77	38.92
	2.5%	35.61	35.63	35.63	35.67	35.73	35.78	35.86	35.92	36.00	36.07	36.13	36.22	36.32	36.42	36.50	36.58	36.65	36.76	36.88	36.96
50-54	97.5%	32.05	32.30	32.53	32.76	32.95	33.18	33.37	33.57	33.79	33.98	34.19	34.38	34.58	34.76	34.94	35.13	35.30	35.51	35.69	35.86
	mean	31.50	31.63	31.76	31.89	32.02	32.15	32.29	32.43	32.56	32.70	32.84	32.98	33.12	33.26	33.39	33.53	33.68	33.82	33.97	34.11
	2.5%	30.93	30.94	30.96	30.99	31.04	31.08	31.16	31.22	31.30	31.36	31.43	31.50	31.59	31.68	31.75	31.84	31.90	32.00	32.11	32.20
55-59	97.5%	27.51	27.75	27.97	28.18	28.38	28.59	28.77	28.94	29.15	29.34	29.53	29.73	29.91	30.08	30.25	30.45	30.61	30.78	30.97	31.15
	mean	26.99	27.11	27.23	27.36	27.48	27.60	27.73	27.86	27.99	28.12	28.25	28.38	28.52	28.65	28.77	28.91	29.04	29.18	29.32	29.45
	2.5%	26.44	26.46	26.46	26.49	26.54	26.59	26.65	26.70	26.78	26.84	26.90	26.98	27.06	27.14	27.20	27.28	27.36	27.45	27.55	27.63
60-64	97.5%	23.15	23.37	23.55	23.76	23.92	24.11	24.29	24.45	24.66	24.83	25.00	25.17	25.35	25.51	25.67	25.85	26.02	26.18	26.35	26.51
	mean	22.64	22.76	22.87	22.98	23.09	23.21	23.32	23.44	23.56	23.68	23.80	23.92	24.05	24.17	24.28	24.41	24.54	24.66	24.79	24.91
	2.5%	22.12	22.14	22.14	22.18	22.22	22.27	22.31	22.36	22.44	22.49	22.55	22.64	22.70	22.78	22.82	22.91	22.96	23.05	23.12	23.20
65-69	97.5%	18.96	19.15	19.31	19.50	19.64	19.81	19.97	20.11	20.30	20.45	20.59	20.74	20.92	21.06	21.19	21.35	21.51	21.65	21.80	21.96
	mean	18.49	18.58	18.68	18.78	18.87	18.97	19.08	19.18	19.28	19.39	19.49	19.60	19.71	19.81	19.91	20.02	20.13	20.25	20.36	20.47
	2.5%	18.00	18.02	18.02	18.04	18.07	18.12	18.16	18.20	18.27	18.30	18.36	18.40	18.46	18.53	18.55	18.62	18.69	18.77	18.82	18.89
70-74	97.5%	15.03	15.20	15.33	15.47	15.59	15.73	15.86	16.00	16.14	16.26	16.38	16.50	16.65	16.76	16.88	17.03	17.15	17.27	17.39	17.53
	mean	14.61	14.69	14.76	14.84	14.91	14.99	15.07	15.15	15.23	15.31	15.39	15.47	15.56	15.64	15.72	15.81	15.90	15.99	16.08	16.17
	2.5%	14.17	14.17	14.16	14.18	14.18	14.19	14.24	14.27	14.32	14.32	14.35	14.39	14.43	14.47	14.50	14.55	14.58	14.64	14.66	14.74
75-79	97.5%	11.44	11.55	11.67	11.78	11.87	11.97	12.07	12.17	12.30	12.39	12.49	12.59	12.70	12.79	12.90	13.0				

Table A.45. The forecasted Spain female life expectancy by model 3

age 0	97.5%	86.14	86.42	86.67	86.80	86.97	87.08	87.15	87.21	87.30	87.39	87.50	87.57	87.58	87.66	87.77	87.79	87.81	87.90	87.93	87.97
	mean	85.26	85.22	85.19	85.16	85.12	85.08	85.03	85.00	84.96	84.92	84.89	84.84	84.80	84.75	84.71	84.66	84.62	84.58	84.54	84.50
	2.5%	84.34	83.90	83.53	83.27	82.96	82.70	82.45	82.28	82.01	81.80	81.59	81.39	81.19	80.98	80.71	80.51	80.32	80.14	79.96	79.78
1-4	97.5%	85.32	85.59	85.83	85.95	86.12	86.23	86.29	86.35	86.44	86.53	86.62	86.68	86.72	86.79	86.89	86.90	86.92	87.01	87.03	87.08
	mean	84.49	84.45	84.42	84.39	84.35	84.31	84.27	84.23	84.20	84.16	84.13	84.08	84.04	84.00	83.96	83.91	83.88	83.84	83.80	83.76
	2.5%	83.59	83.19	82.83	82.59	82.32	82.04	81.81	81.67	81.41	81.21	80.99	80.81	80.62	80.41	80.17	79.99	79.82	79.66	79.45	79.26
5-9	97.5%	81.36	81.63	81.86	81.99	82.15	82.26	82.32	82.38	82.46	82.55	82.64	82.70	82.74	82.81	82.91	82.92	82.95	83.03	83.06	83.10
	mean	80.53	80.49	80.46	80.43	80.40	80.36	80.31	80.28	80.24	80.21	80.17	80.13	80.09	80.05	80.01	79.97	79.93	79.89	79.85	79.81
	2.5%	79.65	79.24	78.90	78.66	78.39	78.13	77.89	77.76	77.49	77.30	77.09	76.91	76.72	76.51	76.29	76.10	75.94	75.79	75.58	75.40
10-14	97.5%	76.38	76.64	76.88	77.00	77.16	77.27	77.33	77.39	77.47	77.57	77.65	77.71	77.75	77.82	77.92	77.93	77.96	78.04	78.07	78.11
	mean	75.56	75.52	75.49	75.46	75.42	75.39	75.34	75.31	75.27	75.24	75.21	75.16	75.13	75.08	75.04	75.00	74.96	74.93	74.89	74.85
	2.5%	74.69	74.28	73.94	73.70	73.44	73.18	72.94	72.81	72.55	72.36	72.16	71.98	71.79	71.59	71.38	71.18	71.02	70.87	70.66	70.49
15-19	97.5%	71.40	71.66	71.90	72.02	72.18	72.29	72.35	72.41	72.49	72.58	72.67	72.73	72.77	72.83	72.94	72.94	72.97	73.05	73.08	73.12
	mean	70.59	70.55	70.52	70.49	70.46	70.42	70.38	70.35	70.31	70.28	70.24	70.20	70.16	70.12	70.08	70.04	70.00	69.97	69.93	69.89
	2.5%	69.73	69.32	68.99	68.76	68.50	68.23	68.02	67.89	67.62	67.43	67.25	67.07	66.87	66.68	66.46	66.28	66.13	65.98	65.79	65.60
20-24	97.5%	66.44	66.69	66.92	67.04	67.20	67.31	67.37	67.43	67.51	67.60	67.69	67.74	67.78	67.85	67.96	67.96	67.99	68.06	68.09	68.13
	mean	65.63	65.59	65.56	65.54	65.50	65.47	65.42	65.39	65.36	65.33	65.29	65.25	65.22	65.18	65.14	65.10	65.06	65.03	64.99	64.95
	2.5%	64.78	64.39	64.05	63.84	63.59	63.33	63.11	62.99	62.72	62.55	62.36	62.19	62.00	61.81	61.58	61.42	61.28	61.13	60.97	60.76
25-29	97.5%	61.47	61.72	61.95	62.07	62.23	62.33	62.39	62.45	62.53	62.62	62.71	62.76	62.80	62.86	62.97	62.98	63.00	63.09	63.11	63.15
	mean	60.68	60.64	60.61	60.59	60.56	60.52	60.48	60.45	60.42	60.39	60.35	60.32	60.28	60.24	60.21	60.16	60.13	60.10	60.06	60.03
	2.5%	59.85	59.46	59.14	58.92	58.68	58.45	58.23	58.11	57.85	57.68	57.50	57.32	57.15	57.00	56.78	56.61	56.47	56.30	56.18	55.97
30-34	97.5%	56.51	56.76	56.98	57.10	57.25	57.36	57.42	57.47	57.56	57.64	57.73	57.78	57.82	57.88	57.99	57.99	58.02	58.10	58.12	58.17
	mean	55.73	55.69	55.67	55.64	55.61	55.58	55.54	55.51	55.48	55.45	55.42	55.38	55.35	55.31	55.28	55.24	55.21	55.18	55.14	55.11
	2.5%	54.91	54.54	54.23	54.02	53.78	53.57	53.35	53.24	53.00	52.83	52.66	52.48	52.33	52.18	51.98	51.82	51.68	51.56	51.41	51.23
35-39	97.5%	51.56	51.80	52.01	52.14	52.30	52.39	52.45	52.51	52.59	52.67	52.76	52.81	52.85	52.91	53.02	53.02	53.04	53.12	53.15	53.19
	mean	50.80	50.76	50.74	50.71	50.68	50.65	50.62	50.59	50.56	50.53	50.50	50.47	50.44	50.40	50.37	50.33	50.30	50.28	50.24	50.21
	2.5%	50.01	49.65	49.36	49.15	48.92	48.71	48.51	48.41	48.17	48.02	47.86	47.70	47.57	47.42	47.22	47.07	46.95	46.84	46.72	46.56
40-44	97.5%	46.64	46.87	47.08	47.20	47.35	47.45	47.50	47.56	47.64	47.72	47.81	47.86	47.90	47.96	48.06	48.06	48.09	48.16	48.19	48.24
	mean	45.90	45.86	45.84	45.82	45.79	45.76	45.72	45.70	45.67	45.65	45.62	45.59	45.56	45.52	45.49	45.46	45.43	45.41	45.38	45.35
	2.5%	45.13	44.79	44.51	44.31	44.10	43.90	43.70	43.62	43.40	43.25	43.09	42.93	42.82	42.67	42.49	42.34	42.24	42.15	42.04	41.88
45-49	97.5%	41.77	41.99	42.20	42.32	42.46	42.55	42.60	42.66	42.74	42.81	42.90	42.95	42.98	43.04	43.14	43.15	43.18	43.25	43.27	43.33
	mean	41.06	41.03	41.00	40.98	40.96	40.93	40.90	40.87	40.85	40.82	40.80	40.77	40.74	40.71	40.68	40.65	40.62	40.60	40.57	40.54
	2.5%	40.32	39.99	39.72	39.53	39.33	39.16	38.96	38.88	38.67	38.53	38.38	38.25	38.14	37.99	37.86	37.72	37.62	37.52	37.41	37.27
50-54	97.5%	36.98	37.20	37.40	37.51	37.65	37.73	37.79	37.85	37.92	37.99	38.08	38.11	38.14	38.21	38.30	38.31	38.34	38.42	38.44	38.48
	mean	36.30	36.27	36.25	36.23	36.20	36.18	36.15	36.12	36.10	36.07	36.05	36.02	36.00	35.97	35.94	35.91	35.88	35.86	35.83	35.81
	2.5%	35.59	35.27	35.02	34.83	34.65	34.48	34.29	34.22	34.02	33.88	33.75	33.62	33.52	33.37	33.25	33.12	33.00	32.93	32.82	32.71
55-59	97.5%	32.27	32.48	32.67	32.77	32.90	32.99	33.03	33.09	33.16	33.23	33.32	33.34	33.37	33.44	33.53	33.54	33.56	33.64	33.65	33.69
	mean	31.63	31.61	31.59	31.57	31.54	31.52	31.49	31.47	31.45	31.42	31.40	31.37	31.35	31.32	31.30	31.27	31.25	31.23	31.20	31.18
	2.5%	30.96	30.67	30.43	30.26	30.09	29.92	29.75	29.69	29.53	29.37	29.26	29.14	29.05	28.91	28.83	28.69	28.57	28.48	28.41	28.28
60-64	97.5%	27.67	27.87	28.05	28.14	28.26	28.34	28.39	28.44	28.50	28.57	28.65	28.68	28.71	28.77	28.85	28.86	28.89	28.95	28.96	29.01
	mean	27.06	27.03	27.01	27.00	26.97	26.95	26.92	26.90	26.88	26.86	26.84	26.81	26.79	26.76	26.74	26.71	26.69	26.67	26.65	26.62
	2.5%	26.42	26.15	25.92	25.77	25.59	25.44	25.28	25.22	25.06	24.93	24.83	24.71	24.61	24.50	24.39	24.26	24.17	24.09	24.01	23.87
65-69	97.5%	23.13	23.31	23.47	23.56	23.67	23.74	23.78	23.83	23.89	23.96	24.02	24.05	24.07	24.13	24.22	24.22	24.24	24.30	24.31	24.36
	mean	22.56	22.54	22.52	22.50	22.48	22.46	22.43	22.42	22.40	22.38	22.36	22.33	22.31	22.29	22.26	22.24	22.22	22.20	22.18	22.16
	2.5%	21.98	21.72	21.50	21.37	21.23	21.08	20.93	20.88	20.73	20.62	20.52	20.40	20.32	20.21	20.14	20.01	19.93	19.86	19.78	19.66
70-74	97.5%	18.71	18.87	19.02	19.09	19.19	19.26	19.30	19.34	19.39	19.45	19.51	19.55	19.56	19.61	19.67	19.68	19.72	19.76	19.78	19.81
	mean	18.17	18.15	18.13	18.12	18.10	18.08	18.05	18.04	18.02	18.00	17.98	17.96	17.94	17.91	17.89	17.87	17.85	17.84	17.81	17.79
	2.5%	17.63	17.39	17.20	17.08	16.95	16.81	16.68	16.61	16.49	16.39	16.29	16.20	16.10	16.01	15.94	15.82	15.77	15.68	15.61	15.50
75-79	97.5%	14.46	14.59	14.71	14.78	14.85	14.90	14.94	14.97	15.01	15.06	15.11	15.15	15.15	15.19	15.25	15.27	15.29	15.33	15.33	15.3

Table A.46. The forecasted Spain male life expectancy by model 3

age 0	97.5%	80.90	81.25	81.59	81.76	81.96	82.09	82.16	82.29	82.38	82.47	82.61	82.70	82.74	82.84	82.91	82.97	83.01	83.10	83.19	83.20
	mean	79.73	79.67	79.61	79.56	79.50	79.44	79.37	79.31	79.25	79.19	79.13	79.06	78.99	78.92	78.85	78.78	78.71	78.65	78.58	78.51
	2.5%	78.48	77.89	77.39	77.07	76.64	76.31	75.88	75.61	75.25	74.96	74.65	74.38	74.07	73.71	73.44	73.10	72.82	72.44	72.23	72.05
1-4	97.5%	80.09	80.44	80.76	80.93	81.13	81.25	81.31	81.43	81.53	81.61	81.75	81.84	81.89	81.97	82.02	82.09	82.14	82.22	82.30	82.32
	mean	78.97	78.91	78.86	78.81	78.75	78.69	78.62	78.56	78.50	78.44	78.38	78.32	78.25	78.18	78.12	78.04	77.98	77.92	77.85	77.78
	2.5%	77.77	77.19	76.71	76.40	76.00	75.67	75.25	75.01	74.67	74.41	74.07	73.83	73.49	73.18	72.88	72.60	72.30	71.96	71.75	71.56
5-9	97.5%	76.13	76.47	76.80	76.96	77.16	77.28	77.34	77.46	77.56	77.64	77.78	77.87	77.91	77.99	78.04	78.11	78.16	78.25	78.32	78.33
	mean	75.02	74.95	74.90	74.85	74.80	74.73	74.67	74.61	74.55	74.49	74.44	74.37	74.31	74.23	74.17	74.10	74.03	73.97	73.91	73.84
	2.5%	73.82	73.26	72.78	72.47	72.07	71.76	71.34	71.10	70.75	70.51	70.17	69.93	69.58	69.30	69.01	68.70	68.42	68.10	67.87	67.68
10-14	97.5%	71.16	71.50	71.81	71.98	72.18	72.30	72.36	72.48	72.57	72.66	72.80	72.89	72.92	73.01	73.05	73.13	73.17	73.26	73.33	73.35
	mean	70.05	69.99	69.94	69.89	69.83	69.77	69.70	69.65	69.59	69.53	69.47	69.40	69.34	69.27	69.21	69.13	69.07	69.01	68.95	68.88
	2.5%	68.86	68.29	67.82	67.52	67.12	66.82	66.39	66.16	65.82	65.57	65.24	65.01	64.66	64.38	64.10	63.77	63.53	63.19	62.96	62.77
15-19	97.5%	66.19	66.52	66.84	67.01	67.20	67.32	67.38	67.50	67.58	67.68	67.82	67.90	67.94	68.03	68.07	68.14	68.19	68.27	68.34	68.37
	mean	65.08	65.02	64.97	64.92	64.86	64.80	64.74	64.68	64.62	64.57	64.51	64.44	64.38	64.31	64.25	64.17	64.11	64.05	63.99	63.92
	2.5%	63.90	63.33	62.87	62.57	62.18	61.88	61.46	61.22	60.89	60.64	60.31	60.08	59.75	59.46	59.17	58.88	58.63	58.27	58.06	57.86
20-24	97.5%	61.24	61.57	61.87	62.04	62.23	62.35	62.41	62.53	62.62	62.71	62.84	62.93	62.97	63.06	63.10	63.17	63.22	63.30	63.37	63.39
	mean	60.15	60.09	60.04	59.99	59.94	59.88	59.81	59.76	59.70	59.65	59.59	59.53	59.47	59.40	59.34	59.27	59.21	59.15	59.09	59.02
	2.5%	58.98	58.44	57.98	57.69	57.30	57.01	56.60	56.39	56.06	55.82	55.50	55.26	54.94	54.68	54.42	54.14	53.84	53.55	53.31	53.13
25-29	97.5%	56.32	56.64	56.94	57.10	57.28	57.41	57.46	57.58	57.66	57.76	57.90	57.98	58.02	58.10	58.15	58.21	58.27	58.34	58.41	58.43
	mean	55.26	55.20	55.15	55.11	55.06	55.00	54.94	54.89	54.84	54.78	54.73	54.67	54.61	54.55	54.49	54.42	54.37	54.32	54.26	54.20
	2.5%	54.13	53.61	53.17	52.89	52.51	52.24	51.86	51.65	51.33	51.10	50.82	50.60	50.32	50.08	49.80	49.54	49.31	48.99	48.80	48.63
30-34	97.5%	51.39	51.71	52.00	52.16	52.34	52.46	52.52	52.63	52.71	52.81	52.94	53.01	53.06	53.14	53.19	53.25	53.30	53.38	53.44	53.46
	mean	50.36	50.31	50.26	50.22	50.18	50.12	50.07	50.02	49.97	49.92	49.87	49.82	49.77	49.71	49.65	49.59	49.54	49.49	49.44	49.39
	2.5%	49.27	48.78	48.37	48.09	47.77	47.47	47.15	46.94	46.66	46.45	46.20	45.98	45.75	45.51	45.30	45.03	44.79	44.56	44.34	44.20
35-39	97.5%	46.49	46.79	47.08	47.23	47.40	47.53	47.58	47.69	47.77	47.87	48.00	48.07	48.11	48.18	48.24	48.31	48.36	48.44	48.49	48.52
	mean	45.49	45.44	45.40	45.37	45.32	45.28	45.22	45.18	45.14	45.10	45.05	45.00	44.96	44.90	44.86	44.80	44.76	44.72	44.67	44.62
	2.5%	44.45	43.98	43.61	43.34	43.04	42.82	42.51	42.30	42.07	41.86	41.63	41.47	41.22	41.03	40.85	40.62	40.44	40.22	40.00	39.93
40-44	97.5%	41.60	41.89	42.17	42.32	42.49	42.62	42.66	42.78	42.85	42.93	43.08	43.14	43.18	43.25	43.31	43.37	43.42	43.50	43.55	43.58
	mean	40.64	40.59	40.56	40.53	40.49	40.44	40.40	40.36	40.32	40.28	40.24	40.19	40.15	40.10	40.06	40.01	39.97	39.93	39.89	39.85
	2.5%	39.64	39.21	38.85	38.60	38.32	38.11	37.82	37.66	37.44	37.24	37.00	36.86	36.69	36.50	36.30	36.12	35.95	35.75	35.60	35.51
45-49	97.5%	36.81	37.09	37.35	37.50	37.65	37.77	37.82	37.93	38.00	38.09	38.22	38.28	38.33	38.39	38.45	38.50	38.55	38.65	38.69	38.73
	mean	35.88	35.84	35.81	35.78	35.75	35.70	35.66	35.63	35.59	35.55	35.52	35.48	35.44	35.39	35.35	35.31	35.27	35.24	35.20	35.16
	2.5%	34.94	34.53	34.20	33.97	33.70	33.51	33.23	33.09	32.89	32.72	32.51	32.36	32.21	32.01	31.85	31.70	31.54	31.36	31.22	31.13
50-54	97.5%	32.16	32.42	32.68	32.82	32.97	33.08	33.13	33.22	33.29	33.38	33.50	33.56	33.60	33.68	33.73	33.78	33.82	33.89	33.95	33.98
	mean	31.29	31.26	31.23	31.20	31.17	31.13	31.09	31.06	31.03	30.99	30.96	30.92	30.89	30.85	30.81	30.77	30.74	30.71	30.67	30.64
	2.5%	30.41	30.04	29.74	29.51	29.27	29.08	28.84	28.71	28.52	28.36	28.19	28.03	27.90	27.76	27.57	27.45	27.31	27.12	27.05	26.94
55-59	97.5%	27.70	27.93	28.16	28.30	28.44	28.53	28.58	28.66	28.74	28.81	28.92	28.98	29.01	29.07	29.14	29.17	29.22	29.29	29.33	29.37
	mean	26.89	26.85	26.83	26.81	26.78	26.74	26.71	26.68	26.65	26.62	26.59	26.56	26.53	26.49	26.46	26.42	26.40	26.37	26.34	26.31
	2.5%	26.07	25.74	25.47	25.28	25.04	24.89	24.65	24.54	24.39	24.24	24.07	23.95	23.83	23.71	23.53	23.46	23.32	23.15	23.08	23.00
60-64	97.5%	23.45	23.66	23.85	23.98	24.12	24.19	24.25	24.32	24.39	24.46	24.55	24.60	24.64	24.67	24.74	24.78	24.83	24.87	24.91	24.99
	mean	22.70	22.67	22.64	22.62	22.60	22.57	22.54	22.51	22.49	22.46	22.44	22.42	22.38	22.35	22.32	22.29	22.27	22.24	22.22	22.19
	2.5%	21.94	21.65	21.41	21.23	21.02	20.89	20.69	20.59	20.46	20.34	20.17	20.06	19.96	19.86	19.72	19.61	19.48	19.38	19.29	19.22
65-69	97.5%	19.41	19.60	19.77	19.89	19.99	20.06	20.11	20.17	20.24	20.28	20.39	20.44	20.47	20.51	20.58	20.59	20.61	20.67	20.72	20.77
	mean	18.74	18.72	18.70	18.68	18.66	18.63	18.61	18.59	18.56	18.54	18.52	18.49	18.47	18.44	18.42	18.39	18.37	18.35	18.33	18.30
	2.5%	18.07	17.82	17.61	17.45	17.29	17.17	17.01	16.90	16.80	16.68	16.53	16.43	16.36	16.25	16.13	16.02	15.95	15.85	15.80	15.71
70-74	97.5%	15.60	15.77	15.92	16.00	16.10	16.16	16.21	16.26	16.32	16.35	16.45	16.49	16.50	16.54	16.60	16.61	16.64	16.69	16.71	16.79
	mean	15.01	14.99	14.97	14.96	14.94	14.92	14.89	14.88	14.86	14.84	14.82	14.79	14.77	14.75	14.73	14.71	14.69	14.67	14.65	14.63
	2.5%	14.42	14.21	14.03	13.90	13.76	13.65	13.51	13.43	13.31	13.19	13.11	13.00	12.93	12.86	12.75	12.67	12.58	12.51	12.49	12.40
75-79	97.5%	12.05	12.19	12.31	12.37	12.44	12.50	12.54	12.57	12.63	12.66	12.73	12.75	12.79	12.80	12.84	12.8				

Table A.47. The forecasted Spain female life expectancy by model 8

age	97.5%	86.12	86.50	86.84	87.16	87.48	87.76	88.04	88.32	88.60	88.89	89.13	89.42	89.65	89.89	90.11	90.37	90.64	90.85	91.07	91.27
0	mean	85.29	85.51	85.72	85.93	86.14	86.35	86.56	86.77	86.98	87.19	87.39	87.59	87.79	87.99	88.18	88.37	88.56	88.75	88.93	89.12
	2.5%	84.17	84.32	84.46	84.51	84.59	84.71	84.81	84.95	85.08	85.20	85.33	85.45	85.58	85.69	85.84	85.98	86.10	86.27	86.35	86.51
1-4	97.5%	85.32	85.68	85.99	86.29	86.59	86.88	87.13	87.41	87.68	87.96	88.19	88.47	88.70	88.95	89.16	89.42	89.68	89.89	90.10	90.30
	mean	84.50	84.71	84.91	85.11	85.31	85.51	85.71	85.91	86.11	86.31	86.50	86.70	86.89	87.08	87.27	87.45	87.64	87.82	88.00	88.18
	2.5%	83.38	83.55	83.69	83.73	83.82	83.92	84.01	84.15	84.28	84.39	84.51	84.62	84.75	84.86	84.99	85.12	85.24	85.39	85.48	85.63
5-9	97.5%	81.35	81.71	82.01	82.31	82.61	82.89	83.15	83.42	83.69	83.97	84.20	84.48	84.71	84.96	85.16	85.42	85.68	85.89	86.10	86.30
	mean	80.54	80.74	80.94	81.14	81.34	81.53	81.74	81.93	82.13	82.33	82.52	82.71	82.91	83.09	83.28	83.47	83.65	83.83	84.01	84.19
	2.5%	79.42	79.59	79.72	79.77	79.85	79.97	80.05	80.18	80.31	80.42	80.54	80.64	80.76	80.88	81.03	81.14	81.26	81.41	81.50	81.64
10-14	97.5%	76.37	76.73	77.03	77.33	77.63	77.91	78.16	78.43	78.70	78.98	79.21	79.49	79.72	79.97	80.17	80.43	80.69	80.90	81.11	81.31
	mean	75.57	75.77	75.97	76.17	76.36	76.56	76.76	76.96	77.15	77.35	77.54	77.73	77.92	78.11	78.29	78.48	78.66	78.84	79.02	79.20
	2.5%	74.45	74.63	74.76	74.81	74.89	75.01	75.08	75.21	75.33	75.45	75.58	75.66	75.79	75.91	76.05	76.17	76.28	76.42	76.53	76.67
15-19	97.5%	71.40	71.75	72.05	72.34	72.64	72.92	73.18	73.44	73.71	73.99	74.21	74.50	74.73	74.97	75.18	75.43	75.69	75.90	76.11	76.31
	mean	70.59	70.80	70.99	71.19	71.38	71.58	71.78	71.97	72.16	72.36	72.55	72.74	72.94	73.12	73.31	73.49	73.68	73.86	74.03	74.21
	2.5%	69.48	69.65	69.78	69.83	69.91	70.03	70.11	70.24	70.36	70.48	70.60	70.70	70.82	70.93	71.08	71.19	71.32	71.45	71.55	71.68
20-24	97.5%	66.43	66.77	67.07	67.37	67.66	67.94	68.19	68.45	68.72	69.00	69.23	69.51	69.74	69.98	70.18	70.44	70.70	70.91	71.12	71.31
	mean	65.63	65.84	66.03	66.22	66.42	66.61	66.81	67.01	67.20	67.39	67.58	67.77	67.96	68.15	68.33	68.52	68.70	68.88	69.06	69.23
	2.5%	64.52	64.70	64.83	64.88	64.96	65.07	65.18	65.30	65.41	65.53	65.65	65.75	65.87	65.98	66.13	66.24	66.36	66.51	66.60	66.72
25-29	97.5%	61.46	61.80	62.10	62.40	62.68	62.96	63.21	63.47	63.74	64.02	64.24	64.52	64.75	64.99	65.19	65.45	65.71	65.91	66.13	66.32
	mean	60.67	60.88	61.07	61.26	61.45	61.65	61.84	62.04	62.23	62.42	62.61	62.80	62.99	63.17	63.36	63.54	63.72	63.90	64.08	64.25
	2.5%	59.56	59.74	59.88	59.92	60.02	60.14	60.23	60.35	60.46	60.58	60.70	60.81	60.93	61.05	61.21	61.30	61.42	61.55	61.64	61.77
30-34	97.5%	56.50	56.83	57.12	57.41	57.71	57.98	58.23	58.49	58.75	59.03	59.25	59.52	59.76	59.99	60.20	60.46	60.71	60.92	61.14	61.33
	mean	55.71	55.91	56.10	56.29	56.49	56.68	56.87	57.06	57.25	57.45	57.63	57.82	58.01	58.19	58.38	58.56	58.74	58.92	59.09	59.27
	2.5%	54.60	54.78	54.92	54.97	55.06	55.17	55.28	55.39	55.50	55.62	55.73	55.85	55.98	56.08	56.25	56.34	56.46	56.58	56.67	56.80
35-39	97.5%	51.55	51.87	52.16	52.44	52.74	53.00	53.25	53.51	53.77	54.05	54.27	54.55	54.77	55.01	55.22	55.47	55.72	55.93	56.15	56.34
	mean	50.77	50.97	51.16	51.34	51.53	51.72	51.92	52.11	52.29	52.49	52.67	52.86	53.05	53.23	53.41	53.59	53.77	53.95	54.12	54.29
	2.5%	49.65	49.85	49.98	50.04	50.13	50.23	50.35	50.47	50.57	50.67	50.80	50.93	51.05	51.15	51.31	51.39	51.53	51.62	51.74	51.85
40-44	97.5%	46.63	46.95	47.24	47.52	47.80	48.06	48.31	48.56	48.82	49.09	49.31	49.58	49.82	50.05	50.25	50.50	50.75	50.95	51.17	51.36
	mean	45.86	46.06	46.24	46.43	46.61	46.80	46.99	47.18	47.36	47.55	47.73	47.92	48.11	48.29	48.46	48.64	48.82	49.00	49.17	49.34
	2.5%	44.76	44.95	45.08	45.14	45.23	45.33	45.44	45.57	45.66	45.77	45.91	46.01	46.13	46.24	46.38	46.47	46.61	46.71	46.80	46.91
45-49	97.5%	41.77	42.06	42.34	42.62	42.89	43.16	43.40	43.64	43.91	44.16	44.38	44.66	44.88	45.12	45.30	45.56	45.80	46.00	46.21	46.40
	mean	41.01	41.20	41.38	41.57	41.75	41.93	42.12	42.31	42.49	42.67	42.85	43.04	43.22	43.40	43.57	43.75	43.92	44.09	44.26	44.43
	2.5%	39.91	40.10	40.24	40.31	40.39	40.50	40.61	40.73	40.83	40.94	41.09	41.17	41.31	41.40	41.54	41.65	41.75	41.87	41.99	42.08
50-54	97.5%	36.98	37.27	37.54	37.81	38.08	38.34	38.58	38.80	39.06	39.31	39.54	39.79	40.01	40.24	40.43	40.67	40.91	41.12	41.32	41.51
	mean	36.25	36.44	36.62	36.80	36.97	37.15	37.34	37.52	37.69	37.88	38.05	38.23	38.41	38.58	38.75	38.92	39.10	39.26	39.43	39.59
	2.5%	35.14	35.35	35.49	35.57	35.65	35.76	35.87	35.98	36.08	36.18	36.33	36.42	36.55	36.65	36.78	36.87	37.00	37.11	37.20	37.31
55-59	97.5%	32.28	32.56	32.81	33.07	33.33	33.57	33.80	34.02	34.28	34.52	34.73	34.97	35.19	35.40	35.60	35.84	36.06	36.26	36.48	36.66
	mean	31.58	31.76	31.94	32.11	32.28	32.46	32.63	32.81	32.98	33.16	33.33	33.50	33.68	33.85	34.01	34.18	34.35	34.51	34.67	34.83
	2.5%	30.48	30.69	30.85	30.93	31.03	31.13	31.24	31.36	31.43	31.57	31.67	31.77	31.92	32.00	32.15	32.25	32.34	32.45	32.53	32.65
60-64	97.5%	27.67	27.93	28.18	28.44	28.67	28.90	29.12	29.33	29.56	29.80	30.01	30.24	30.47	30.66	30.86	31.09	31.30	31.49	31.69	31.87
	mean	27.00	27.17	27.34	27.51	27.67	27.84	28.01	28.17	28.34	28.51	28.68	28.84	29.01	29.17	29.33	29.49	29.65	29.80	29.96	30.11
	2.5%	25.87	26.10	26.26	26.34	26.44	26.55	26.65	26.76	26.83	26.96	27.07	27.16	27.32	27.39	27.52	27.60	27.71	27.82	27.88	28.00
65-69	97.5%	23.13	23.38	23.60	23.83	24.05	24.27	24.47	24.67	24.89	25.10	25.31	25.52	25.73	25.92	26.11	26.33	26.51	26.72	26.92	27.07
	mean	22.49	22.65	22.81	22.96	23.12	23.27	23.43	23.59	23.74	23.91	24.06	24.21	24.37	24.52	24.67	24.82	24.97	25.12	25.26	25.41
	2.5%	21.35	21.57	21.72	21.80	21.89	22.01	22.10	22.23	22.30	22.41	22.54	22.57	22.75	22.83	22.92	23.02	23.10	23.23	23.30	23.40
70-74	97.5%	18.71	18.92	19.13	19.33	19.52	19.72	19.91	20.09	20.29	20.49	20.68	20.86	21.06	21.23	21.41	21.61	21.79	21.97	22.16	22.31
	mean	18.08	18.23	18.37	18.50	18.64	18.78	18.92	19.06	19.20	19.35	19.48	19.62	19.76	19.90	20.04	20.17	20.31	20.44	20.57	20.70
	2.5%	16.92	17.13	17.26	17.32	17.43	17.52	17.60	17.69	17.77	17.87	17.98	18.04	18.19	18.22	18.33	18.41	18.48	18.58	18.64	18.74
75-79	97.5%	14.44	14.64	14.81	14.98	15.14	15.32	15.48	15.64	15.82	16.01	16.16	16.32	16.50	16.65	16.82	16.98	17.14	17.30	17.48	17.63
	mean	13.88	14.01	14.12	14.23	14.35	14.46	14.58	14.70	14.81	14.94	15.05	15.16	15.28	15.40	15.51	15.63	15.75	15.86	15.97	16.08
	2.5%	12.68	12.88	12.98	13.03	13.13	13.21	13.30	13.35	13.39	13.48	13.55	13.65	13.75	13.75	13.81	13.89	13.97	14.02	14.08	14.17
80-84	97.5%	10.62	10.76	10.90	11.02	11.16	11.27	11.42	11.54	11.66	11.80	11.94	12.04	12.20	12.32	12.44	12.57	12.71	12.84	12.99	13.11
	mean	10.11	10.10	10.27	10.34	10.43	10.51	10.59	10.66	10.74	10.83	10.91	10.99	11.07	11.15	11.23	11.31	11.40	11.48	11.56	11.64
	2.5%	8.78	8.96	9.02	9.05	9.11	9.18	9.21	9.25	9.28	9.35	9.38	9.43	9.46	9.50	9.53	9.60	9.62	9.64	9.71	9.71
85-89	97.5%	7.41	7.52	7.60	7.69	7.76	7.85	7.92	7.99	8.06	8.16	8.23	8.30	8.39	8.47	8.54	8.63	8.71	8.79	8.88	8.97
	mean	7.00	7.04	7.07	7.09	7.12	7.15	7.18	7.21	7.24	7.28	7.31	7.34	7.37	7.40	7.43	7.47	7.50	7.54	7.57	7.61

Table A.48. The forecasted Spain male life expectancy by model 8

age	97.5%	81.04	81.50	81.92	82.34	82.68	83.03	83.40	83.74	84.11	84.48	84.81	85.16	85.49	85.83	86.20	86.56	86.83	87.17	87.47	87.84
	mean	80.01	80.25	80.48	80.74	80.98	81.22	81.47	81.72	81.97	82.23	82.48	82.73	82.99	83.24	83.49	83.75	84.00	84.26	84.52	84.76
	2.5%	78.87	78.87	78.88	78.89	78.95	79.05	79.14	79.28	79.38	79.51	79.58	79.64	79.83	79.94	80.12	80.17	80.47	80.55	80.67	80.79
1-4	97.5%	80.26	80.70	81.08	81.49	81.83	82.16	82.51	82.85	83.21	83.57	83.88	84.22	84.56	84.88	85.25	85.61	85.87	86.21	86.51	86.86
	mean	79.25	79.48	79.70	79.94	80.17	80.41	80.64	80.88	81.12	81.37	81.61	81.85	82.10	82.35	82.59	82.84	83.09	83.34	83.59	83.83
	2.5%	78.15	78.13	78.15	78.17	78.22	78.30	78.39	78.50	78.62	78.72	78.78	78.84	79.01	79.16	79.29	79.36	79.60	79.71	79.81	79.90
5-9	97.5%	76.29	76.73	77.11	77.51	77.85	78.17	78.53	78.87	79.22	79.58	79.90	80.23	80.57	80.89	81.26	81.62	81.88	82.22	82.51	82.87
	mean	75.29	75.52	75.74	75.97	76.20	76.43	76.67	76.91	77.15	77.39	77.63	77.87	78.12	78.37	78.60	78.85	79.10	79.35	79.60	79.84
	2.5%	74.19	74.18	74.20	74.21	74.26	74.34	74.43	74.53	74.65	74.74	74.81	74.87	75.05	75.19	75.32	75.38	75.62	75.72	75.84	75.93
10-14	97.5%	71.32	71.75	72.13	72.53	72.86	73.19	73.54	73.88	74.24	74.59	74.91	75.24	75.58	75.89	76.26	76.63	76.88	77.22	77.52	77.88
	mean	70.32	70.54	70.76	71.00	71.22	71.45	71.69	71.93	72.16	72.41	72.64	72.89	73.13	73.38	73.61	73.86	74.11	74.36	74.61	74.85
	2.5%	69.23	69.21	69.22	69.25	69.29	69.38	69.45	69.56	69.68	69.77	69.83	69.90	70.07	70.22	70.35	70.40	70.63	70.75	70.86	70.94
15-19	97.5%	66.35	66.78	67.15	67.55	67.88	68.20	68.55	68.89	69.25	69.60	69.91	70.25	70.59	70.90	71.27	71.64	71.89	72.23	72.52	72.88
	mean	65.35	65.57	65.79	66.02	66.25	66.48	66.71	66.95	67.18	67.42	67.66	67.90	68.15	68.39	68.63	68.88	69.13	69.37	69.62	69.86
	2.5%	64.26	64.25	64.25	64.29	64.32	64.41	64.49	64.59	64.71	64.79	64.87	64.94	65.10	65.25	65.38	65.43	65.66	65.78	65.87	65.96
20-24	97.5%	61.40	61.82	62.20	62.58	62.91	63.24	63.58	63.92	64.27	64.62	64.93	65.26	65.60	65.91	66.29	66.64	66.90	67.24	67.53	67.89
	mean	60.41	60.64	60.85	61.08	61.31	61.53	61.76	62.00	62.23	62.47	62.71	62.95	63.19	63.44	63.67	63.92	64.17	64.41	64.66	64.90
	2.5%	59.34	59.33	59.34	59.39	59.43	59.50	59.59	59.69	59.79	59.90	59.96	60.04	60.19	60.32	60.50	60.54	60.77	60.86	60.95	61.08
25-29	97.5%	56.46	56.87	57.24	57.62	57.95	58.27	58.61	58.93	59.28	59.65	59.95	60.27	60.61	60.92	61.29	61.65	61.91	62.25	62.55	62.89
	mean	55.49	55.71	55.92	56.15	56.37	56.59	56.82	57.05	57.28	57.52	57.75	57.99	58.24	58.48	58.71	58.96	59.20	59.44	59.69	59.94
	2.5%	54.43	54.44	54.45	54.49	54.54	54.60	54.70	54.80	54.92	54.99	55.07	55.14	55.31	55.42	55.59	55.63	55.85	55.99	56.08	56.18
30-34	97.5%	51.53	51.93	52.29	52.66	52.99	53.30	53.64	53.97	54.32	54.67	54.97	55.30	55.64	55.94	56.31	56.66	56.92	57.26	57.57	57.92
	mean	50.58	50.80	51.01	51.23	51.45	51.68	51.90	52.13	52.36	52.60	52.83	53.07	53.31	53.56	53.79	54.03	54.28	54.52	54.77	55.01
	2.5%	49.54	49.57	49.58	49.63	49.69	49.77	49.86	49.97	50.10	50.21	50.25	50.35	50.52	50.61	50.79	50.84	51.10	51.22	51.31	51.39
35-39	97.5%	46.63	47.02	47.37	47.73	48.04	48.36	48.69	49.01	49.36	49.71	50.00	50.33	50.67	50.97	51.34	51.69	51.95	52.29	52.60	52.94
	mean	45.69	45.91	46.11	46.34	46.56	46.78	47.01	47.24	47.47	47.70	47.94	48.17	48.42	48.66	48.89	49.14	49.38	49.62	49.87	50.11
	2.5%	44.68	44.72	44.75	44.80	44.87	44.98	45.11	45.20	45.34	45.43	45.51	45.61	45.78	45.86	46.05	46.08	46.35	46.46	46.61	46.70
40-44	97.5%	41.73	42.12	42.45	42.82	43.12	43.42	43.75	44.07	44.40	44.75	45.04	45.38	45.71	46.01	46.37	46.72	46.99	47.32	47.62	47.96
	mean	40.82	41.04	41.24	41.46	41.68	41.90	42.12	42.35	42.58	42.81	43.04	43.27	43.51	43.75	43.98	44.23	44.47	44.71	44.96	45.20
	2.5%	39.84	39.89	39.92	39.98	40.05	40.16	40.26	40.39	40.51	40.59	40.72	40.80	41.01	41.06	41.23	41.31	41.52	41.67	41.80	41.90
45-49	97.5%	36.94	37.29	37.62	37.96	38.27	38.57	38.88	39.21	39.53	39.87	40.16	40.49	40.81	41.10	41.45	41.81	42.07	42.41	42.70	43.03
	mean	36.06	36.27	36.47	36.69	36.90	37.12	37.34	37.56	37.79	38.01	38.24	38.48	38.71	38.95	39.18	39.42	39.66	39.90	40.14	40.38
	2.5%	35.10	35.17	35.24	35.29	35.38	35.50	35.60	35.72	35.83	35.95	36.07	36.16	36.36	36.47	36.59	36.75	36.90	37.06	37.18	37.27
50-54	97.5%	32.30	32.64	32.95	33.28	33.55	33.87	34.15	34.47	34.78	35.11	35.39	35.71	36.01	36.31	36.65	36.97	37.23	37.57	37.89	38.21
	mean	31.46	31.66	31.86	32.07	32.28	32.49	32.71	32.93	33.15	33.37	33.60	33.83	34.06	34.29	34.51	34.75	34.99	35.22	35.46	35.70
	2.5%	30.55	30.63	30.70	30.77	30.87	31.00	31.12	31.26	31.35	31.45	31.60	31.69	31.86	32.00	32.10	32.25	32.40	32.56	32.67	32.80
55-59	97.5%	27.85	28.15	28.44	28.75	29.03	29.30	29.59	29.89	30.18	30.50	30.79	31.07	31.37	31.67	31.98	32.31	32.57	32.91	33.19	33.49
	mean	27.06	27.25	27.45	27.65	27.86	28.06	28.27	28.49	28.70	28.92	29.14	29.36	29.59	29.81	30.03	30.27	30.50	30.73	30.96	31.19
	2.5%	26.20	26.30	26.39	26.47	26.58	26.71	26.82	26.96	27.09	27.19	27.32	27.45	27.62	27.74	27.82	28.02	28.19	28.32	28.49	28.57
60-64	97.5%	23.60	23.88	24.13	24.43	24.69	24.94	25.21	25.48	25.76	26.06	26.33	26.63	26.92	27.20	27.50	27.81	28.05	28.38	28.65	28.93
	mean	22.85	23.04	23.23	23.43	23.63	23.83	24.03	24.24	24.45	24.66	24.87	25.09	25.31	25.53	25.74	25.97	26.19	26.41	26.64	26.86
	2.5%	22.03	22.14	22.24	22.37	22.46	22.61	22.71	22.86	23.04	23.12	23.27	23.40	23.51	23.67	23.77	23.96	24.11	24.28	24.40	24.49
65-69	97.5%	19.59	19.83	20.08	20.34	20.58	20.82	21.07	21.31	21.57	21.81	22.09	22.38	22.62	22.91	23.19	23.46	23.72	24.00	24.25	24.53
	mean	18.90	19.07	19.24	19.43	19.61	19.79	19.98	20.17	20.36	20.55	20.75	20.95	21.14	21.35	21.55	21.75	21.96	22.16	22.37	22.57
	2.5%	18.16	18.26	18.37	18.47	18.57	18.69	18.81	18.97	19.05	19.16	19.31	19.41	19.52	19.65	19.78	19.92	20.05	20.17	20.31	20.44
70-74	97.5%	15.73	15.95	16.17	16.38	16.60	16.81	17.02	17.23	17.45	17.68	17.92	18.16	18.38	18.62	18.88	19.13	19.34	19.61	19.85	20.08
	mean	15.08	15.22	15.37	15.52	15.68	15.83	15.99	16.14	16.30	16.46	16.63	16.80	16.96	17.14	17.30	17.48	17.66	17.83	18.01	18.18
	2.5%	14.37	14.46	14.51	14.61	14.68	14.77	14.84	14.94	15.05	15.12	15.23	15.33	15.38	15.52	15.59	15.69	15.79	15.87	15.99	16.08
75-79	97.5%	12.16	12.33	12.52	12.69	12.86	13.03	13.22	13.38	13.57	13.75	13.94	14.16	14.35	14.57	14.77	14.96				

Table A.49. The forecasted Sweden female life expectancy by model 3

age 0	97.5%	84.59	84.86	85.13	85.28	85.45	85.53	85.59	85.68	85.76	85.84	85.95	86.00	86.05	86.12	86.20	86.21	86.25	86.35	86.37	86.39
	mean	83.71	83.66	83.62	83.58	83.54	83.49	83.43	83.39	83.34	83.29	83.24	83.18	83.13	83.07	83.01	82.95	82.90	82.84	82.78	82.72
	2.5%	82.74	82.25	81.84	81.55	81.21	80.91	80.57	80.37	80.09	79.75	79.52	79.19	78.95	78.66	78.40	78.19	77.93	77.51	77.27	77.04
1-4	97.5%	83.73	83.99	84.26	84.40	84.57	84.63	84.69	84.79	84.87	84.94	85.04	85.08	85.15	85.20	85.29	85.30	85.34	85.43	85.46	85.48
	mean	82.88	82.83	82.79	82.75	82.71	82.66	82.60	82.56	82.51	82.46	82.42	82.36	82.31	82.25	82.19	82.13	82.08	82.03	81.97	81.91
	2.5%	81.94	81.47	81.07	80.78	80.46	80.17	79.83	79.64	79.36	79.04	78.83	78.52	78.26	77.99	77.73	77.52	77.27	76.86	76.64	76.41
5-9	97.5%	79.76	80.01	80.27	80.41	80.58	80.64	80.71	80.80	80.88	80.95	81.05	81.09	81.16	81.22	81.30	81.31	81.35	81.44	81.47	81.49
	mean	78.92	78.87	78.83	78.79	78.75	78.70	78.65	78.61	78.56	78.52	78.47	78.42	78.37	78.31	78.26	78.20	78.15	78.10	78.04	77.99
	2.5%	77.99	77.53	77.14	76.85	76.56	76.28	75.94	75.78	75.50	75.20	75.00	74.69	74.44	74.18	73.93	73.75	73.52	73.13	72.96	72.72
10-14	97.5%	74.78	75.03	75.29	75.43	75.60	75.65	75.72	75.81	75.89	75.96	76.06	76.11	76.17	76.22	76.31	76.32	76.36	76.45	76.47	76.50
	mean	73.95	73.90	73.87	73.83	73.79	73.74	73.69	73.66	73.61	73.57	73.53	73.48	73.43	73.38	73.33	73.27	73.23	73.18	73.13	73.08
	2.5%	73.04	72.60	72.23	71.96	71.67	71.39	71.09	70.93	70.66	70.37	70.20	69.94	69.68	69.46	69.22	69.05	68.86	68.49	68.34	68.16
15-19	97.5%	69.79	70.04	70.30	70.44	70.60	70.66	70.73	70.82	70.90	70.97	71.07	71.12	71.17	71.23	71.31	71.33	71.37	71.45	71.48	71.50
	mean	68.97	68.93	68.89	68.86	68.82	68.77	68.72	68.69	68.64	68.60	68.56	68.51	68.47	68.41	68.37	68.31	68.27	68.22	68.17	68.13
	2.5%	68.07	67.64	67.27	67.01	66.73	66.45	66.16	66.00	65.74	65.46	65.31	65.03	64.80	64.57	64.36	64.19	63.99	63.67	63.53	63.33
20-24	97.5%	64.84	65.08	65.33	65.47	65.63	65.70	65.75	65.84	65.92	65.99	66.09	66.14	66.19	66.26	66.34	66.36	66.39	66.47	66.50	66.52
	mean	64.03	63.99	63.96	63.92	63.88	63.84	63.79	63.76	63.72	63.68	63.64	63.59	63.55	63.50	63.46	63.41	63.36	63.32	63.27	63.23
	2.5%	63.16	62.73	62.38	62.12	61.86	61.60	61.31	61.16	60.91	60.65	60.51	60.25	60.02	59.83	59.59	59.45	59.23	58.93	58.81	58.65
25-29	97.5%	59.89	60.12	60.38	60.51	60.67	60.73	60.79	60.87	60.95	61.03	61.13	61.17	61.22	61.29	61.37	61.38	61.41	61.50	61.53	61.55
	mean	59.11	59.07	59.03	59.00	58.97	58.92	58.88	58.85	58.81	58.77	58.73	58.69	58.65	58.60	58.56	58.51	58.47	58.43	58.39	58.34
	2.5%	58.26	57.85	57.51	57.26	57.01	56.76	56.49	56.34	56.11	55.86	55.71	55.47	55.27	55.05	54.86	54.71	54.49	54.21	54.14	53.96
30-34	97.5%	54.95	55.18	55.42	55.55	55.71	55.77	55.83	55.92	55.99	56.06	56.16	56.21	56.26	56.32	56.40	56.41	56.44	56.53	56.56	56.58
	mean	54.19	54.15	54.12	54.09	54.05	54.02	53.97	53.94	53.90	53.87	53.83	53.79	53.75	53.71	53.67	53.62	53.59	53.55	53.51	53.47
	2.5%	53.36	52.97	52.64	52.41	52.16	51.94	51.68	51.54	51.30	51.09	50.95	50.72	50.52	50.33	50.14	50.00	49.82	49.53	49.47	49.30
35-39	97.5%	50.03	50.25	50.49	50.62	50.77	50.83	50.88	50.97	51.04	51.11	51.20	51.25	51.31	51.37	51.44	51.45	51.48	51.57	51.60	51.62
	mean	49.29	49.25	49.22	49.19	49.16	49.12	49.08	49.05	49.02	48.98	48.95	48.91	48.87	48.83	48.79	48.75	48.72	48.68	48.64	48.61
	2.5%	48.48	48.12	47.81	47.58	47.35	47.13	46.89	46.75	46.51	46.32	46.19	45.97	45.79	45.60	45.43	45.31	45.12	44.89	44.78	44.64
40-44	97.5%	45.11	45.33	45.56	45.68	45.84	45.89	45.94	46.02	46.10	46.17	46.26	46.30	46.36	46.43	46.49	46.50	46.53	46.62	46.65	46.67
	mean	44.39	44.35	44.32	44.30	44.27	44.23	44.19	44.16	44.13	44.10	44.06	44.03	43.99	43.95	43.91	43.87	43.84	43.81	43.77	43.73
	2.5%	43.60	43.25	42.95	42.73	42.51	42.31	42.07	41.93	41.72	41.52	41.39	41.18	41.00	40.84	40.67	40.55	40.36	40.14	40.05	39.90
45-49	97.5%	40.23	40.44	40.67	40.78	40.93	40.98	41.04	41.11	41.19	41.26	41.34	41.39	41.44	41.50	41.57	41.58	41.62	41.69	41.72	41.74
	mean	39.54	39.51	39.48	39.46	39.43	39.39	39.36	39.33	39.30	39.27	39.24	39.20	39.17	39.13	39.10	39.06	39.03	39.00	38.96	38.93
	2.5%	38.79	38.46	38.17	37.96	37.75	37.56	37.33	37.22	37.02	36.83	36.70	36.51	36.35	36.17	36.03	35.91	35.75	35.55	35.46	35.33
50-54	97.5%	35.42	35.62	35.83	35.94	36.08	36.13	36.19	36.25	36.32	36.39	36.46	36.51	36.56	36.63	36.69	36.69	36.74	36.81	36.83	36.85
	mean	34.77	34.74	34.72	34.70	34.67	34.64	34.61	34.58	34.56	34.53	34.50	34.47	34.44	34.41	34.38	34.35	34.32	34.29	34.26	34.23
	2.5%	34.07	33.77	33.51	33.32	33.12	32.96	32.76	32.64	32.46	32.32	32.19	32.02	31.87	31.71	31.56	31.48	31.36	31.16	31.08	30.95
55-59	97.5%	30.70	30.88	31.06	31.17	31.30	31.34	31.40	31.46	31.52	31.58	31.67	31.69	31.74	31.80	31.87	31.88	31.92	31.98	32.00	32.03
	mean	30.11	30.08	30.06	30.05	30.03	30.00	29.98	29.96	29.94	29.92	29.90	29.87	29.85	29.82	29.80	29.77	29.75	29.73	29.70	29.68
	2.5%	29.48	29.22	29.00	28.84	28.68	28.53	28.37	28.26	28.12	27.99	27.88	27.75	27.63	27.50	27.39	27.35	27.20	27.08	27.01	26.91
60-64	97.5%	26.09	26.26	26.42	26.52	26.63	26.67	26.73	26.78	26.84	26.89	26.95	27.01	27.04	27.10	27.14	27.15	27.20	27.26	27.27	27.30
	mean	25.57	25.55	25.54	25.53	25.51	25.49	25.47	25.46	25.44	25.43	25.41	25.39	25.38	25.36	25.34	25.32	25.30	25.29	25.27	25.25
	2.5%	25.02	24.80	24.62	24.49	24.35	24.24	24.11	24.03	23.91	23.80	23.71	23.61	23.52	23.40	23.33	23.27	23.18	23.08	23.04	22.97
65-69	97.5%	21.67	21.82	21.94	22.05	22.14	22.19	22.23	22.28	22.33	22.37	22.43	22.47	22.51	22.55	22.58	22.61	22.66	22.70	22.73	22.74
	mean	21.20	21.19	21.18	21.17	21.15	21.14	21.12	21.11	21.10	21.08	21.07	21.05	21.04	21.02	21.01	20.99	20.98	20.96	20.95	20.93
	2.5%	20.72	20.53	20.38	20.26	20.14	20.04	19.94	19.87	19.76	19.69	19.60	19.51	19.46	19.36	19.28	19.23	19.16	19.08	19.04	18.97
70-74	97.5%	17.48	17.60	17.73	17.80	17.88	17.93	17.97	18.01	18.05	18.09	18.16	18.19	18.23	18.27	18.29	18.31	18.34	18.38	18.42	18.43
	mean	17.06	17.05	17.04	17.03	17.02	17.01	16.99	16.98	16.97	16.96	16.94	16.93	16.92	16.90	16.89	16.87	16.86	16.85	16.84	16.83
	2.5%	16.63	16.47	16.33	16.23	16.13	16.04	15.94	15.89	15.80	15.74	15.66	15.59	15.53	15.43	15.37	15.32	15.27	15.21	15.16	15.11
75-79	97.5%	13.50	13.61	13.71	13.78	13.85	13.89	13.91	13.95	13.99	14.03	14.09	14.11	14.14	14.15	14.18	14.21	14.23	14.27	14.29	14

Table A.50. The forecasted Sweden male life expectancy by model 3

age 0	97.5%	81.18	81.50	81.79	82.00	82.18	82.27	82.35	82.46	82.56	82.66	82.74	82.81	82.85	82.96	83.04	83.07	83.10	83.21	83.24	83.27
	mean	80.10	80.02	79.96	79.90	79.83	79.75	79.67	79.60	79.53	79.45	79.38	79.30	79.21	79.13	79.04	78.95	78.87	78.79	78.70	78.61
	2.5%	78.87	78.24	77.73	77.32	76.86	76.42	76.02	75.70	75.25	74.97	74.55	74.13	73.86	73.37	73.04	72.71	72.33	71.86	71.45	71.18
1-4	97.5%	80.36	80.67	80.95	81.16	81.34	81.42	81.48	81.60	81.71	81.79	81.88	81.94	81.98	82.08	82.16	82.19	82.22	82.33	82.36	82.39
	mean	79.31	79.24	79.18	79.12	79.05	78.97	78.89	78.82	78.75	78.68	78.60	78.52	78.44	78.35	78.27	78.18	78.10	78.02	77.93	77.85
	2.5%	78.11	77.49	77.00	76.61	76.15	75.73	75.33	75.03	74.57	74.28	73.91	73.48	73.21	72.75	72.41	72.08	71.71	71.24	70.85	70.57
5-9	97.5%	76.38	76.70	76.97	77.18	77.36	77.44	77.50	77.61	77.72	77.80	77.89	77.95	77.99	78.10	78.17	78.20	78.23	78.34	78.37	78.40
	mean	75.35	75.27	75.21	75.16	75.09	75.01	74.93	74.87	74.80	74.73	74.65	74.57	74.50	74.41	74.33	74.24	74.16	74.09	74.00	73.92
	2.5%	74.17	73.56	73.08	72.70	72.24	71.84	71.44	71.15	70.70	70.43	70.08	69.66	69.40	68.93	68.62	68.25	67.94	67.51	67.12	66.86
10-14	97.5%	71.39	71.71	71.98	72.19	72.37	72.45	72.51	72.62	72.72	72.81	72.90	72.96	73.00	73.10	73.18	73.20	73.24	73.35	73.38	73.40
	mean	70.37	70.30	70.24	70.18	70.12	70.04	69.97	69.90	69.83	69.76	69.69	69.61	69.54	69.46	69.38	69.29	69.22	69.14	69.06	68.98
	2.5%	69.21	68.61	68.14	67.76	67.31	66.91	66.53	66.25	65.81	65.53	65.21	64.80	64.56	64.06	63.81	63.47	63.16	62.78	62.40	62.14
15-19	97.5%	66.42	66.73	67.00	67.20	67.38	67.46	67.52	67.63	67.74	67.82	67.91	67.97	68.01	68.11	68.18	68.21	68.25	68.36	68.39	68.41
	mean	65.40	65.33	65.27	65.22	65.15	65.08	65.01	64.95	64.87	64.81	64.74	64.66	64.59	64.51	64.43	64.35	64.28	64.20	64.12	64.04
	2.5%	64.25	63.67	63.20	62.83	62.39	62.00	61.64	61.36	60.92	60.68	60.36	59.96	59.73	59.26	59.00	58.66	58.37	58.01	57.60	57.36
20-24	97.5%	61.49	61.79	62.05	62.25	62.43	62.51	62.57	62.67	62.78	62.87	62.95	63.01	63.05	63.15	63.22	63.25	63.29	63.39	63.42	63.46
	mean	60.50	60.43	60.38	60.33	60.26	60.20	60.12	60.07	60.00	59.94	59.87	59.80	59.73	59.66	59.58	59.51	59.44	59.37	59.29	59.22
	2.5%	59.39	58.82	58.37	58.02	57.60	57.23	56.88	56.67	56.21	55.96	55.68	55.29	55.08	54.64	54.39	54.09	53.78	53.45	53.11	52.91
25-29	97.5%	56.63	56.91	57.17	57.36	57.53	57.61	57.67	57.77	57.87	57.95	58.04	58.10	58.14	58.24	58.30	58.32	58.36	58.47	58.50	58.53
	mean	55.69	55.63	55.58	55.53	55.48	55.42	55.35	55.30	55.24	55.18	55.12	55.06	55.00	54.93	54.86	54.79	54.73	54.67	54.60	54.54
	2.5%	54.64	54.11	53.72	53.38	53.01	52.65	52.37	52.10	51.76	51.50	51.24	50.89	50.71	50.33	50.09	49.84	49.52	49.23	48.98	48.75
30-34	97.5%	51.77	52.04	52.29	52.47	52.63	52.71	52.76	52.86	52.96	53.04	53.13	53.18	53.23	53.32	53.37	53.41	53.44	53.54	53.57	53.60
	mean	50.89	50.83	50.79	50.75	50.70	50.64	50.59	50.54	50.49	50.44	50.39	50.33	50.28	50.22	50.16	50.10	50.05	49.99	49.93	49.88
	2.5%	49.90	49.42	49.05	48.76	48.43	48.12	47.87	47.64	47.32	47.10	46.86	46.57	46.36	46.04	45.86	45.61	45.38	45.14	44.90	44.75
35-39	97.5%	46.92	47.17	47.41	47.59	47.73	47.81	47.88	47.96	48.06	48.15	48.22	48.27	48.32	48.40	48.46	48.49	48.52	48.62	48.65	48.68
	mean	46.09	46.04	46.00	45.97	45.92	45.88	45.82	45.79	45.74	45.70	45.65	45.60	45.56	45.50	45.46	45.40	45.36	45.31	45.26	45.21
	2.5%	45.17	44.74	44.42	44.14	43.84	43.56	43.33	43.13	42.85	42.64	42.47	42.20	42.05	41.76	41.60	41.38	41.19	40.97	40.83	40.63
40-44	97.5%	42.05	42.29	42.52	42.69	42.83	42.92	42.97	43.05	43.15	43.23	43.31	43.35	43.40	43.48	43.54	43.57	43.59	43.69	43.73	43.75
	mean	41.26	41.22	41.18	41.15	41.11	41.07	41.02	40.99	40.95	40.91	40.87	40.83	40.78	40.74	40.70	40.65	40.61	40.57	40.52	40.48
	2.5%	40.40	40.01	39.70	39.45	39.18	38.94	38.72	38.54	38.28	38.08	37.95	37.71	37.56	37.30	37.16	37.01	36.79	36.62	36.48	36.30
45-49	97.5%	37.22	37.45	37.66	37.83	37.97	38.05	38.10	38.17	38.27	38.35	38.41	38.46	38.51	38.59	38.65	38.67	38.70	38.79	38.83	38.85
	mean	36.47	36.44	36.41	36.38	36.34	36.30	36.26	36.23	36.19	36.16	36.12	36.08	36.05	36.00	35.97	35.92	35.89	35.85	35.81	35.77
	2.5%	35.67	35.31	35.02	34.77	34.55	34.34	34.13	33.95	33.73	33.56	33.41	33.23	33.06	32.86	32.71	32.54	32.40	32.23	32.10	31.93
50-54	97.5%	32.47	32.68	32.89	33.04	33.17	33.24	33.30	33.37	33.45	33.53	33.60	33.65	33.70	33.76	33.84	33.84	33.86	33.96	33.98	34.00
	mean	31.78	31.75	31.72	31.70	31.67	31.63	31.60	31.57	31.54	31.51	31.48	31.44	31.41	31.37	31.34	31.30	31.27	31.24	31.21	31.17
	2.5%	31.04	30.71	30.46	30.24	30.03	29.84	29.66	29.53	29.33	29.18	29.04	28.85	28.73	28.57	28.43	28.30	28.23	28.02	27.89	27.78
55-59	97.5%	27.84	28.04	28.21	28.36	28.46	28.53	28.60	28.66	28.74	28.80	28.87	28.91	28.96	29.03	29.08	29.10	29.13	29.20	29.22	29.26
	mean	27.22	27.20	27.17	27.16	27.13	27.10	27.07	27.05	27.02	27.00	26.98	26.95	26.92	26.89	26.87	26.83	26.81	26.79	26.76	26.73
	2.5%	26.57	26.28	26.07	25.89	25.72	25.55	25.38	25.27	25.12	24.98	24.86	24.73	24.65	24.47	24.38	24.27	24.19	24.04	23.95	23.83
60-64	97.5%	23.38	23.55	23.71	23.85	23.95	24.00	24.06	24.10	24.17	24.24	24.31	24.34	24.39	24.45	24.50	24.53	24.56	24.62	24.64	24.67
	mean	22.83	22.81	22.79	22.78	22.76	22.73	22.71	22.69	22.67	22.65	22.63	22.60	22.58	22.56	22.53	22.51	22.49	22.47	22.45	22.42
	2.5%	22.26	22.01	21.82	21.66	21.52	21.38	21.24	21.14	20.99	20.90	20.80	20.66	20.60	20.49	20.42	20.32	20.24	20.11	20.05	19.95
65-69	97.5%	19.16	19.31	19.44	19.55	19.64	19.69	19.75	19.79	19.84	19.89	19.95	20.00	20.02	20.07	20.12	20.14	20.17	20.22	20.26	20.29
	mean	18.70	18.68	18.66	18.65	18.64	18.62	18.60	18.58	18.57	18.55	18.54	18.52	18.50	18.48	18.47	18.45	18.43	18.42	18.40	18.38
	2.5%	18.21	18.01	17.86	17.74	17.61	17.50	17.40	17.30	17.20	17.12	17.03	16.94	16.87	16.78	16.71	16.62	16.57	16.48	16.41	16.34
70-74	97.5%	15.22	15.34	15.45	15.55	15.62	15.66	15.71	15.74	15.78	15.82	15.87	15.91	15.93	15.97	16.02	16.02	16.07	16.11	16.13	16.16
	mean	14.83	14.81	14.80	14.79	14.78	14.77	14.75	14.74	14.73	14.72	14.71	14.69	14.68	14.66	14.65	14.64	14.62	14.61	14.60	14.59
	2.5%	14.43	14.26	14.14	14.05	13.95	13.85	13.77	13.71	13.63	13.55	13.49	13.42	13.36	13.28	13.23	13.18	13.12	13.05	13.01	12.98
75-79	97.5%	11.58	11.68	11.76	11.83	11.90	11.91	11.97	11.99	12.02	12.06	12.10	12.13	12.14	12.18	12.22	12.21				

Table A.51. The forecasted Sweden female life expectancy by model 8

age 0	97.5%	84.32	84.58	84.82	85.03	85.15	85.28	85.40	85.52	85.67	85.78	85.86	86.01	86.13	86.19	86.35	86.40	86.44	86.61	86.62	86.72
	mean	83.36	83.36	83.38	83.40	83.40	83.41	83.42	83.44	83.45	83.46	83.47	83.47	83.49	83.49	83.49	83.50	83.51	83.52	83.53	83.54
	2.5%	82.30	82.04	81.81	81.61	81.44	81.27	81.16	81.03	80.87	80.78	80.67	80.61	80.48	80.38	80.30	80.16	80.10	80.07	80.03	79.87
1-4	97.5%	83.48	83.73	83.94	84.14	84.27	84.38	84.51	84.61	84.76	84.87	84.93	85.08	85.19	85.25	85.41	85.46	85.49	85.66	85.66	85.77
	mean	82.53	82.53	82.54	82.55	82.55	82.55	82.57	82.57	82.57	82.58	82.58	82.58	82.59	82.59	82.59	82.59	82.60	82.60	82.61	82.62
	2.5%	81.49	81.24	81.00	80.80	80.64	80.46	80.32	80.19	80.04	79.95	79.82	79.78	79.64	79.54	79.44	79.29	79.24	79.20	79.16	79.00
5-9	97.5%	79.51	79.75	79.97	80.16	80.29	80.40	80.53	80.62	80.77	80.88	80.94	81.09	81.21	81.26	81.42	81.48	81.50	81.66	81.67	81.77
	mean	78.57	78.57	78.57	78.59	78.58	78.58	78.58	78.60	78.60	78.61	78.61	78.61	78.62	78.62	78.61	78.61	78.62	78.63	78.63	78.64
	2.5%	77.54	77.29	77.05	76.86	76.68	76.51	76.37	76.24	76.09	76.01	75.85	75.82	75.69	75.59	75.48	75.34	75.29	75.24	75.20	75.05
10-14	97.5%	74.53	74.77	74.99	75.18	75.31	75.41	75.54	75.64	75.78	75.89	75.96	76.10	76.22	76.27	76.42	76.48	76.51	76.67	76.68	76.78
	mean	73.59	73.60	73.60	73.61	73.61	73.60	73.61	73.62	73.62	73.63	73.63	73.63	73.64	73.64	73.63	73.63	73.64	73.65	73.65	73.66
	2.5%	72.57	72.32	72.09	71.89	71.72	71.56	71.42	71.29	71.14	71.05	70.90	70.87	70.73	70.63	70.51	70.40	70.32	70.30	70.25	70.11
15-19	97.5%	69.55	69.79	70.00	70.19	70.32	70.42	70.55	70.65	70.79	70.90	70.96	71.11	71.23	71.28	71.43	71.49	71.52	71.67	71.68	71.78
	mean	68.62	68.62	68.62	68.63	68.62	68.62	68.63	68.64	68.64	68.65	68.64	68.64	68.65	68.65	68.64	68.65	68.65	68.66	68.66	68.67
	2.5%	67.60	67.35	67.12	66.91	66.74	66.58	66.45	66.31	66.16	66.08	65.93	65.90	65.75	65.66	65.54	65.42	65.35	65.32	65.28	65.13
20-24	97.5%	64.60	64.84	65.05	65.23	65.36	65.46	65.59	65.68	65.83	65.93	65.99	66.13	66.25	66.30	66.45	66.51	66.54	66.69	66.70	66.80
	mean	63.68	63.68	63.68	63.69	63.68	63.68	63.68	63.69	63.69	63.70	63.70	63.70	63.71	63.70	63.69	63.70	63.70	63.71	63.71	63.71
	2.5%	62.68	62.43	62.21	62.00	61.84	61.67	61.55	61.40	61.26	61.17	61.02	60.99	60.84	60.76	60.64	60.53	60.47	60.41	60.38	60.21
25-29	97.5%	59.66	59.89	60.09	60.27	60.40	60.50	60.62	60.71	60.85	60.96	61.02	61.16	61.28	61.32	61.46	61.53	61.56	61.70	61.71	61.82
	mean	58.75	58.75	58.75	58.76	58.75	58.74	58.74	58.75	58.75	58.76	58.75	58.75	58.76	58.76	58.75	58.75	58.75	58.76	58.76	58.76
	2.5%	57.76	57.53	57.30	57.09	56.93	56.76	56.65	56.50	56.38	56.28	56.14	56.09	55.96	55.88	55.75	55.64	55.60	55.53	55.48	55.33
30-34	97.5%	54.73	54.96	55.15	55.33	55.45	55.55	55.67	55.77	55.90	56.00	56.06	56.20	56.31	56.35	56.50	56.56	56.59	56.74	56.74	56.84
	mean	53.83	53.83	53.83	53.84	53.83	53.82	53.82	53.83	53.83	53.83	53.83	53.82	53.83	53.83	53.81	53.82	53.82	53.82	53.82	53.82
	2.5%	52.87	52.63	52.41	52.19	52.05	51.87	51.76	51.62	51.50	51.42	51.25	51.21	51.07	50.98	50.86	50.77	50.72	50.66	50.57	50.46
35-39	97.5%	49.73	49.96	50.15	50.33	50.45	50.55	50.67	50.77	50.90	51.00	51.06	51.20	51.31	51.35	51.50	51.56	51.59	51.74	51.74	51.84
	mean	48.83	48.83	48.83	48.84	48.83	48.82	48.82	48.83	48.83	48.83	48.83	48.82	48.83	48.83	48.81	48.82	48.82	48.82	48.82	48.82
	2.5%	47.87	47.63	47.41	47.19	47.05	46.87	46.76	46.62	46.50	46.42	46.25	46.21	46.07	45.98	45.86	45.77	45.72	45.66	45.57	45.46
40-44	97.5%	44.82	45.05	45.24	45.41	45.53	45.63	45.73	45.83	45.97	46.06	46.12	46.25	46.36	46.39	46.54	46.61	46.64	46.77	46.77	46.87
	mean	43.94	43.93	43.93	43.94	43.92	43.91	43.91	43.92	43.91	43.91	43.91	43.90	43.91	43.90	43.89	43.89	43.89	43.89	43.89	43.89
	2.5%	42.98	42.75	42.52	42.30	42.17	41.99	41.89	41.73	41.62	41.52	41.39	41.31	41.19	41.09	40.97	40.87	40.83	40.77	40.68	40.56
45-49	97.5%	39.94	40.17	40.34	40.51	40.63	40.73	40.82	40.92	41.05	41.14	41.19	41.31	41.42	41.48	41.60	41.67	41.71	41.82	41.83	41.92
	mean	39.08	39.07	39.06	39.07	39.05	39.04	39.03	39.04	39.03	39.03	39.02	39.01	39.01	39.00	38.99	38.99	38.99	38.98	38.98	38.98
	2.5%	38.15	37.91	37.70	37.47	37.34	37.17	37.04	36.91	36.78	36.68	36.54	36.47	36.35	36.25	36.14	36.03	35.93	35.91	35.85	35.72
50-54	97.5%	35.16	35.37	35.53	35.69	35.79	35.89	35.99	36.07	36.19	36.27	36.32	36.45	36.55	36.61	36.71	36.78	36.81	36.94	36.95	37.02
	mean	34.32	34.30	34.29	34.30	34.28	34.27	34.26	34.26	34.25	34.25	34.23	34.23	34.23	34.22	34.20	34.19	34.19	34.19	34.19	34.18
	2.5%	33.43	33.18	32.96	32.77	32.64	32.46	32.35	32.21	32.07	31.98	31.86	31.81	31.68	31.58	31.47	31.38	31.28	31.28	31.17	31.07
55-59	97.5%	30.46	30.66	30.81	30.96	31.06	31.16	31.24	31.31	31.45	31.53	31.58	31.69	31.80	31.83	31.95	32.00	32.06	32.15	32.16	32.27
	mean	29.67	29.66	29.65	29.66	29.64	29.63	29.63	29.63	29.62	29.62	29.61	29.61	29.61	29.60	29.59	29.58	29.58	29.58	29.58	29.57
	2.5%	28.84	28.62	28.40	28.25	28.10	27.97	27.85	27.73	27.62	27.53	27.43	27.35	27.24	27.18	27.06	26.98	26.93	26.85	26.81	26.70
60-64	97.5%	25.91	26.09	26.22	26.35	26.45	26.55	26.62	26.70	26.81	26.90	26.98	27.06	27.14	27.19	27.31	27.38	27.41	27.49	27.51	27.59
	mean	25.17	25.17	25.16	25.16	25.15	25.15	25.15	25.15	25.15	25.15	25.15	25.14	25.14	25.14	25.13	25.13	25.13	25.12	25.13	25.13
	2.5%	24.40	24.20	24.04	23.90	23.74	23.62	23.53	23.45	23.35	23.26	23.18	23.09	23.01	22.95	22.88	22.79	22.71	22.70	22.64	22.52
65-69	97.5%	21.49	21.65	21.76	21.88	21.98	22.06	22.13	22.22	22.30	22.37	22.45	22.53	22.62	22.69	22.76	22.81	22.86	22.95	22.96	23.02
	mean	20.82	20.81	20.81	20.81	20.81	20.81	20.80	20.80	20.80	20.80	20.80	20.80	20.80	20.80	20.79	20.79	20.79	20.79	20.79	20.79
	2.5%	20.11	19.95	19.81	19.66	19.53	19.45	19.36	19.30	19.22	19.14	19.05	18.99	18.88	18.86	18.79	18.69	18.63	18.64	18.58	18.50
70-74	97.5%	17.32	17.47	17.57	17.66	17.75	17.83	17.90	17.96	18.05	18.13	18.18	18.28	18.34	18.39	18.44	18.51	18.54	18.63	18.66	18.70
	mean	16.70	16.70	16.69	16.70	16.69	16.69	16.69	16.68	16.68	16.68	16.68	16.68	16.68	16.67	16.67	16.67	16.67	16.66	16.66	16.66
	2.5%	16.07	15.91	15.78	15.66	15.56	15.47	15.36	15.32	15.22	15.17	15.10	15.05	14.96	14.91	14.83	14.77	14.71	14.69	14.69	14.62
75-79	97.5%	13.41	13.53	13.62	13.70	13.77	13.86	13.91	13.96	14.04	14.11	14.14	14.22	14.29	14.32	14.40	14.43	14.47	14.52	14.56	14.6

Table A.52. The forecasted Sweden male life expectancy by model 8

age 0	97.5%	80.86	81.10	81.33	81.52	81.70	81.84	81.93	82.05	82.18	82.36	82.46	82.54	82.68	82.77	82.88	82.92	83.02	83.09	83.17	83.25
	mean	79.44	79.44	79.42	79.43	79.42	79.41	79.38	79.38	79.38	79.39	79.36	79.35	79.34	79.34	79.32	79.30	79.29	79.29	79.29	79.28
	2.5%	77.58	77.33	77.10	76.93	76.69	76.57	76.34	76.18	76.11	75.92	75.78	75.52	75.47	75.40	75.23	75.05	74.92	74.96	74.82	74.69
1-4	97.5%	80.05	80.28	80.48	80.68	80.85	80.97	81.05	81.17	81.29	81.47	81.55	81.63	81.76	81.84	81.95	81.99	82.09	82.16	82.23	82.29
	mean	78.65	78.65	78.61	78.61	78.60	78.58	78.55	78.54	78.53	78.50	78.48	78.47	78.46	78.44	78.41	78.40	78.39	78.39	78.39	78.37
	2.5%	76.82	76.57	76.33	76.15	75.90	75.79	75.56	75.38	75.29	75.13	74.99	74.72	74.67	74.58	74.41	74.23	74.07	74.10	73.95	73.83
5-9	97.5%	76.08	76.31	76.51	76.71	76.88	76.99	77.08	77.19	77.31	77.48	77.57	77.64	77.78	77.86	77.96	78.00	78.11	78.17	78.24	78.31
	mean	74.70	74.69	74.66	74.66	74.64	74.62	74.59	74.58	74.56	74.57	74.54	74.51	74.50	74.49	74.47	74.44	74.42	74.42	74.42	74.40
	2.5%	72.86	72.62	72.39	72.20	71.95	71.84	71.62	71.43	71.35	71.19	71.03	70.76	70.72	70.63	70.46	70.28	70.14	70.14	69.99	69.88
10-14	97.5%	71.10	71.33	71.53	71.72	71.89	72.00	72.09	72.20	72.32	72.49	72.59	72.65	72.78	72.87	72.97	73.01	73.11	73.17	73.25	73.32
	mean	69.73	69.71	69.68	69.68	69.66	69.63	69.60	69.59	69.58	69.58	69.55	69.53	69.51	69.50	69.48	69.45	69.43	69.43	69.43	69.40
	2.5%	67.91	67.64	67.41	67.23	66.98	66.86	66.64	66.44	66.37	66.21	66.05	65.78	65.73	65.66	65.47	65.30	65.15	65.15	65.01	64.89
15-19	97.5%	66.13	66.36	66.56	66.74	66.92	67.03	67.12	67.23	67.34	67.51	67.60	67.67	67.80	67.89	67.98	68.02	68.12	68.18	68.25	68.32
	mean	64.76	64.75	64.71	64.71	64.69	64.67	64.64	64.62	64.61	64.61	64.58	64.55	64.54	64.53	64.50	64.48	64.46	64.45	64.45	64.43
	2.5%	62.95	62.69	62.47	62.28	62.03	61.91	61.68	61.48	61.41	61.25	61.09	60.82	60.77	60.69	60.50	60.34	60.19	60.20	60.06	59.92
20-24	97.5%	61.22	61.45	61.64	61.83	61.99	62.10	62.18	62.28	62.40	62.58	62.65	62.71	62.85	62.93	63.02	63.07	63.17	63.22	63.30	63.35
	mean	59.87	59.86	59.82	59.82	59.80	59.77	59.73	59.72	59.70	59.70	59.66	59.64	59.62	59.61	59.58	59.56	59.53	59.52	59.52	59.49
	2.5%	58.06	57.82	57.60	57.41	57.17	57.02	56.82	56.62	56.54	56.38	56.22	55.95	55.92	55.82	55.64	55.45	55.31	55.32	55.15	55.03
25-29	97.5%	56.46	56.72	56.95	57.13	57.29	57.44	57.54	57.61	57.76	57.93	58.03	58.11	58.23	58.30	58.41	58.42	58.55	58.63	58.71	58.77
	mean	55.39	55.36	55.33	55.33	55.32	55.30	55.27	55.26	55.25	55.25	55.22	55.20	55.20	55.19	55.16	55.14	55.13	55.12	55.13	55.11
	2.5%	54.24	53.93	53.60	53.39	53.20	52.97	52.76	52.58	52.40	52.29	52.10	51.93	51.77	51.72	51.60	51.38	51.36	51.25	51.11	50.96
30-34	97.5%	51.64	51.89	52.09	52.27	52.44	52.57	52.68	52.75	52.90	53.04	53.13	53.22	53.33	53.39	53.54	53.53	53.66	53.72	53.81	53.87
	mean	50.60	50.58	50.54	50.55	50.53	50.51	50.48	50.47	50.46	50.46	50.44	50.42	50.41	50.40	50.38	50.36	50.34	50.34	50.34	50.33
	2.5%	49.50	49.20	48.87	48.69	48.51	48.28	48.08	47.93	47.73	47.61	47.45	47.31	47.16	47.09	46.99	46.75	46.71	46.60	46.53	46.35
35-39	97.5%	46.80	47.04	47.22	47.40	47.57	47.69	47.79	47.86	48.01	48.16	48.23	48.32	48.42	48.49	48.63	48.65	48.74	48.82	48.90	48.94
	mean	45.79	45.77	45.74	45.74	45.73	45.70	45.68	45.66	45.65	45.65	45.63	45.61	45.60	45.60	45.57	45.55	45.54	45.53	45.54	45.52
	2.5%	44.74	44.45	44.14	43.96	43.77	43.55	43.36	43.21	43.07	42.94	42.79	42.61	42.49	42.41	42.32	42.10	42.06	42.00	41.90	41.77
40-44	97.5%	41.95	42.18	42.35	42.53	42.68	42.79	42.88	42.95	43.09	43.24	43.32	43.40	43.50	43.56	43.69	43.70	43.80	43.87	43.94	43.99
	mean	40.97	40.94	40.91	40.90	40.89	40.86	40.83	40.82	40.80	40.80	40.77	40.75	40.74	40.73	40.70	40.68	40.66	40.66	40.65	40.64
	2.5%	39.96	39.67	39.36	39.19	39.00	38.79	38.59	38.45	38.30	38.17	38.03	37.86	37.72	37.64	37.59	37.32	37.29	37.21	37.11	37.00
45-49	97.5%	36.95	37.18	37.35	37.53	37.68	37.79	37.88	37.95	38.09	38.24	38.32	38.40	38.50	38.56	38.69	38.70	38.80	38.87	38.94	38.99
	mean	35.97	35.94	35.91	35.90	35.89	35.86	35.83	35.82	35.80	35.80	35.77	35.75	35.74	35.73	35.70	35.68	35.66	35.66	35.65	35.64
	2.5%	34.96	34.67	34.36	34.19	34.00	33.79	33.59	33.45	33.30	33.17	33.03	32.86	32.72	32.64	32.59	32.32	32.29	32.21	32.11	32.00
50-54	97.5%	32.23	32.45	32.60	32.78	32.93	33.03	33.13	33.18	33.31	33.45	33.52	33.60	33.69	33.76	33.85	33.88	33.98	34.06	34.11	34.16
	mean	31.30	31.27	31.23	31.22	31.21	31.18	31.15	31.13	31.12	31.11	31.09	31.06	31.05	31.04	31.01	30.99	30.97	30.96	30.96	30.94
	2.5%	30.34	30.05	29.76	29.58	29.43	29.22	28.99	28.89	28.74	28.62	28.48	28.29	28.23	28.15	28.03	27.85	27.83	27.75	27.61	27.46
55-59	97.5%	27.68	27.86	28.01	28.18	28.32	28.42	28.49	28.56	28.70	28.82	28.90	28.96	29.04	29.11	29.20	29.26	29.32	29.41	29.48	29.50
	mean	26.80	26.78	26.74	26.74	26.71	26.68	26.67	26.66	26.66	26.64	26.62	26.61	26.61	26.58	26.56	26.55	26.54	26.55	26.55	26.53
	2.5%	25.90	25.64	25.41	25.23	25.10	24.93	24.77	24.64	24.50	24.42	24.27	24.14	24.04	23.97	23.86	23.73	23.66	23.64	23.54	23.40
60-64	97.5%	23.26	23.43	23.59	23.71	23.84	23.95	24.02	24.10	24.21	24.33	24.40	24.46	24.55	24.59	24.68	24.76	24.82	24.90	24.94	24.98
	mean	22.46	22.45	22.42	22.42	22.41	22.39	22.38	22.37	22.36	22.36	22.35	22.34	22.33	22.32	22.31	22.28	22.28	22.28	22.28	22.27
	2.5%	21.63	21.41	21.21	21.06	20.94	20.80	20.67	20.53	20.41	20.35	20.21	20.12	20.05	19.96	19.87	19.74	19.71	19.66	19.60	19.50
65-69	97.5%	19.11	19.24	19.39	19.51	19.61	19.72	19.79	19.86	19.96	20.06	20.14	20.23	20.29	20.34	20.44	20.50	20.54	20.62	20.69	20.75
	mean	18.38	18.39	18.37	18.38	18.37	18.36	18.36	18.35	18.35	18.35	18.35	18.35	18.35	18.34	18.33	18.33	18.32	18.32	18.33	18.32
	2.5%	17.66	17.51	17.34	17.21	17.11	17.00	16.89	16.78	16.67	16.61	16.52	16.43	16.40	16.33	16.24	16.17	16.12	16.06	16.04	15.94
70-74	97.5%	15.17	15.30	15.43	15.54	15.62	15.71	15.76	15.85	15.92	16.00	16.07	16.14	16.22	16.26	16.33	16.39	16.46	16.51	16.56	16.59
	mean	14.53	14.54	14.53	14.53	14.52	14.52	14.52	14.52	14.51	14.52	14.51	14.51	14.51	14.51	14.50	14.50	14.50	14.50	14.50	14.50
	2.5%	13.91	13.77	13.63	13.54	13.44	13.34	13.25	13.19	13.07	13.02	12.95	12.88	12.83	12.81	12.75	12.66	12.60	12.56	12.51	12.45
75-79	97.5%	11.62	11.73	11.81	11.92	11.99	12.07	12.12	12.19	12.26	12.32	12.38	12.43	12.49	12.54	12.56	12.63				

Table A.53. The forecasted Switzerland female life expectancy by model 3

age 0	97.5%	85.19	85.42	85.61	85.78	85.91	85.98	86.06	86.12	86.20	86.28	86.37	86.44	86.49	86.55	86.64	86.66	86.73	86.78	86.82	86.85
	mean	84.50	84.48	84.46	84.44	84.42	84.40	84.37	84.35	84.33	84.31	84.29	84.26	84.24	84.21	84.18	84.15	84.13	84.11	84.08	84.05
	2.5%	83.76	83.43	83.15	82.97	82.75	82.55	82.38	82.23	82.04	81.88	81.72	81.55	81.44	81.25	81.09	80.97	80.85	80.64	80.49	80.40
1-4	97.5%	84.46	84.68	84.85	85.01	85.13	85.20	85.27	85.34	85.41	85.49	85.57	85.63	85.68	85.74	85.82	85.84	85.91	85.97	86.00	86.02
	mean	83.80	83.78	83.77	83.75	83.73	83.71	83.68	83.66	83.64	83.62	83.60	83.58	83.55	83.53	83.50	83.47	83.45	83.43	83.41	83.38
	2.5%	83.11	82.80	82.54	82.36	82.15	81.98	81.82	81.68	81.49	81.36	81.18	81.03	80.93	80.77	80.59	80.49	80.38	80.23	80.06	79.96
5-9	97.5%	80.49	80.71	80.87	81.03	81.15	81.22	81.29	81.36	81.43	81.50	81.59	81.65	81.70	81.76	81.84	81.86	81.93	81.98	82.02	82.03
	mean	79.84	79.82	79.81	79.79	79.77	79.75	79.72	79.71	79.69	79.67	79.65	79.62	79.60	79.58	79.55	79.53	79.50	79.48	79.46	79.44
	2.5%	79.17	78.86	78.60	78.43	78.22	78.06	77.89	77.77	77.58	77.45	77.28	77.13	77.04	76.88	76.72	76.59	76.50	76.37	76.19	76.10
10-14	97.5%	75.50	75.72	75.89	76.04	76.16	76.23	76.31	76.37	76.44	76.52	76.60	76.67	76.71	76.77	76.85	76.87	76.94	76.99	77.03	77.04
	mean	74.87	74.84	74.83	74.82	74.80	74.78	74.75	74.73	74.71	74.69	74.68	74.65	74.63	74.60	74.58	74.55	74.53	74.51	74.49	74.47
	2.5%	74.20	73.89	73.64	73.46	73.26	73.11	72.94	72.81	72.63	72.50	72.34	72.19	72.10	71.94	71.78	71.66	71.57	71.43	71.27	71.19
15-19	97.5%	70.53	70.74	70.91	71.06	71.18	71.25	71.33	71.39	71.46	71.53	71.62	71.68	71.73	71.79	71.86	71.89	71.95	72.01	72.04	72.05
	mean	69.90	69.87	69.86	69.85	69.83	69.81	69.78	69.77	69.75	69.73	69.71	69.69	69.66	69.64	69.62	69.59	69.57	69.55	69.53	69.51
	2.5%	69.24	68.93	68.69	68.52	68.32	68.16	68.00	67.87	67.70	67.57	67.41	67.26	67.18	67.01	66.87	66.74	66.66	66.51	66.35	66.26
20-24	97.5%	65.57	65.78	65.93	66.09	66.21	66.27	66.35	66.41	66.48	66.55	66.64	66.71	66.75	66.81	66.88	66.91	66.97	67.02	67.06	67.07
	mean	64.94	64.92	64.91	64.89	64.88	64.85	64.83	64.82	64.80	64.78	64.76	64.74	64.72	64.69	64.67	64.65	64.63	64.61	64.58	64.56
	2.5%	64.29	64.00	63.75	63.58	63.40	63.24	63.08	62.95	62.78	62.65	62.52	62.37	62.28	62.12	61.98	61.87	61.79	61.66	61.49	61.41
25-29	97.5%	60.61	60.81	60.97	61.12	61.23	61.30	61.38	61.44	61.51	61.58	61.67	61.73	61.77	61.83	61.90	61.93	61.99	62.04	62.08	62.09
	mean	60.00	59.98	59.96	59.95	59.94	59.92	59.89	59.88	59.86	59.85	59.83	59.81	59.79	59.77	59.75	59.72	59.71	59.69	59.67	59.65
	2.5%	59.36	59.08	58.85	58.68	58.51	58.36	58.22	58.09	57.92	57.82	57.68	57.54	57.45	57.30	57.17	57.08	57.01	56.87	56.76	56.66
30-34	97.5%	55.65	55.85	56.01	56.16	56.27	56.33	56.41	56.46	56.54	56.61	56.70	56.76	56.80	56.85	56.93	56.95	57.01	57.07	57.10	57.11
	mean	55.06	55.04	55.03	55.02	55.00	54.98	54.96	54.95	54.93	54.92	54.90	54.89	54.87	54.85	54.83	54.81	54.79	54.78	54.76	54.74
	2.5%	54.44	54.17	53.95	53.80	53.64	53.50	53.36	53.24	53.08	52.98	52.87	52.73	52.65	52.53	52.40	52.31	52.25	52.12	52.00	51.93
35-39	97.5%	50.71	50.90	51.06	51.20	51.31	51.37	51.45	51.50	51.57	51.64	51.73	51.80	51.83	51.88	51.96	51.98	52.04	52.10	52.13	52.14
	mean	50.13	50.12	50.11	50.10	50.08	50.07	50.05	50.04	50.02	50.01	49.99	49.98	49.96	49.94	49.93	49.91	49.89	49.88	49.86	49.85
	2.5%	49.54	49.28	49.07	48.92	48.77	48.65	48.53	48.39	48.26	48.17	48.06	47.93	47.85	47.73	47.64	47.55	47.48	47.36	47.26	47.23
40-44	97.5%	45.79	45.98	46.13	46.27	46.37	46.44	46.51	46.57	46.63	46.71	46.79	46.85	46.90	46.93	47.01	47.04	47.09	47.14	47.18	47.20
	mean	45.23	45.22	45.21	45.20	45.19	45.17	45.15	45.14	45.13	45.12	45.10	45.09	45.07	45.06	45.04	45.02	45.01	45.00	44.98	44.97
	2.5%	44.66	44.41	44.21	44.07	43.93	43.80	43.68	43.57	43.43	43.36	43.25	43.12	43.04	42.94	42.85	42.78	42.70	42.58	42.51	42.47
45-49	97.5%	40.92	41.10	41.25	41.39	41.48	41.54	41.62	41.67	41.73	41.80	41.88	41.95	41.99	42.03	42.11	42.12	42.18	42.24	42.26	42.29
	mean	40.38	40.36	40.35	40.34	40.33	40.32	40.30	40.29	40.28	40.26	40.25	40.24	40.22	40.21	40.19	40.17	40.16	40.15	40.14	40.12
	2.5%	39.81	39.58	39.39	39.25	39.11	39.00	38.88	38.78	38.65	38.56	38.47	38.34	38.28	38.17	38.08	38.02	37.93	37.83	37.76	37.70
50-54	97.5%	36.11	36.29	36.43	36.56	36.66	36.72	36.79	36.84	36.89	36.97	37.04	37.12	37.14	37.20	37.26	37.28	37.33	37.39	37.42	37.43
	mean	35.58	35.57	35.56	35.55	35.54	35.53	35.51	35.50	35.49	35.48	35.47	35.45	35.44	35.42	35.41	35.39	35.38	35.37	35.35	35.34
	2.5%	35.04	34.81	34.64	34.50	34.37	34.26	34.13	34.04	33.92	33.83	33.75	33.63	33.56	33.47	33.38	33.33	33.24	33.14	33.06	33.02
55-59	97.5%	31.40	31.57	31.71	31.83	31.92	31.98	32.05	32.09	32.15	32.22	32.30	32.36	32.39	32.44	32.50	32.52	32.58	32.63	32.65	32.67
	mean	30.89	30.88	30.87	30.86	30.85	30.84	30.82	30.81	30.80	30.79	30.78	30.76	30.75	30.74	30.72	30.71	30.70	30.68	30.67	30.66
	2.5%	30.36	30.15	29.98	29.85	29.72	29.61	29.50	29.42	29.30	29.21	29.13	29.01	28.95	28.86	28.78	28.72	28.65	28.55	28.47	28.45
60-64	97.5%	26.81	26.97	27.10	27.21	27.30	27.35	27.41	27.46	27.51	27.59	27.66	27.71	27.74	27.79	27.85	27.87	27.91	27.96	28.00	28.02
	mean	26.32	26.31	26.30	26.30	26.29	26.27	26.26	26.25	26.24	26.23	26.22	26.20	26.19	26.18	26.16	26.15	26.14	26.13	26.11	26.10
	2.5%	25.82	25.61	25.47	25.34	25.22	25.12	25.00	24.93	24.82	24.74	24.65	24.54	24.49	24.41	24.32	24.28	24.21	24.11	24.06	24.01
65-69	97.5%	22.33	22.47	22.60	22.70	22.78	22.83	22.89	22.93	22.98	23.05	23.11	23.17	23.20	23.24	23.29	23.31	23.37	23.40	23.43	23.45
	mean	21.87	21.86	21.85	21.85	21.84	21.82	21.81	21.80	21.79	21.78	21.77	21.76	21.75	21.73	21.72	21.71	21.70	21.69	21.67	21.66
	2.5%	21.39	21.20	21.06	20.95	20.83	20.75	20.64	20.58	20.47	20.39	20.31	20.21	20.17	20.08	20.01	19.97	19.90	19.80	19.75	19.70
70-74	97.5%	18.01	18.13	18.25	18.33	18.41	18.45	18.50	18.55	18.59	18.64	18.69	18.75	18.78	18.82	18.87	18.88	18.93	18.96	18.99	19.01
	mean	17.60	17.59	17.58	17.58	17.57	17.56	17.54	17.54	17.53	17.52	17.51	17.50	17.49	17.48	17.47	17.45	17.44	17.44	17.43	17.42
	2.5%	17.18	17.02	16.89	16.79	16.69	16.62	16.52	16.45	16.37	16.30	16.23	16.15	16.10	16.02	15.95	15.93	15.85	15.80	15.76	15.71
75-79	97.5%	13.86	13.97	14.06	14.14	14.21	14.24	14.28	14.32	14.36	14.41	14.45	14.49	14.52	14.57	14.59	14.61	14.65	14.67	14.70	14.7

Table A.54. The forecasted Switzerland male life expectancy by model 3

age	97.5%	81.13	81.39	81.64	81.79	81.97	82.04	82.14	82.24	82.33	82.43	82.51	82.59	82.65	82.71	82.81	82.82	82.88	82.97	82.98	83.06
	mean	80.28	80.24	80.21	80.18	80.15	80.11	80.06	80.03	79.99	79.95	79.91	79.87	79.83	79.78	79.74	79.69	79.64	79.60	79.55	79.51
	2.5%	79.37	78.94	78.55	78.29	78.00	77.70	77.45	77.22	76.97	76.75	76.50	76.29	75.97	75.77	75.54	75.27	75.08	74.93	74.62	74.38
1-4	97.5%	80.44	80.67	80.91	81.05	81.23	81.29	81.39	81.48	81.58	81.66	81.74	81.82	81.90	81.93	82.02	82.06	82.09	82.18	82.19	82.26
	mean	79.62	79.59	79.56	79.53	79.49	79.46	79.41	79.38	79.34	79.31	79.27	79.23	79.19	79.14	79.10	79.05	79.01	78.97	78.92	78.88
	2.5%	78.76	78.35	77.98	77.74	77.45	77.19	76.91	76.73	76.48	76.26	76.01	75.84	75.52	75.35	75.11	74.84	74.68	74.52	74.23	74.00
5-9	97.5%	76.47	76.70	76.93	77.07	77.25	77.31	77.41	77.50	77.60	77.68	77.77	77.84	77.91	77.95	78.04	78.07	78.10	78.20	78.21	78.28
	mean	75.66	75.63	75.60	75.57	75.54	75.50	75.46	75.43	75.39	75.35	75.32	75.27	75.23	75.19	75.15	75.10	75.06	75.02	74.97	74.93
	2.5%	74.81	74.41	74.05	73.80	73.52	73.25	72.99	72.80	72.56	72.34	72.11	71.93	71.61	71.45	71.23	70.94	70.78	70.64	70.35	70.10
10-14	97.5%	71.49	71.72	71.95	72.09	72.27	72.33	72.43	72.51	72.62	72.69	72.78	72.86	72.93	72.96	73.05	73.09	73.12	73.20	73.22	73.29
	mean	70.69	70.65	70.63	70.60	70.57	70.53	70.49	70.46	70.42	70.38	70.35	70.31	70.27	70.22	70.18	70.13	70.09	70.05	70.01	69.96
	2.5%	69.84	69.45	69.09	68.85	68.56	68.30	68.05	67.86	67.61	67.39	67.17	66.98	66.68	66.51	66.30	66.02	65.85	65.69	65.43	65.17
15-19	97.5%	66.51	66.75	66.97	67.11	67.28	67.35	67.45	67.53	67.64	67.71	67.79	67.87	67.94	67.97	68.06	68.10	68.13	68.22	68.23	68.30
	mean	65.72	65.68	65.66	65.63	65.60	65.56	65.52	65.49	65.45	65.42	65.38	65.34	65.30	65.25	65.21	65.17	65.13	65.09	65.04	65.00
	2.5%	64.88	64.49	64.13	63.89	63.61	63.34	63.10	62.92	62.66	62.45	62.23	62.04	61.74	61.58	61.38	61.10	60.92	60.80	60.52	60.25
20-24	97.5%	61.59	61.82	62.04	62.18	62.34	62.40	62.50	62.59	62.70	62.76	62.84	62.92	62.98	63.02	63.11	63.14	63.17	63.26	63.27	63.35
	mean	60.82	60.79	60.76	60.74	60.71	60.67	60.63	60.60	60.57	60.54	60.50	60.46	60.43	60.39	60.35	60.30	60.27	60.23	60.19	60.15
	2.5%	60.01	59.63	59.29	59.06	58.79	58.55	58.31	58.13	57.89	57.69	57.46	57.30	57.04	56.85	56.69	56.43	56.23	56.12	55.88	55.61
25-29	97.5%	56.67	56.89	57.11	57.24	57.40	57.47	57.56	57.63	57.75	57.81	57.88	57.97	58.02	58.07	58.16	58.17	58.21	58.29	58.32	58.38
	mean	55.93	55.90	55.88	55.86	55.83	55.80	55.77	55.74	55.71	55.68	55.66	55.62	55.59	55.55	55.52	55.48	55.45	55.42	55.38	55.34
	2.5%	55.16	54.82	54.51	54.28	54.04	53.82	53.60	53.44	53.23	53.05	52.84	52.68	52.50	52.31	52.16	51.94	51.80	51.66	51.46	51.27
30-34	97.5%	51.77	51.98	52.18	52.31	52.46	52.52	52.62	52.69	52.80	52.86	52.94	53.02	53.07	53.11	53.21	53.22	53.27	53.34	53.37	53.43
	mean	51.07	51.04	51.03	51.01	50.99	50.96	50.94	50.92	50.89	50.87	50.85	50.82	50.80	50.77	50.74	50.71	50.69	50.67	50.64	50.61
	2.5%	50.34	50.03	49.76	49.57	49.38	49.19	49.01	48.87	48.69	48.54	48.37	48.28	48.10	47.96	47.82	47.68	47.56	47.43	47.31	47.13
35-39	97.5%	46.86	47.06	47.26	47.39	47.53	47.60	47.68	47.75	47.85	47.92	48.00	48.08	48.12	48.16	48.25	48.27	48.32	48.39	48.43	48.47
	mean	46.20	46.18	46.17	46.16	46.14	46.12	46.10	46.08	46.06	46.05	46.03	46.01	45.99	45.97	45.95	45.92	45.90	45.89	45.86	45.84
	2.5%	45.53	45.25	45.03	44.85	44.68	44.50	44.36	44.23	44.09	43.97	43.83	43.74	43.62	43.47	43.36	43.27	43.19	43.07	42.95	42.81
40-44	97.5%	41.99	42.18	42.37	42.49	42.63	42.69	42.77	42.83	42.94	43.00	43.08	43.16	43.19	43.24	43.32	43.34	43.39	43.46	43.50	43.56
	mean	41.36	41.34	41.33	41.32	41.31	41.29	41.27	41.26	41.24	41.23	41.21	41.19	41.18	41.16	41.14	41.12	41.11	41.09	41.07	41.04
	2.5%	40.72	40.46	40.25	40.09	39.94	39.80	39.66	39.56	39.44	39.31	39.20	39.10	39.02	38.87	38.78	38.69	38.63	38.54	38.44	38.37
45-49	97.5%	37.19	37.38	37.55	37.67	37.80	37.85	37.92	37.99	38.09	38.15	38.23	38.31	38.34	38.38	38.46	38.48	38.54	38.60	38.64	38.69
	mean	36.58	36.57	36.56	36.55	36.53	36.52	36.50	36.49	36.47	36.46	36.45	36.43	36.41	36.40	36.38	36.36	36.35	36.33	36.32	36.30
	2.5%	35.97	35.73	35.53	35.37	35.24	35.10	34.98	34.88	34.76	34.65	34.54	34.45	34.37	34.22	34.16	34.08	34.01	33.97	33.84	33.77
50-54	97.5%	32.47	32.65	32.81	32.93	33.06	33.11	33.17	33.25	33.34	33.39	33.47	33.54	33.57	33.62	33.70	33.71	33.79	33.83	33.87	33.90
	mean	31.89	31.87	31.86	31.85	31.84	31.82	31.81	31.80	31.78	31.77	31.76	31.74	31.73	31.71	31.69	31.67	31.66	31.65	31.63	31.62
	2.5%	31.29	31.06	30.86	30.72	30.60	30.47	30.35	30.24	30.14	30.03	29.93	29.84	29.77	29.65	29.56	29.50	29.43	29.36	29.26	29.20
55-59	97.5%	27.90	28.07	28.22	28.34	28.45	28.50	28.58	28.64	28.73	28.79	28.85	28.92	28.96	29.00	29.07	29.09	29.14	29.20	29.24	29.27
	mean	27.34	27.32	27.31	27.31	27.29	27.28	27.26	27.25	27.24	27.23	27.21	27.20	27.18	27.17	27.15	27.13	27.12	27.11	27.09	27.08
	2.5%	26.76	26.55	26.36	26.23	26.10	25.98	25.87	25.78	25.67	25.57	25.47	25.38	25.31	25.19	25.12	25.04	24.99	24.92	24.83	24.76
60-64	97.5%	23.53	23.70	23.83	23.94	24.04	24.10	24.17	24.22	24.30	24.35	24.42	24.48	24.51	24.54	24.61	24.63	24.68	24.72	24.76	24.80
	mean	23.01	23.00	22.99	22.98	22.97	22.96	22.94	22.93	22.92	22.91	22.90	22.88	22.87	22.86	22.84	22.83	22.82	22.80	22.79	22.78
	2.5%	22.48	22.29	22.12	21.99	21.88	21.78	21.66	21.58	21.49	21.40	21.30	21.24	21.15	21.05	20.98	20.91	20.87	20.80	20.72	20.64
65-69	97.5%	19.38	19.53	19.64	19.75	19.83	19.89	19.93	19.99	20.06	20.11	20.18	20.23	20.26	20.30	20.34	20.37	20.42	20.45	20.49	20.52
	mean	18.91	18.90	18.89	18.88	18.87	18.86	18.85	18.84	18.83	18.82	18.81	18.79	18.78	18.77	18.76	18.74	18.73	18.72	18.71	18.70
	2.5%	18.43	18.25	18.10	17.99	17.88	17.80	17.69	17.62	17.54	17.47	17.37	17.31	17.24	17.17	17.10	17.05	17.01	16.93	16.87	16.83
70-74	97.5%	15.45	15.57	15.67	15.75	15.83	15.89	15.93	15.97	16.03	16.08	16.12	16.16	16.18	16.24	16.27	16.30	16.33	16.37	16.40	16.40
	mean	15.03	15.02	15.01	15.01	15.00	14.99	14.98	14.97	14.96	14.95	14.94	14.93	14.92	14.91	14.90	14.89	14.88	14.87	14.86	14.85
	2.5%	14.61	14.46	14.33	14.24	14.15	14.07	13.97	13.92	13.85	13.78	13.72	13.66	13.60	13.55	13.50	13.42	13.38	13.35	13.29	13.23
75-79	97.5%	11.72	11.83	11.91	11.98	12.05	12.08	12.13	12.17	12.21	12.25	12.30	12.32	12.35	12.38	12.43	12.44</				

Table A.55. The forecasted Switzerland female life expectancy by model 8

age	97.5%	85.05	85.37	85.66	85.96	86.25	86.51	86.79	87.05	87.31	87.60	87.85	88.09	88.35	88.61	88.86	89.10	89.34	89.58	89.80	90.04
0	mean	84.48	84.70	84.92	85.14	85.36	85.58	85.80	86.03	86.25	86.46	86.68	86.91	87.13	87.34	87.56	87.78	87.99	88.21	88.42	88.63
	2.5%	83.88	83.98	84.12	84.24	84.37	84.51	84.68	84.86	85.02	85.19	85.35	85.51	85.70	85.88	86.08	86.27	86.42	86.62	86.80	86.98
1-4	97.5%	84.34	84.62	84.91	85.20	85.45	85.71	85.98	86.23	86.49	86.75	87.00	87.24	87.49	87.73	87.97	88.20	88.45	88.68	88.89	89.13
	mean	83.79	84.00	84.21	84.42	84.63	84.84	85.05	85.27	85.48	85.69	85.90	86.12	86.33	86.53	86.74	86.95	87.16	87.37	87.58	87.78
	2.5%	83.23	83.33	83.47	83.59	83.73	83.85	84.02	84.18	84.34	84.51	84.67	84.83	85.00	85.17	85.36	85.53	85.71	85.91	86.05	86.22
5-9	97.5%	80.37	80.66	80.93	81.22	81.47	81.73	81.99	82.25	82.50	82.77	83.01	83.25	83.50	83.74	83.98	84.20	84.46	84.69	84.90	85.14
	mean	79.83	80.04	80.25	80.46	80.66	80.87	81.08	81.30	81.51	81.71	81.92	82.14	82.35	82.55	82.76	82.97	83.18	83.39	83.59	83.80
	2.5%	79.28	79.38	79.51	79.63	79.77	79.90	80.07	80.23	80.38	80.55	80.71	80.87	81.03	81.21	81.39	81.56	81.73	81.93	82.07	82.25
10-14	97.5%	75.39	75.68	75.95	76.24	76.49	76.75	77.00	77.26	77.51	77.78	78.02	78.26	78.51	78.75	78.99	79.21	79.46	79.69	79.90	80.14
	mean	74.86	75.07	75.27	75.48	75.69	75.89	76.10	76.32	76.52	76.73	76.94	77.15	77.36	77.57	77.77	77.98	78.19	78.40	78.61	78.81
	2.5%	74.31	74.41	74.54	74.67	74.80	74.93	75.10	75.26	75.41	75.58	75.74	75.90	76.06	76.24	76.42	76.59	76.75	76.96	77.09	77.27
15-19	97.5%	70.43	70.71	70.98	71.26	71.51	71.77	72.02	72.28	72.53	72.79	73.03	73.27	73.52	73.76	74.00	74.22	74.47	74.70	74.91	75.15
	mean	69.90	70.11	70.31	70.51	70.72	70.92	71.13	71.35	71.55	71.76	71.97	72.18	72.39	72.59	72.80	73.01	73.21	73.42	73.63	73.83
	2.5%	69.36	69.46	69.59	69.71	69.85	69.98	70.15	70.31	70.46	70.63	70.79	70.94	71.11	71.29	71.46	71.63	71.80	72.00	72.14	72.31
20-24	97.5%	65.47	65.74	66.01	66.29	66.54	66.79	67.05	67.30	67.55	67.81	68.05	68.28	68.53	68.77	69.01	69.23	69.49	69.71	69.92	70.16
	mean	64.94	65.15	65.35	65.56	65.76	65.97	66.17	66.38	66.59	66.79	67.00	67.21	67.42	67.62	67.83	68.04	68.24	68.45	68.65	68.85
	2.5%	64.41	64.52	64.65	64.77	64.91	65.05	65.22	65.37	65.52	65.69	65.84	66.00	66.16	66.35	66.52	66.69	66.86	67.05	67.20	67.36
25-29	97.5%	60.52	60.79	61.05	61.33	61.58	61.83	62.08	62.33	62.57	62.83	63.07	63.30	63.55	63.79	64.03	64.25	64.50	64.72	64.93	65.17
	mean	60.00	60.21	60.41	60.61	60.81	61.01	61.22	61.43	61.64	61.84	62.05	62.26	62.46	62.66	62.87	63.07	63.28	63.49	63.69	63.89
	2.5%	59.48	59.60	59.72	59.85	59.98	60.13	60.29	60.45	60.60	60.77	60.93	61.08	61.24	61.44	61.60	61.79	61.94	62.12	62.28	62.44
30-34	97.5%	55.56	55.82	56.09	56.36	56.61	56.85	57.10	57.35	57.59	57.85	58.09	58.32	58.57	58.80	59.05	59.26	59.51	59.73	59.94	60.18
	mean	55.05	55.26	55.45	55.65	55.86	56.06	56.26	56.47	56.67	56.88	57.08	57.29	57.49	57.70	57.90	58.10	58.31	58.51	58.71	58.91
	2.5%	54.54	54.66	54.78	54.90	55.05	55.20	55.35	55.52	55.66	55.83	55.99	56.13	56.30	56.50	56.66	56.84	56.99	57.18	57.33	57.50
35-39	97.5%	50.62	50.88	51.14	51.40	51.65	51.90	52.14	52.38	52.62	52.88	53.12	53.35	53.60	53.83	54.07	54.28	54.53	54.75	54.97	55.20
	mean	50.12	50.32	50.52	50.72	50.92	51.12	51.32	51.53	51.73	51.93	52.13	52.34	52.54	52.74	52.94	53.15	53.35	53.55	53.75	53.95
	2.5%	49.62	49.74	49.87	49.99	50.14	50.28	50.43	50.60	50.75	50.91	51.08	51.21	51.39	51.58	51.74	51.92	52.07	52.25	52.40	52.56
40-44	97.5%	45.70	45.96	46.21	46.47	46.72	46.96	47.20	47.44	47.68	47.93	48.17	48.39	48.64	48.88	49.11	49.32	49.57	49.78	50.00	50.23
	mean	45.22	45.42	45.61	45.81	46.00	46.20	46.40	46.60	46.80	47.00	47.20	47.41	47.61	47.81	48.01	48.21	48.41	48.61	48.81	49.01
	2.5%	44.73	44.85	44.98	45.10	45.25	45.39	45.54	45.71	45.85	46.02	46.18	46.31	46.49	46.67	46.83	47.01	47.16	47.33	47.50	47.66
45-49	97.5%	40.84	41.09	41.33	41.59	41.83	42.06	42.30	42.53	42.77	43.02	43.25	43.47	43.71	43.95	44.18	44.39	44.63	44.85	45.06	45.29
	mean	40.36	40.56	40.75	40.94	41.14	41.33	41.52	41.72	41.92	42.12	42.31	42.52	42.71	42.91	43.10	43.30	43.50	43.70	43.90	44.09
	2.5%	39.88	40.01	40.13	40.26	40.40	40.55	40.71	40.86	41.01	41.17	41.33	41.46	41.64	41.81	41.97	42.14	42.31	42.48	42.62	42.79
50-54	97.5%	36.04	36.29	36.52	36.76	37.00	37.22	37.46	37.69	37.91	38.16	38.38	38.60	38.83	39.06	39.29	39.50	39.73	39.95	40.16	40.38
	mean	35.57	35.77	35.95	36.14	36.32	36.51	36.70	36.90	37.09	37.28	37.47	37.67	37.86	38.06	38.25	38.44	38.64	38.83	39.02	39.21
	2.5%	35.11	35.23	35.35	35.48	35.61	35.76	35.91	36.06	36.21	36.37	36.52	36.66	36.82	37.00	37.15	37.32	37.47	37.64	37.79	37.95
55-59	97.5%	31.32	31.56	31.78	32.02	32.24	32.46	32.68	32.92	33.14	33.37	33.59	33.79	34.02	34.24	34.47	34.67	34.89	35.10	35.30	35.53
	mean	30.87	31.06	31.23	31.41	31.60	31.78	31.96	32.15	32.34	32.52	32.71	32.90	33.08	33.27	33.46	33.65	33.83	34.02	34.21	34.39
	2.5%	30.41	30.53	30.65	30.77	30.90	31.04	31.20	31.34	31.48	31.63	31.78	31.91	32.08	32.25	32.40	32.57	32.72	32.88	33.03	33.19
60-64	97.5%	26.75	26.97	27.18	27.41	27.62	27.84	28.04	28.27	28.47	28.69	28.89	29.10	29.32	29.53	29.75	29.94	30.16	30.36	30.55	30.76
	mean	26.31	26.49	26.66	26.83	27.01	27.18	27.36	27.54	27.71	27.89	28.07	28.25	28.43	28.61	28.79	28.97	29.15	29.33	29.51	29.69
	2.5%	25.87	26.00	26.11	26.22	26.36	26.49	26.63	26.77	26.92	27.05	27.20	27.32	27.48	27.65	27.79	27.95	28.09	28.25	28.38	28.55
65-69	97.5%	22.29	22.49	22.69	22.90	23.10	23.29	23.50	23.70	23.90	24.10	24.30	24.49	24.69	24.89	25.10	25.28	25.49	25.67	25.86	26.05
	mean	21.87	22.04	22.19	22.35	22.52	22.68	22.85	23.02	23.18	23.35	23.52	23.69	23.86	24.03	24.19	24.37	24.54	24.71	24.88	25.04
	2.5%	21.44	21.56	21.67	21.78	21.92	22.03	22.16	22.30	22.44	22.57	22.70	22.82	22.97	23.12	23.26	23.40	23.53	23.69	23.81	23.97
70-74	97.5%	18.02	18.20	18.39	18.57	18.76	18.95	19.11	19.30	19.47	19.65	19.84	20.01	20.19	20.39	20.57	20.74	20.93	21.11	21.28	21.46
	mean	17.63	17.79	17.93	18.07	18.22	18.37	18.52	18.67	18.83	18.98	19.13	19.29	19.44	19.59	19.75	19.90	20.06	20.21	20.37	20.52
	2.5%	17.24	17.35	17.44	17.55	17.67	17.78	17.90	18.02	18.13	18.26	18.39	18.51	18.63	18.76	18.89	19.02	19.14	19.27	19.39	19.52
75-79	97.5%	13.94	14.10	14.24	14.40	14.57	14.72	14.87	15.04	15.19	15.34	15.51	15.66	15.81	15.97	16.14	16.29	16.45	16.60	16.78	16.92
	mean	13.59	13.72	13.84	13.96	14.09	14.21	14.34	14.47	14.60	14.73	14.86	15.00	15.13	15.26	15.39	15.53	15.66	15.80	15.93	16.06
	2.5%	13.23	13.34	13.41	13.50	13.60	13.68	13.78	13.89	13.98	14.09	14.19	14.29	14.40	14.48	14.62	14.72	14.81	14.93	15.02	15.15
80-84	97.5%	10.17	10.31	10.43	10.54	10.66	10.79	10.91	11.03	11.16	11.29	11.42	11.54	11.67	11.79	11.92	12.04	12.18	12.30	12.44	12.56
	mean	9.89	9.99	10.07	10.17	10.26	10.36	10.45	10.55	10.65	10.75	10.85	10.96	11.06	11.16	11.26	11.36	11.47	11.58	11.68	11.78
	2.5%	9.59	9.67	9.72	9.77	9.86	9.91	9.97	10.06	10.12	10.19	10.27	10.34	10.42	10.47	10.57	10.65	10.72	10.80	10.87	10.97
85-89	97.5%	6.97	7.07	7.14	7.21	7.30	7.38	7.44	7.53	7.62	7.71	7.78	7.87	7.95	8.04	8.13	8.22	8.31	8.40	8.50	8.58
	mean	6.71	6.78	6.83	6.89	6.95	7.01	7.07	7.13	7.20	7.25	7.32	7.39	7.45	7.51	7.58	7.65	7.72	7.79	7.85	

Table A.56. The forecasted Switzerland male life expectancy by model 8

age	97.5%	81.61	81.98	82.38	82.75	83.11	83.48	83.86	84.20	84.57	84.92	85.27	85.64	86.01	86.34	86.71	87.05	87.42	87.76	88.11	88.44
	mean	81.03	81.34	81.66	81.98	82.30	82.62	82.95	83.27	83.60	83.93	84.25	84.58	84.91	85.24	85.57	85.90	86.23	86.56	86.88	87.21
	2.5%	80.40	80.64	80.89	81.13	81.39	81.68	81.94	82.23	82.51	82.79	83.05	83.36	83.66	83.92	84.24	84.52	84.84	85.16	85.45	85.77
1-4	97.5%	80.96	81.30	81.67	82.02	82.37	82.72	83.08	83.40	83.77	84.11	84.45	84.80	85.17	85.49	85.85	86.19	86.54	86.88	87.23	87.55
	mean	80.39	80.70	81.00	81.31	81.61	81.92	82.24	82.56	82.87	83.19	83.51	83.83	84.14	84.46	84.78	85.10	85.42	85.74	86.06	86.38
	2.5%	79.80	80.03	80.29	80.53	80.78	81.05	81.32	81.60	81.87	82.13	82.41	82.70	82.99	83.26	83.58	83.83	84.13	84.46	84.74	85.03
5-9	97.5%	77.00	77.33	77.70	78.04	78.40	78.74	79.10	79.43	79.79	80.13	80.46	80.82	81.19	81.51	81.86	82.20	82.55	82.88	83.24	83.55
	mean	76.44	76.74	77.04	77.35	77.65	77.96	78.27	78.58	78.90	79.21	79.53	79.85	80.17	80.48	80.80	81.12	81.44	81.76	82.08	82.39
	2.5%	75.85	76.08	76.34	76.58	76.83	77.09	77.37	77.64	77.91	78.17	78.45	78.73	79.02	79.29	79.60	79.86	80.16	80.48	80.77	81.05
10-14	97.5%	72.02	72.35	72.72	73.06	73.41	73.76	74.11	74.44	74.80	75.14	75.47	75.83	76.19	76.51	76.86	77.20	77.56	77.89	78.24	78.56
	mean	71.46	71.76	72.06	72.37	72.67	72.98	73.29	73.60	73.91	74.23	74.54	74.86	75.18	75.50	75.81	76.13	76.45	76.77	77.08	77.40
	2.5%	70.88	71.11	71.36	71.61	71.85	72.12	72.39	72.66	72.94	73.19	73.47	73.75	74.05	74.30	74.62	74.88	75.18	75.49	75.78	76.06
15-19	97.5%	67.04	67.38	67.74	68.08	68.43	68.77	69.13	69.45	69.82	70.16	70.48	70.84	71.20	71.52	71.87	72.21	72.56	72.89	73.25	73.56
	mean	66.49	66.79	67.09	67.39	67.69	68.00	68.31	68.62	68.93	69.24	69.56	69.88	70.19	70.51	70.82	71.14	71.46	71.78	72.09	72.41
	2.5%	65.91	66.14	66.39	66.64	66.89	67.15	67.42	67.69	67.96	68.22	68.50	68.77	69.07	69.32	69.64	69.90	70.19	70.51	70.80	71.07
20-24	97.5%	62.13	62.47	62.82	63.16	63.50	63.84	64.19	64.51	64.88	65.21	65.53	65.88	66.25	66.57	66.92	67.25	67.60	67.93	68.28	68.59
	mean	61.60	61.90	62.19	62.49	62.79	63.09	63.40	63.71	64.02	64.33	64.64	64.95	65.27	65.58	65.89	66.21	66.53	66.84	67.15	67.47
	2.5%	61.04	61.28	61.52	61.76	62.01	62.28	62.54	62.81	63.08	63.34	63.61	63.89	64.20	64.44	64.75	65.00	65.31	65.61	65.91	66.19
25-29	97.5%	57.23	57.55	57.90	58.23	58.57	58.90	59.25	59.57	59.92	60.26	60.58	60.92	61.28	61.60	61.95	62.28	62.63	62.95	63.30	63.61
	mean	56.71	57.00	57.30	57.59	57.88	58.18	58.49	58.79	59.09	59.40	59.71	60.02	60.33	60.64	60.95	61.27	61.58	61.89	62.20	62.51
	2.5%	56.18	56.41	56.65	56.90	57.14	57.40	57.68	57.94	58.21	58.46	58.73	59.01	59.30	59.55	59.85	60.11	60.41	60.71	61.00	61.26
30-34	97.5%	52.31	52.64	52.97	53.30	53.64	53.97	54.31	54.62	54.97	55.30	55.63	55.96	56.32	56.64	56.98	57.32	57.66	57.98	58.33	58.64
	mean	51.82	52.11	52.40	52.69	52.98	53.28	53.58	53.88	54.18	54.48	54.79	55.10	55.41	55.72	56.02	56.33	56.65	56.96	57.27	57.57
	2.5%	51.31	51.55	51.79	52.03	52.28	52.54	52.81	53.08	53.34	53.60	53.87	54.15	54.43	54.69	54.99	55.25	55.53	55.83	56.12	56.39
35-39	97.5%	47.41	47.73	48.06	48.38	48.71	49.04	49.37	49.68	50.03	50.36	50.68	51.01	51.37	51.68	52.02	52.36	52.70	53.02	53.36	53.67
	mean	46.94	47.23	47.51	47.80	48.09	48.38	48.68	48.98	49.28	49.58	49.88	50.19	50.49	50.80	51.11	51.42	51.73	52.03	52.34	52.65
	2.5%	46.45	46.68	46.93	47.18	47.43	47.68	47.96	48.23	48.48	48.74	49.03	49.30	49.59	49.85	50.14	50.41	50.68	50.98	51.28	51.53
40-44	97.5%	42.55	42.85	43.18	43.50	43.82	44.14	44.47	44.77	45.12	45.44	45.76	46.09	46.44	46.74	47.09	47.41	47.76	48.07	48.41	48.72
	mean	42.09	42.37	42.65	42.93	43.22	43.51	43.80	44.10	44.39	44.69	44.99	45.30	45.60	45.90	46.20	46.51	46.82	47.12	47.43	47.73
	2.5%	41.61	41.85	42.10	42.35	42.59	42.84	43.12	43.39	43.64	43.90	44.18	44.45	44.74	45.00	45.28	45.55	45.82	46.10	46.41	46.66
45-49	97.5%	37.76	38.05	38.37	38.67	38.99	39.30	39.62	39.93	40.26	40.58	40.90	41.22	41.56	41.86	42.20	42.52	42.85	43.18	43.51	43.81
	mean	37.31	37.59	37.86	38.14	38.42	38.71	39.00	39.29	39.58	39.87	40.17	40.46	40.76	41.06	41.36	41.66	41.96	42.26	42.56	42.86
	2.5%	36.85	37.09	37.34	37.58	37.82	38.07	38.35	38.61	38.87	39.11	39.40	39.67	39.95	40.21	40.49	40.74	41.02	41.30	41.60	41.84
50-54	97.5%	33.05	33.33	33.64	33.93	34.24	34.53	34.85	35.14	35.47	35.78	36.09	36.40	36.73	37.04	37.36	37.68	38.00	38.32	38.65	38.95
	mean	32.61	32.88	33.14	33.42	33.69	33.97	34.25	34.53	34.81	35.10	35.39	35.68	35.97	36.27	36.56	36.85	37.15	37.44	37.74	38.03
	2.5%	32.16	32.39	32.64	32.87	33.11	33.36	33.63	33.89	34.13	34.38	34.65	34.92	35.19	35.45	35.72	35.97	36.26	36.52	36.80	37.06
55-59	97.5%	28.48	28.74	29.04	29.31	29.60	29.89	30.20	30.48	30.79	31.10	31.39	31.69	32.02	32.31	32.61	32.93	33.25	33.55	33.87	34.16
	mean	28.04	28.30	28.56	28.82	29.08	29.35	29.62	29.89	30.17	30.44	30.72	31.01	31.29	31.57	31.85	32.14	32.42	32.71	32.99	33.28
	2.5%	27.60	27.82	28.07	28.29	28.53	28.76	29.02	29.28	29.51	29.75	30.02	30.27	30.53	30.80	31.05	31.29	31.56	31.82	32.09	32.34
60-64	97.5%	24.07	24.33	24.60	24.86	25.12	25.40	25.68	25.95	26.24	26.53	26.81	27.09	27.39	27.67	27.96	28.26	28.55	28.85	29.15	29.44
	mean	23.66	23.90	24.14	24.38	24.63	24.88	25.13	25.39	25.64	25.90	26.16	26.43	26.69	26.96	27.22	27.49	27.77	28.04	28.30	28.57
	2.5%	23.22	23.44	23.66	23.87	24.08	24.32	24.56	24.80	25.02	25.24	25.48	25.73	25.97	26.21	26.45	26.68	26.93	27.18	27.43	27.67
65-69	97.5%	19.88	20.12	20.37	20.59	20.84	21.09	21.35	21.59	21.86	22.12	22.38	22.64	22.92	23.19	23.46	23.73	24.00	24.27	24.55	24.82
	mean	19.49	19.71	19.93	20.15	20.37	20.60	20.84	21.07	21.30	21.54	21.78	22.03	22.27	22.52	22.76	23.01	23.26	23.51	23.76	24.01
	2.5%	19.09	19.30	19.49	19.68	19.88	20.09	20.30	20.51	20.73	20.93	21.16	21.38	21.60	21.81	22.05	22.26	22.47	22.70	22.93	23.15
70-74	97.5%	15.89	16.10	16.30	16.50	16.71	16.94	17.16	17.37	17.61	17.83	18.08	18.31	18.54	18.78	19.01	19.27	19.50	19.75	20.00	20.23
	mean	15.54	15.73	15.91	16.10	16.29	16.49	16.69	16.89	17.10	17.30	17.51	17.72	17.93	18.15	18.36	18.58	18.80	19.02	19.25	19.47
	2.5%	15.17	15.35	15.49	15.66	15.84	16.01	16.20	16.39	16.56	16.74	16.95	17.12	17.30	17.49	17.69	17.86	18.05	18.27	18.46	18.65
75-79	97.5%	12.15	12.32	12.48	12.65	12.81	13.00	13.19	13.35	13.55	13.74	13.93	14.12	14.32	14.52	14.72	14.92				

Table A.57. The forecasted U.K. female life expectancy by model 3

age 0	97.5%	83.12	83.35	83.56	83.72	83.84	83.92	84.00	84.09	84.15	84.21	84.33	84.38	84.41	84.51	84.61	84.62	84.67	84.77	84.79	84.79
	mean	82.37	82.35	82.33	82.31	82.28	82.26	82.23	82.20	82.18	82.15	82.13	82.10	82.07	82.04	82.02	81.99	81.96	81.94	81.91	81.88
	2.5%	81.57	81.27	81.00	80.80	80.63	80.42	80.21	80.12	79.94	79.80	79.61	79.49	79.34	79.18	79.07	78.95	78.80	78.72	78.59	78.49
1-4	97.5%	82.39	82.61	82.80	82.95	83.07	83.14	83.21	83.30	83.36	83.42	83.53	83.60	83.61	83.71	83.80	83.82	83.85	83.95	83.97	83.96
	mean	81.67	81.65	81.63	81.61	81.59	81.56	81.53	81.51	81.49	81.47	81.44	81.42	81.39	81.36	81.34	81.31	81.29	81.26	81.24	81.21
	2.5%	80.92	80.63	80.38	80.21	80.03	79.84	79.66	79.55	79.39	79.26	79.09	78.98	78.85	78.71	78.56	78.46	78.32	78.26	78.13	78.03
5-9	97.5%	78.43	78.64	78.83	78.98	79.10	79.17	79.24	79.33	79.38	79.44	79.55	79.62	79.64	79.73	79.83	79.84	79.88	79.97	79.99	79.98
	mean	77.73	77.70	77.68	77.67	77.64	77.62	77.59	77.57	77.55	77.53	77.51	77.48	77.46	77.43	77.40	77.37	77.35	77.33	77.31	77.28
	2.5%	76.98	76.70	76.46	76.29	76.12	75.94	75.77	75.66	75.52	75.38	75.22	75.11	74.99	74.85	74.72	74.61	74.50	74.42	74.31	74.23
10-14	97.5%	73.45	73.67	73.85	74.00	74.12	74.19	74.27	74.34	74.40	74.46	74.57	74.63	74.65	74.75	74.84	74.85	74.89	74.98	75.00	75.00
	mean	72.75	72.73	72.71	72.70	72.67	72.65	72.62	72.60	72.58	72.56	72.54	72.51	72.49	72.46	72.44	72.41	72.39	72.37	72.34	72.32
	2.5%	72.02	71.74	71.50	71.33	71.17	70.98	70.81	70.71	70.57	70.43	70.28	70.17	70.06	69.91	69.78	69.70	69.57	69.49	69.38	69.31
15-19	97.5%	68.48	68.69	68.88	69.03	69.14	69.21	69.29	69.36	69.42	69.48	69.59	69.65	69.67	69.76	69.86	69.87	69.91	70.00	70.02	70.01
	mean	67.78	67.76	67.74	67.73	67.71	67.68	67.65	67.63	67.61	67.59	67.57	67.54	67.52	67.49	67.47	67.44	67.42	67.40	67.38	67.35
	2.5%	67.05	66.78	66.54	66.37	66.21	66.04	65.86	65.76	65.62	65.48	65.34	65.23	65.11	64.96	64.84	64.75	64.63	64.55	64.45	64.37
20-24	97.5%	63.52	63.73	63.92	64.06	64.18	64.24	64.32	64.40	64.45	64.51	64.62	64.68	64.70	64.79	64.89	64.90	64.93	65.03	65.05	65.04
	mean	62.84	62.81	62.80	62.78	62.76	62.74	62.71	62.69	62.67	62.65	62.63	62.60	62.58	62.55	62.53	62.50	62.48	62.46	62.44	62.41
	2.5%	62.12	61.84	61.62	61.45	61.29	61.11	60.95	60.84	60.71	60.58	60.44	60.33	60.20	60.07	59.95	59.88	59.73	59.67	59.57	59.49
25-29	97.5%	58.58	58.79	58.97	59.11	59.22	59.29	59.36	59.44	59.50	59.55	59.66	59.72	59.74	59.83	59.92	59.94	59.97	60.07	60.09	60.08
	mean	57.90	57.88	57.86	57.85	57.83	57.80	57.78	57.76	57.74	57.72	57.70	57.67	57.65	57.62	57.60	57.57	57.55	57.53	57.51	57.49
	2.5%	57.19	56.92	56.70	56.53	56.38	56.21	56.04	55.94	55.81	55.69	55.54	55.44	55.32	55.19	55.06	55.00	54.86	54.80	54.71	54.63
30-34	97.5%	53.65	53.86	54.03	54.18	54.29	54.35	54.42	54.50	54.55	54.61	54.72	54.77	54.79	54.88	54.97	54.99	55.02	55.12	55.13	55.13
	mean	52.98	52.96	52.94	52.93	52.91	52.89	52.86	52.84	52.82	52.80	52.78	52.76	52.73	52.71	52.69	52.66	52.64	52.62	52.60	52.58
	2.5%	52.28	52.02	51.80	51.64	51.48	51.32	51.15	51.06	50.93	50.80	50.66	50.57	50.45	50.32	50.20	50.13	50.00	49.93	49.86	49.76
35-39	97.5%	48.75	48.96	49.13	49.27	49.38	49.44	49.51	49.59	49.64	49.70	49.80	49.85	49.87	49.96	50.05	50.07	50.10	50.19	50.21	50.20
	mean	48.10	48.07	48.06	48.04	48.02	48.00	47.98	47.96	47.94	47.92	47.90	47.87	47.85	47.83	47.81	47.78	47.76	47.74	47.72	47.70
	2.5%	47.41	47.15	46.94	46.78	46.63	46.47	46.29	46.20	46.09	45.96	45.83	45.74	45.63	45.49	45.37	45.30	45.17	45.11	45.04	44.94
40-44	97.5%	43.90	44.10	44.26	44.40	44.50	44.57	44.64	44.71	44.76	44.82	44.91	44.97	44.99	45.08	45.15	45.18	45.22	45.31	45.32	45.31
	mean	43.25	43.23	43.22	43.20	43.18	43.16	43.14	43.12	43.10	43.08	43.06	43.04	43.02	42.99	42.97	42.95	42.93	42.91	42.89	42.87
	2.5%	42.58	42.33	42.12	41.97	41.82	41.68	41.50	41.41	41.30	41.17	41.04	40.95	40.86	40.71	40.60	40.54	40.42	40.36	40.27	40.20
45-49	97.5%	39.10	39.30	39.46	39.59	39.69	39.76	39.82	39.89	39.93	39.99	40.09	40.14	40.16	40.25	40.33	40.35	40.38	40.47	40.48	40.48
	mean	38.48	38.45	38.44	38.43	38.41	38.39	38.36	38.34	38.32	38.31	38.29	38.27	38.25	38.22	38.20	38.18	38.16	38.14	38.12	38.10
	2.5%	37.82	37.57	37.37	37.23	37.09	36.95	36.77	36.70	36.58	36.45	36.33	36.24	36.14	36.02	35.91	35.85	35.73	35.67	35.59	35.53
50-54	97.5%	34.38	34.57	34.72	34.86	34.95	35.02	35.08	35.14	35.19	35.24	35.33	35.38	35.42	35.49	35.56	35.60	35.62	35.70	35.72	35.71
	mean	33.78	33.75	33.74	33.73	33.71	33.69	33.66	33.65	33.63	33.61	33.59	33.57	33.55	33.53	33.51	33.49	33.47	33.45	33.43	33.41
	2.5%	33.13	32.90	32.71	32.57	32.44	32.30	32.13	32.06	31.95	31.83	31.70	31.63	31.53	31.41	31.30	31.23	31.13	31.08	31.00	30.92
55-59	97.5%	29.78	29.96	30.11	30.23	30.32	30.38	30.43	30.50	30.55	30.60	30.69	30.73	30.76	30.84	30.90	30.93	30.96	31.04	31.05	31.04
	mean	29.19	29.17	29.16	29.14	29.13	29.11	29.08	29.07	29.05	29.03	29.02	28.99	28.98	28.95	28.93	28.91	28.90	28.88	28.86	28.84
	2.5%	28.56	28.35	28.16	28.03	27.90	27.78	27.62	27.54	27.44	27.33	27.21	27.13	27.04	26.91	26.81	26.77	26.65	26.60	26.51	26.46
60-64	97.5%	25.32	25.49	25.63	25.74	25.83	25.88	25.93	26.00	26.04	26.09	26.16	26.21	26.24	26.31	26.36	26.40	26.42	26.50	26.52	26.51
	mean	24.75	24.73	24.72	24.70	24.69	24.67	24.65	24.63	24.61	24.60	24.58	24.56	24.54	24.52	24.50	24.48	24.46	24.45	24.43	24.41
	2.5%	24.15	23.94	23.77	23.65	23.53	23.41	23.25	23.18	23.09	22.97	22.86	22.78	22.71	22.59	22.49	22.45	22.35	22.28	22.21	22.15
65-69	97.5%	21.02	21.17	21.29	21.39	21.48	21.54	21.58	21.63	21.67	21.70	21.78	21.82	21.85	21.91	21.98	21.98	22.04	22.08	22.11	22.11
	mean	20.49	20.48	20.46	20.45	20.44	20.42	20.40	20.39	20.37	20.36	20.34	20.32	20.31	20.29	20.27	20.25	20.24	20.23	20.21	20.19
	2.5%	19.95	19.76	19.60	19.49	19.38	19.27	19.16	19.06	18.98	18.88	18.80	18.72	18.65	18.53	18.46	18.41	18.31	18.26	18.19	18.16
70-74	97.5%	16.92	17.05	17.15	17.25	17.31	17.37	17.42	17.46	17.49	17.52	17.59	17.63	17.66	17.71	17.75	17.78	17.80	17.84	17.88	17.89
	mean	16.44	16.43	16.42	16.41	16.39	16.38	16.36	16.35	16.34	16.32	16.31	16.29	16.28	16.26	16.25	16.23	16.22	16.21	16.19	16.18
	2.5%	15.95	15.79	15.65	15.55	15.46	15.38	15.26	15.18	15.13	15.04	14.95	14.88	14.82	14.72	14.67	14.61	14.54	14.50	14.45	14.41
75-79	97.5%	13.11	13.22	13.31	13.38	13.45	13.49	13.53	13.57	13.59	13.62	13.69	13.72	13.74	13.77	13.81	13.84	13.87	13.90	13.92	13.943

Table A.58. The forecasted U.K. male life expectancy by model 3

age 0	97.5%	79.58	79.84	80.06	80.26	80.53	80.52	80.64	80.77	80.81	80.81	80.84	80.90	81.02	81.09	81.16	81.32	81.41	81.48	81.57	81.56
	mean	78.62	78.57	78.54	78.51	78.48	78.43	78.38	78.34	78.29	78.24	78.19	78.16	78.14	78.11	78.06	78.03	78.00	77.98	77.92	77.89
	2.5%	77.65	77.18	76.86	76.60	76.44	76.14	75.91	75.79	75.60	75.40	75.06	75.00	74.79	74.60	74.42	74.23	74.13	74.10	73.91	73.62
1-4	97.5%	78.89	79.15	79.35	79.51	79.81	79.79	79.91	80.02	80.03	80.07	80.12	80.14	80.27	80.31	80.39	80.57	80.64	80.69	80.78	80.77
	mean	77.98	77.93	77.90	77.87	77.84	77.80	77.75	77.71	77.66	77.61	77.57	77.53	77.52	77.49	77.45	77.41	77.38	77.37	77.31	77.27
	2.5%	77.07	76.62	76.32	76.05	75.88	75.61	75.41	75.30	75.09	74.92	74.58	74.59	74.35	74.20	74.02	73.83	73.74	73.68	73.56	73.31
5-9	97.5%	74.93	75.19	75.39	75.55	75.84	75.82	75.95	76.05	76.07	76.10	76.15	76.18	76.29	76.33	76.41	76.60	76.67	76.72	76.80	76.80
	mean	74.03	73.99	73.96	73.93	73.90	73.86	73.81	73.77	73.72	73.68	73.63	73.60	73.59	73.55	73.51	73.48	73.45	73.44	73.38	73.35
	2.5%	73.14	72.70	72.41	72.14	71.97	71.70	71.51	71.42	71.21	71.04	70.73	70.71	70.49	70.37	70.19	69.97	69.88	69.84	69.72	69.46
10-14	97.5%	69.96	70.21	70.41	70.57	70.86	70.85	70.97	71.07	71.08	71.12	71.17	71.20	71.31	71.36	71.42	71.62	71.69	71.74	71.82	71.82
	mean	69.07	69.02	68.99	68.97	68.94	68.89	68.85	68.81	68.76	68.71	68.67	68.64	68.62	68.59	68.55	68.51	68.49	68.48	68.42	68.39
	2.5%	68.18	67.75	67.46	67.19	67.02	66.75	66.56	66.48	66.28	66.10	65.79	65.76	65.56	65.44	65.24	65.04	64.96	64.92	64.80	64.53
15-19	97.5%	64.99	65.24	65.44	65.59	65.89	65.88	65.99	66.10	66.11	66.14	66.20	66.22	66.33	66.38	66.44	66.64	66.71	66.76	66.84	66.84
	mean	64.10	64.06	64.03	64.00	63.97	63.93	63.88	63.84	63.79	63.75	63.70	63.67	63.66	63.63	63.59	63.55	63.53	63.52	63.46	63.42
	2.5%	63.22	62.80	62.51	62.24	62.07	61.80	61.61	61.53	61.32	61.15	60.84	60.82	60.62	60.50	60.30	60.09	60.01	59.98	59.87	59.60
20-24	97.5%	60.08	60.32	60.52	60.67	60.96	60.95	61.06	61.17	61.18	61.20	61.26	61.29	61.40	61.45	61.51	61.71	61.77	61.82	61.90	61.91
	mean	59.20	59.16	59.13	59.11	59.08	59.03	58.99	58.95	58.90	58.86	58.81	58.78	58.77	58.74	58.70	58.67	58.64	58.63	58.58	58.54
	2.5%	58.33	57.93	57.63	57.38	57.20	56.94	56.76	56.67	56.47	56.32	56.01	55.98	55.80	55.69	55.48	55.29	55.21	55.16	55.06	54.81
25-29	97.5%	55.20	55.43	55.63	55.78	56.06	56.04	56.15	56.27	56.28	56.29	56.35	56.38	56.49	56.54	56.62	56.78	56.86	56.90	56.98	56.98
	mean	54.35	54.31	54.29	54.26	54.24	54.20	54.15	54.12	54.07	54.03	53.99	53.96	53.95	53.92	53.89	53.85	53.83	53.82	53.77	53.74
	2.5%	53.50	53.14	52.85	52.61	52.44	52.21	52.04	51.91	51.76	51.62	51.34	51.31	51.13	51.02	50.82	50.68	50.62	50.51	50.43	50.20
30-34	97.5%	50.35	50.57	50.78	50.91	51.17	51.16	51.27	51.37	51.38	51.42	51.46	51.48	51.59	51.66	51.73	51.88	51.96	52.00	52.08	52.09
	mean	49.53	49.49	49.46	49.44	49.42	49.38	49.34	49.30	49.26	49.22	49.18	49.16	49.15	49.12	49.09	49.06	49.04	49.03	48.98	48.95
	2.5%	48.70	48.36	48.09	47.87	47.70	47.46	47.30	47.21	47.04	46.93	46.65	46.66	46.45	46.35	46.17	46.04	45.96	45.89	45.84	45.63
35-39	97.5%	45.52	45.74	45.93	46.06	46.31	46.33	46.42	46.52	46.53	46.55	46.59	46.63	46.74	46.80	46.85	47.00	47.08	47.12	47.18	47.20
	mean	44.73	44.69	44.67	44.65	44.63	44.59	44.55	44.52	44.48	44.44	44.40	44.38	44.37	44.34	44.31	44.28	44.26	44.25	44.21	44.18
	2.5%	43.92	43.60	43.33	43.12	42.97	42.76	42.60	42.49	42.36	42.23	41.99	41.97	41.78	41.69	41.53	41.38	41.33	41.25	41.19	40.95
40-44	97.5%	40.76	40.97	41.15	41.29	41.53	41.54	41.61	41.71	41.73	41.75	41.81	41.84	41.93	41.99	42.05	42.19	42.26	42.31	42.35	42.40
	mean	40.00	39.96	39.94	39.92	39.90	39.86	39.82	39.79	39.75	39.72	39.68	39.66	39.65	39.62	39.59	39.57	39.55	39.54	39.49	39.47
	2.5%	39.21	38.90	38.65	38.46	38.32	38.11	37.96	37.85	37.73	37.61	37.37	37.33	37.16	37.09	36.95	36.78	36.76	36.67	36.58	36.41
45-49	97.5%	36.08	36.28	36.45	36.59	36.82	36.82	36.90	37.00	37.01	37.04	37.09	37.12	37.21	37.29	37.32	37.47	37.52	37.58	37.60	37.65
	mean	35.34	35.31	35.28	35.26	35.24	35.21	35.17	35.14	35.10	35.07	35.04	35.01	35.00	34.98	34.95	34.92	34.91	34.90	34.85	34.83
	2.5%	34.60	34.27	34.03	33.85	33.71	33.53	33.39	33.27	33.18	33.06	32.82	32.76	32.60	32.54	32.35	32.25	32.21	32.13	32.06	31.89
50-54	97.5%	31.48	31.68	31.83	31.98	32.18	32.19	32.26	32.35	32.38	32.40	32.45	32.49	32.57	32.67	32.67	32.82	32.88	32.93	32.93	32.99
	mean	30.77	30.73	30.71	30.69	30.67	30.64	30.60	30.57	30.54	30.50	30.47	30.45	30.44	30.42	30.39	30.36	30.34	30.33	30.29	30.27
	2.5%	30.04	29.73	29.52	29.34	29.19	29.01	28.86	28.76	28.68	28.57	28.34	28.30	28.13	28.06	27.90	27.79	27.76	27.68	27.60	27.44
55-59	97.5%	27.01	27.21	27.34	27.48	27.67	27.69	27.77	27.84	27.87	27.88	27.93	27.98	28.05	28.10	28.14	28.29	28.33	28.38	28.34	28.42
	mean	26.32	26.28	26.26	26.24	26.22	26.19	26.16	26.13	26.09	26.06	26.03	26.01	26.00	25.98	25.95	25.92	25.91	25.90	25.86	25.83
	2.5%	25.62	25.32	25.10	24.94	24.80	24.63	24.49	24.40	24.35	24.20	24.01	23.94	23.81	23.72	23.60	23.46	23.41	23.34	23.31	23.13
60-64	97.5%	22.74	22.94	23.04	23.17	23.36	23.39	23.44	23.50	23.53	23.55	23.57	23.62	23.69	23.74	23.80	23.93	23.93	24.04	23.99	24.05
	mean	22.09	22.06	22.04	22.03	22.01	21.98	21.95	21.92	21.89	21.86	21.83	21.81	21.80	21.78	21.75	21.73	21.72	21.71	21.67	21.65
	2.5%	21.43	21.18	20.96	20.81	20.70	20.54	20.42	20.33	20.25	20.16	19.99	19.90	19.80	19.68	19.60	19.47	19.44	19.38	19.33	19.18
65-69	97.5%	18.71	18.88	18.98	19.10	19.26	19.29	19.35	19.41	19.41	19.43	19.48	19.50	19.57	19.63	19.66	19.74	19.79	19.89	19.85	19.89
	mean	18.11	18.09	18.07	18.05	18.04	18.01	17.98	17.96	17.93	17.91	17.88	17.86	17.86	17.84	17.81	17.79	17.78	17.77	17.74	17.72
	2.5%	17.52	17.29	17.07	16.96	16.89	16.73	16.64	16.54	16.46	16.36	16.23	16.16	16.05	15.95	15.90	15.79	15.76	15.72	15.62	15.52
70-74	97.5%	14.92	15.06	15.16	15.25	15.39	15.41	15.48	15.51	15.54	15.57	15.61	15.62	15.64	15.71	15.72	15.83	15.88	15.91	15.91	15.99
	mean	14.39	14.37	14.35	14.34	14.32	14.30	14.28	14.26	14.23	14.21	14.19	14.17	14.15	14.13	14.11	14.11	14.10	14.09	14.07	14.05
	2.5%	13.88	13.68	13.51	13.40	13.31	13.17	13.10	13.02	12.98	12.87	12.75	12.71	12.64	12.55	12.52	12.36	12.36	12.29	12.27	12.15
75-79	97.5%	11.49	11.60	11.67	11.77	11.88	11.91	11.95	11.98	11.97	12.03	12.05	12.04	12.11	12.12	12.13	12.21				

Table A.59. The forecasted U.K. female life expectancy by model 8

age	2.5%	83.22	83.52	83.85	84.13	84.45	84.74	85.05	85.37	85.68	86.01	86.34	86.66	87.00	87.33	87.67	88.00	88.35	88.70	89.04	89.39
0	mean	82.60	82.85	83.10	83.36	83.62	83.88	84.15	84.43	84.71	84.99	85.28	85.57	85.87	86.16	86.46	86.76	87.07	87.38	87.69	87.99
	97.5%	81.96	82.13	82.35	82.53	82.75	82.97	83.18	83.42	83.65	83.86	84.09	84.40	84.60	84.83	85.10	85.38	85.67	85.92	86.16	86.42
1-4	2.5%	82.51	82.81	83.14	83.42	83.75	84.05	84.36	84.68	84.99	85.31	85.65	85.97	86.32	86.64	87.00	87.33	87.68	88.02	88.37	88.72
	mean	81.90	82.15	82.41	82.67	82.93	83.20	83.47	83.75	84.03	84.32	84.61	84.90	85.20	85.50	85.80	86.10	86.42	86.73	87.04	87.35
	97.5%	81.28	81.46	81.66	81.85	82.07	82.30	82.50	82.74	82.98	83.20	83.44	83.72	83.94	84.17	84.46	84.74	85.02	85.29	85.53	85.79
5-9	2.5%	78.56	78.86	79.17	79.46	79.77	80.08	80.38	80.71	81.01	81.33	81.66	81.99	82.34	82.66	83.02	83.34	83.69	84.03	84.38	84.73
	mean	77.95	78.20	78.45	78.71	78.97	79.24	79.51	79.79	80.07	80.35	80.64	80.94	81.23	81.53	81.83	82.13	82.44	82.75	83.06	83.37
	97.5%	77.33	77.51	77.72	77.91	78.13	78.36	78.55	78.80	79.03	79.26	79.50	79.77	79.99	80.23	80.51	80.79	81.07	81.35	81.58	81.84
10-14	2.5%	73.59	73.88	74.20	74.48	74.79	75.10	75.40	75.72	76.02	76.34	76.67	77.00	77.35	77.67	78.02	78.35	78.70	79.04	79.39	79.74
	mean	72.98	73.23	73.48	73.74	74.00	74.26	74.53	74.81	75.09	75.37	75.66	75.95	76.25	76.55	76.85	77.15	77.46	77.77	78.08	78.39
	97.5%	72.36	72.54	72.75	72.94	73.16	73.39	73.59	73.83	74.06	74.29	74.52	74.80	75.02	75.26	75.54	75.81	76.09	76.37	76.61	76.86
15-19	2.5%	68.62	68.90	69.22	69.50	69.81	70.11	70.41	70.74	71.04	71.35	71.69	72.01	72.36	72.68	73.04	73.36	73.71	74.05	74.39	74.75
	mean	68.01	68.25	68.51	68.76	69.02	69.29	69.56	69.83	70.11	70.39	70.68	70.97	71.27	71.56	71.86	72.16	72.48	72.78	73.09	73.40
	97.5%	67.39	67.57	67.78	67.97	68.19	68.42	68.61	68.86	69.09	69.32	69.55	69.83	70.04	70.29	70.57	70.84	71.12	71.40	71.63	71.89
20-24	2.5%	63.66	63.95	64.26	64.54	64.85	65.15	65.45	65.77	66.07	66.39	66.71	67.04	67.38	67.70	68.06	68.38	68.73	69.07	69.41	69.77
	mean	63.06	63.31	63.56	63.81	64.07	64.33	64.60	64.87	65.15	65.43	65.72	66.01	66.31	66.60	66.90	67.20	67.51	67.81	68.13	68.43
	97.5%	62.45	62.63	62.84	63.03	63.25	63.48	63.67	63.92	64.16	64.38	64.61	64.90	65.10	65.35	65.63	65.91	66.18	66.45	66.68	66.94
25-29	2.5%	58.72	59.00	59.32	59.59	59.89	60.19	60.49	60.80	61.11	61.42	61.75	62.07	62.42	62.73	63.09	63.40	63.75	64.09	64.44	64.78
	mean	58.13	58.37	58.62	58.87	59.13	59.39	59.66	59.93	60.21	60.49	60.77	61.06	61.35	61.65	61.95	62.24	62.55	62.86	63.17	63.47
	97.5%	57.52	57.70	57.92	58.10	58.32	58.55	58.75	59.00	59.23	59.46	59.69	59.97	60.18	60.43	60.70	60.97	61.25	61.53	61.76	62.00
30-34	2.5%	53.80	54.08	54.38	54.65	54.96	55.25	55.54	55.86	56.16	56.47	56.80	57.11	57.46	57.77	58.12	58.45	58.79	59.13	59.47	59.82
	mean	53.21	53.45	53.70	53.95	54.20	54.46	54.73	55.00	55.27	55.55	55.84	56.13	56.42	56.71	57.01	57.30	57.61	57.91	58.22	58.53
	97.5%	52.60	52.79	53.01	53.19	53.42	53.65	53.84	54.09	54.32	54.55	54.78	55.06	55.27	55.52	55.78	56.05	56.34	56.59	56.84	57.08
35-39	2.5%	48.90	49.18	49.48	49.75	50.04	50.33	50.63	50.93	51.23	51.54	51.87	52.18	52.52	52.83	53.18	53.50	53.85	54.18	54.52	54.87
	mean	48.32	48.56	48.80	49.05	49.30	49.56	49.83	50.09	50.37	50.65	50.93	51.22	51.50	51.80	52.09	52.38	52.69	52.99	53.30	53.60
	97.5%	47.72	47.90	48.12	48.31	48.53	48.76	48.96	49.20	49.44	49.66	49.90	50.17	50.38	50.63	50.88	51.16	51.45	51.69	51.96	52.19
40-44	2.5%	44.05	44.32	44.62	44.88	45.17	45.45	45.75	46.05	46.35	46.65	46.97	47.29	47.62	47.94	48.28	48.61	48.94	49.28	49.60	49.95
	mean	43.48	43.71	43.95	44.20	44.45	44.71	44.97	45.23	45.51	45.78	46.06	46.35	46.63	46.92	47.21	47.51	47.81	48.11	48.41	48.72
	97.5%	42.88	43.07	43.28	43.48	43.70	43.93	44.12	44.37	44.61	44.83	45.07	45.33	45.55	45.79	46.04	46.33	46.62	46.85	47.10	47.36
45-49	2.5%	39.28	39.54	39.83	40.09	40.37	40.65	40.94	41.23	41.53	41.82	42.14	42.45	42.79	43.09	43.43	43.75	44.08	44.42	44.74	45.08
	mean	38.71	38.94	39.18	39.43	39.67	39.93	40.19	40.45	40.72	40.99	41.26	41.55	41.83	42.12	42.40	42.69	42.99	43.29	43.59	43.89
	97.5%	38.13	38.31	38.52	38.72	38.93	39.17	39.37	39.61	39.86	40.07	40.30	40.56	40.78	41.03	41.27	41.55	41.84	42.08	42.32	42.57
50-54	2.5%	34.59	34.84	35.12	35.37	35.64	35.93	36.21	36.49	36.78	37.06	37.38	37.68	38.01	38.31	38.63	38.97	39.27	39.61	39.92	40.26
	mean	34.03	34.26	34.49	34.73	34.97	35.22	35.47	35.73	35.99	36.26	36.53	36.81	37.08	37.37	37.65	37.93	38.23	38.52	38.81	39.11
	97.5%	33.46	33.64	33.84	34.05	34.26	34.49	34.69	34.92	35.16	35.38	35.61	35.86	36.08	36.32	36.56	36.82	37.11	37.35	37.58	37.83
55-59	2.5%	30.00	30.25	30.52	30.78	31.03	31.30	31.57	31.84	32.12	32.40	32.71	33.01	33.31	33.62	33.94	34.25	34.56	34.88	35.19	35.52
	mean	29.47	29.69	29.92	30.15	30.38	30.62	30.87	31.12	31.37	31.63	31.89	32.17	32.43	32.71	32.98	33.26	33.55	33.83	34.12	34.41
	97.5%	28.90	29.09	29.29	29.49	29.71	29.92	30.12	30.34	30.57	30.79	31.00	31.25	31.48	31.71	31.94	32.19	32.47	32.70	32.93	33.16
60-64	2.5%	25.53	25.77	26.03	26.27	26.50	26.76	27.02	27.28	27.55	27.81	28.12	28.41	28.69	28.98	29.28	29.60	29.89	30.21	30.51	30.82
	mean	25.01	25.22	25.44	25.66	25.89	26.12	26.35	26.59	26.84	27.08	27.34	27.60	27.86	28.12	28.39	28.66	28.93	29.20	29.48	29.76
	97.5%	24.46	24.64	24.83	25.03	25.24	25.45	25.65	25.85	26.07	26.29	26.50	26.73	26.94	27.17	27.38	27.64	27.89	28.11	28.32	28.56
65-69	2.5%	21.15	21.37	21.60	21.82	22.03	22.27	22.51	22.75	23.00	23.26	23.54	23.82	24.10	24.35	24.63	24.92	25.21	25.49	25.79	26.08
	mean	20.65	20.84	21.04	21.25	21.45	21.67	21.88	22.10	22.33	22.56	22.80	23.04	23.28	23.53	23.77	24.02	24.28	24.54	24.80	25.06
	97.5%	20.12	20.29	20.47	20.64	20.83	21.03	21.22	21.39	21.61	21.80	22.00	22.22	22.41	22.61	22.82	23.06	23.29	23.48	23.69	23.91
70-74	2.5%	16.95	17.14	17.34	17.54	17.74	17.95	18.16	18.38	18.60	18.82	19.08	19.32	19.58	19.81	20.06	20.33	20.58	20.86	21.14	21.40
	mean	16.47	16.64	16.82	17.00	17.18	17.37	17.56	17.75	17.95	18.16	18.37	18.58	18.80	19.02	19.24	19.47	19.70	19.94	20.17	20.41
	97.5%	15.98	16.12	16.28	16.43	16.59	16.77	16.93	17.09	17.28	17.44	17.62	17.80	17.96	18.15	18.34	18.53	18.74	18.91	19.11	19.30
75-79	2.5%	13.09	13.24	13.40	13.57	13.73	13.91	14.09	14.27	14.46	14.66	14.86	15.08	15.30	15.50	15.72	15.95	16.19	16.43	16.67	16.92
	mean	12.63	12.77	12.92	13.07	13.21	13.37	13.53	13.69	13.86	14.03	14.21	14.39	14.58	14.77	14.96	15.15	15.35	15.56	15.76	15.97
	97.5%	12.18	12.30	12.44	12.55	12.68	12.82	12.97	13.09	13.22	13.37	13.53	13.67	13.81	13.95	14.12	14.27	14.45	14.61	14.75	14.91
80-8	2.5%	9.66	9.78	9.89	10.02	10.14	10.28	10.42	10.57	10.70	10.86	11.02	11.19	11.37	11.53	11.71	11.90	12.08	12.26	12.48	12.68
	mean	9.26	9.36	9.47	9.58	9.69	9.81	9.93	10.06	10.19	10.32	10.45	10.59	10.74	10.88	11.03	11.19	11.34	11.50	11.67	11.83
	97.5%	8.86	8.94	9.05	9.15	9.22	9.32	9.43	9.53	9.64	9.74	9.83	9.94	10.04	10.16	10.28	10.38	10.51	10.61	10.74	10.85
85-89	2.5%	6.84	6.91	7.01	7.08	7.16	7.25	7.34	7.44	7.53	7.62	7.75	7.86	7.98	8.09	8.23	8.35	8.48	8.61	8.77	8.91
	mean	6.50	6.57	6.64	6.70	6.78	6.85	6.93	7.01	7.09	7.18	7.27	7.36	7.45	7.55	7.65	7.75	7.86	7.96	8.07	8.18

Table A.60. The forecasted U.K. male life expectancy by model 8

age 0	97.5%	79.83	80.35	80.91	81.43	81.98	82.52	83.09	83.66	84.23	84.81	85.38	85.96	86.54	87.12	87.71	88.26	88.82	89.35	89.86	90.38
	mean	79.15	79.59	80.05	80.52	81.00	81.50	82.00	82.51	83.03	83.56	84.10	84.63	85.18	85.72	86.25	86.78	87.31	87.82	88.33	88.82
	2.5%	78.40	78.75	79.10	79.47	79.89	80.30	80.72	81.17	81.61	82.02	82.50	82.97	83.46	83.96	84.43	84.93	85.38	85.84	86.31	86.74
1-4	97.5%	79.18	79.67	80.19	80.71	81.25	81.77	82.31	82.87	83.42	83.99	84.55	85.13	85.69	86.27	86.84	87.40	87.93	88.47	88.97	89.48
	mean	78.52	78.95	79.39	79.85	80.32	80.80	81.29	81.79	82.29	82.81	83.34	83.86	84.39	84.92	85.44	85.97	86.49	86.99	87.49	87.97
	2.5%	77.80	78.15	78.50	78.87	79.27	79.67	80.07	80.52	80.95	81.37	81.84	82.31	82.75	83.26	83.73	84.21	84.65	85.10	85.58	85.97
5-9	97.5%	75.23	75.72	76.24	76.75	77.28	77.80	78.34	78.90	79.44	80.01	80.57	81.15	81.71	82.28	82.85	83.41	83.94	84.48	84.98	85.50
	mean	74.57	75.00	75.44	75.90	76.36	76.84	77.33	77.83	78.33	78.85	79.37	79.90	80.43	80.96	81.47	82.00	82.52	83.02	83.51	84.00
	2.5%	73.86	74.22	74.57	74.93	75.33	75.73	76.15	76.59	77.01	77.45	77.90	78.36	78.82	79.32	79.79	80.26	80.70	81.16	81.63	82.03
10-14	97.5%	70.26	70.75	71.27	71.77	72.31	72.83	73.36	73.91	74.45	75.03	75.58	76.16	76.72	77.30	77.86	78.42	78.95	79.49	79.99	80.50
	mean	69.61	70.03	70.47	70.93	71.39	71.87	72.36	72.85	73.36	73.88	74.40	74.92	75.45	75.98	76.49	77.01	77.53	78.04	78.53	79.01
	2.5%	68.90	69.25	69.61	69.96	70.37	70.77	71.18	71.62	72.05	72.48	72.93	73.40	73.85	74.35	74.82	75.29	75.74	76.19	76.65	77.06
15-19	97.5%	65.29	65.78	66.29	66.80	67.33	67.84	68.38	68.93	69.48	70.04	70.60	71.18	71.74	72.31	72.87	73.43	73.96	74.50	75.00	75.51
	mean	64.64	65.06	65.50	65.95	66.42	66.90	67.38	67.88	68.38	68.90	69.42	69.94	70.47	70.99	71.51	72.03	72.55	73.05	73.54	74.03
	2.5%	63.93	64.29	64.64	65.00	65.40	65.80	66.22	66.66	67.08	67.51	67.97	68.42	68.88	69.37	69.84	70.31	70.76	71.21	71.67	72.09
20-24	97.5%	60.39	60.86	61.37	61.87	62.40	62.91	63.44	63.99	64.53	65.10	65.65	66.22	66.78	67.35	67.91	68.48	68.99	69.52	70.03	70.54
	mean	59.74	60.16	60.60	61.05	61.51	61.99	62.47	62.96	63.46	63.98	64.50	65.01	65.54	66.07	66.58	67.10	67.61	68.11	68.61	69.09
	2.5%	59.04	59.40	59.76	60.12	60.53	60.93	61.34	61.78	62.21	62.63	63.09	63.54	64.00	64.48	64.95	65.42	65.87	66.30	66.79	67.18
25-29	97.5%	55.52	55.98	56.48	56.97	57.49	58.01	58.53	59.08	59.61	60.17	60.73	61.28	61.85	62.41	62.98	63.54	64.06	64.58	65.09	65.61
	mean	54.89	55.31	55.75	56.20	56.66	57.13	57.62	58.10	58.60	59.12	59.63	60.15	60.68	61.20	61.72	62.23	62.75	63.25	63.74	64.22
	2.5%	54.20	54.59	54.96	55.33	55.73	56.14	56.58	57.02	57.44	57.87	58.35	58.80	59.23	59.74	60.21	60.69	61.14	61.59	62.03	62.47
30-34	97.5%	50.67	51.12	51.62	52.10	52.61	53.11	53.65	54.18	54.72	55.27	55.82	56.38	56.94	57.49	58.06	58.61	59.15	59.66	60.18	60.68
	mean	50.06	50.48	50.91	51.36	51.82	52.29	52.78	53.27	53.77	54.28	54.79	55.31	55.84	56.36	56.87	57.39	57.90	58.40	58.89	59.37
	2.5%	49.39	49.79	50.16	50.55	50.96	51.38	51.80	52.25	52.68	53.11	53.61	54.07	54.51	55.03	55.50	55.96	56.45	56.88	57.34	57.75
35-39	97.5%	45.85	46.30	46.79	47.27	47.77	48.27	48.79	49.32	49.86	50.41	50.97	51.50	52.06	52.62	53.18	53.73	54.26	54.78	55.30	55.80
	mean	45.26	45.68	46.11	46.56	47.02	47.49	47.98	48.46	48.96	49.48	49.99	50.51	51.03	51.56	52.07	52.59	53.10	53.60	54.09	54.56
	2.5%	44.61	45.02	45.39	45.80	46.22	46.63	47.08	47.51	47.95	48.41	48.88	49.35	49.82	50.31	50.79	51.28	51.75	52.18	52.65	53.07
40-44	97.5%	41.11	41.56	42.03	42.51	43.02	43.48	44.01	44.54	45.08	45.62	46.16	46.72	47.28	47.81	48.38	48.92	49.46	49.98	50.48	50.98
	mean	40.54	40.96	41.39	41.84	42.30	42.77	43.26	43.75	44.24	44.76	45.27	45.79	46.31	46.84	47.35	47.87	48.38	48.88	49.36	49.84
	2.5%	39.92	40.33	40.73	41.12	41.55	41.97	42.42	42.86	43.33	43.77	44.26	44.72	45.21	45.68	46.17	46.65	47.12	47.56	48.03	48.48
45-49	97.5%	36.46	36.90	37.37	37.84	38.36	38.80	39.33	39.84	40.38	40.92	41.46	42.01	42.56	43.11	43.64	44.20	44.73	45.23	45.75	46.23
	mean	35.91	36.33	36.76	37.21	37.67	38.14	38.62	39.11	39.61	40.12	40.63	41.15	41.67	42.19	42.71	43.22	43.73	44.23	44.71	45.19
	2.5%	35.31	35.71	36.12	36.51	36.96	37.37	37.84	38.28	38.76	39.20	39.69	40.16	40.64	41.13	41.60	42.07	42.54	43.00	43.48	43.91
50-54	97.5%	31.91	32.33	32.79	33.25	33.75	34.20	34.72	35.23	35.76	36.30	36.83	37.36	37.91	38.46	38.98	39.52	40.04	40.55	41.05	41.54
	mean	31.37	31.78	32.21	32.65	33.11	33.57	34.05	34.53	35.03	35.53	36.04	36.55	37.07	37.58	38.09	38.60	39.10	39.59	40.07	40.54
	2.5%	30.78	31.19	31.59	31.99	32.44	32.83	33.30	33.76	34.22	34.67	35.14	35.61	36.08	36.56	37.03	37.49	37.97	38.42	38.88	39.33
55-59	97.5%	27.50	27.91	28.35	28.80	29.28	29.72	30.22	30.71	31.23	31.75	32.28	32.79	33.33	33.85	34.39	34.90	35.40	35.89	36.39	36.85
	mean	26.97	27.38	27.79	28.22	28.66	29.12	29.59	30.05	30.54	31.03	31.52	32.02	32.52	33.03	33.52	34.01	34.50	34.98	35.44	35.90
	2.5%	26.40	26.80	27.20	27.60	28.02	28.41	28.86	29.31	29.77	30.20	30.66	31.13	31.58	32.04	32.51	32.95	33.40	33.84	34.29	34.73
60-64	97.5%	23.21	23.60	24.01	24.43	24.88	25.31	25.77	26.25	26.74	27.22	27.72	28.22	28.73	29.23	29.73	30.23	30.71	31.18	31.65	32.08
	mean	22.72	23.10	23.49	23.89	24.31	24.74	25.18	25.63	26.08	26.55	27.02	27.49	27.97	28.45	28.92	29.39	29.85	30.30	30.74	31.17
	2.5%	22.19	22.57	22.93	23.31	23.72	24.11	24.51	24.93	25.36	25.78	26.21	26.65	27.09	27.50	27.94	28.37	28.80	29.22	29.64	30.04
65-69	97.5%	19.11	19.46	19.83	20.21	20.62	21.01	21.45	21.89	22.34	22.79	23.25	23.72	24.19	24.65	25.12	25.58	26.04	26.48	26.90	27.33
	mean	18.65	19.00	19.36	19.73	20.11	20.50	20.91	21.32	21.74	22.17	22.60	23.04	23.48	23.92	24.36	24.80	25.23	25.65	26.06	26.46
	2.5%	18.16	18.51	18.84	19.20	19.57	19.92	20.31	20.68	21.07	21.45	21.84	22.24	22.65	23.04	23.42	23.83	24.21	24.61	24.99	25.36
70-74	97.5%	15.25	15.54	15.87	16.21	16.56	16.91	17.30	17.68	18.07	18.47	18.90	19.32	19.74	20.16	20.58	21.00	21.42	21.82	22.21	22.60
	mean	14.83	15.13	15.44	15.76	16.10	16.44	16.80	17.16	17.53	17.91	18.30	18.69	19.08	19.48	19.87	20.26	20.65	21.03	21.40	21.77
	2.5%	14.38	14.69	14.97	15.27	15.58	15.90	16.23	16.56	16.91	17.22	17.56	17.92	18.27	18.61	18.97	19.32	19.65	20.02	20.36	20.68
75-79	97.5%	11.74	11.97	12.24	12.51	12.80	13.09	13.41	13.72	14.06	14.40	14.76	15.11	15.46	15.82	16.19	16.55</				

Table A.61. The forecasted U.S.A. female life expectancy by model 3

age	97.5%	81.96	82.21	82.46	82.58	82.75	82.82	82.90	82.99	83.06	83.16	83.23	83.31	83.36	83.42	83.53	83.54	83.57	83.67	83.70	83.75
0	mean	81.21	81.18	81.16	81.14	81.12	81.09	81.06	81.04	81.01	80.99	80.96	80.93	80.91	80.88	80.85	80.82	80.79	80.77	80.74	80.71
	2.5%	80.43	80.08	79.80	79.60	79.39	79.20	78.97	78.86	78.67	78.51	78.36	78.27	78.14	77.94	77.78	77.68	77.51	77.44	77.31	77.23
1-4	97.5%	81.34	81.58	81.80	81.92	82.07	82.15	82.22	82.30	82.36	82.47	82.54	82.61	82.65	82.71	82.82	82.84	82.85	82.95	82.98	83.02
	mean	80.64	80.61	80.59	80.58	80.55	80.53	80.50	80.48	80.46	80.43	80.41	80.38	80.36	80.33	80.31	80.28	80.25	80.23	80.21	80.18
	2.5%	79.90	79.58	79.31	79.15	78.94	78.77	78.58	78.49	78.31	78.16	78.02	77.91	77.81	77.60	77.47	77.37	77.27	77.14	77.05	76.98
5-9	97.5%	77.39	77.64	77.85	77.98	78.12	78.20	78.26	78.35	78.41	78.52	78.59	78.65	78.69	78.75	78.86	78.88	78.89	78.99	79.02	79.07
	mean	76.71	76.68	76.66	76.65	76.62	76.60	76.57	76.55	76.53	76.51	76.49	76.46	76.43	76.41	76.38	76.35	76.33	76.31	76.28	76.26
	2.5%	75.99	75.66	75.40	75.24	75.03	74.86	74.67	74.59	74.41	74.27	74.14	74.02	73.91	73.72	73.59	73.48	73.39	73.28	73.18	73.10
10-14	97.5%	72.43	72.67	72.89	73.01	73.15	73.23	73.29	73.38	73.44	73.55	73.62	73.68	73.72	73.78	73.89	73.91	73.92	74.02	74.04	74.09
	mean	71.74	71.72	71.70	71.69	71.66	71.64	71.61	71.59	71.57	71.55	71.52	71.50	71.47	71.45	71.42	71.39	71.37	71.35	71.32	71.30
	2.5%	71.03	70.71	70.45	70.29	70.08	69.91	69.73	69.63	69.46	69.33	69.19	69.07	68.97	68.78	68.65	68.55	68.45	68.34	68.24	68.16
15-19	97.5%	67.47	67.70	67.91	68.04	68.18	68.26	68.32	68.41	68.46	68.57	68.64	68.71	68.75	68.80	68.91	68.93	68.94	69.04	69.07	69.12
	mean	66.78	66.76	66.74	66.73	66.70	66.68	66.65	66.63	66.61	66.59	66.57	66.54	66.52	66.49	66.47	66.44	66.41	66.39	66.37	66.34
	2.5%	66.07	65.76	65.50	65.34	65.13	64.97	64.78	64.70	64.53	64.38	64.26	64.13	64.04	63.85	63.72	63.62	63.51	63.42	63.31	63.24
20-24	97.5%	62.55	62.78	62.99	63.11	63.25	63.33	63.39	63.47	63.53	63.64	63.71	63.77	63.81	63.86	63.97	64.00	64.00	64.12	64.12	64.18
	mean	61.88	61.85	61.83	61.82	61.80	61.77	61.75	61.73	61.70	61.68	61.66	61.64	61.61	61.59	61.56	61.53	61.51	61.49	61.47	61.44
	2.5%	61.17	60.87	60.61	60.46	60.25	60.09	59.91	59.83	59.66	59.51	59.40	59.27	59.18	58.99	58.87	58.77	58.67	58.56	58.47	58.39
25-29	97.5%	57.67	57.90	58.10	58.22	58.36	58.43	58.49	58.57	58.63	58.74	58.80	58.87	58.91	58.96	59.06	59.09	59.10	59.19	59.21	59.27
	mean	57.01	56.99	56.97	56.96	56.94	56.91	56.89	56.87	56.85	56.83	56.81	56.78	56.76	56.73	56.71	56.68	56.66	56.64	56.62	56.59
	2.5%	56.33	56.03	55.77	55.63	55.43	55.27	55.09	55.01	54.84	54.71	54.60	54.48	54.38	54.20	54.09	53.99	53.90	53.79	53.70	53.62
30-34	97.5%	52.82	53.04	53.24	53.35	53.49	53.56	53.62	53.70	53.75	53.86	53.92	53.99	54.02	54.08	54.17	54.20	54.21	54.30	54.33	54.38
	mean	52.18	52.15	52.14	52.12	52.10	52.08	52.05	52.03	52.01	51.99	51.97	51.95	51.93	51.90	51.88	51.85	51.83	51.81	51.79	51.77
	2.5%	51.51	51.22	50.97	50.84	50.64	50.49	50.31	50.23	50.07	49.95	49.83	49.72	49.62	49.44	49.33	49.25	49.15	49.04	48.97	48.89
35-39	97.5%	48.01	48.21	48.41	48.52	48.66	48.72	48.78	48.87	48.90	49.01	49.07	49.14	49.17	49.22	49.32	49.34	49.36	49.45	49.47	49.52
	mean	47.38	47.35	47.34	47.32	47.30	47.28	47.26	47.24	47.22	47.20	47.18	47.16	47.14	47.11	47.09	47.07	47.05	47.03	47.01	46.98
	2.5%	46.73	46.45	46.21	46.08	45.89	45.74	45.57	45.49	45.34	45.21	45.11	45.02	44.91	44.73	44.63	44.54	44.45	44.34	44.29	44.21
40-44	97.5%	43.23	43.43	43.62	43.73	43.86	43.92	43.98	44.06	44.11	44.21	44.27	44.32	44.35	44.41	44.51	44.52	44.54	44.63	44.65	44.71
	mean	42.62	42.60	42.59	42.57	42.55	42.53	42.51	42.49	42.47	42.45	42.43	42.41	42.39	42.37	42.35	42.32	42.30	42.29	42.26	42.24
	2.5%	42.00	41.72	41.50	41.37	41.19	41.05	40.88	40.80	40.66	40.54	40.44	40.34	40.23	40.08	39.97	39.90	39.81	39.71	39.64	39.57
45-49	97.5%	38.52	38.71	38.89	38.99	39.11	39.18	39.25	39.31	39.36	39.45	39.50	39.56	39.60	39.66	39.74	39.76	39.77	39.85	39.87	39.92
	mean	37.94	37.92	37.90	37.89	37.87	37.85	37.83	37.81	37.80	37.78	37.76	37.74	37.72	37.70	37.68	37.66	37.64	37.62	37.60	37.58
	2.5%	37.34	37.08	36.87	36.75	36.58	36.46	36.29	36.23	36.10	35.98	35.87	35.78	35.69	35.54	35.45	35.37	35.27	35.20	35.12	35.07
50-54	97.5%	33.93	34.11	34.27	34.38	34.49	34.55	34.60	34.67	34.73	34.81	34.86	34.92	34.94	35.00	35.08	35.10	35.11	35.18	35.20	35.24
	mean	33.38	33.36	33.35	33.33	33.32	33.30	33.28	33.26	33.25	33.23	33.22	33.20	33.18	33.16	33.14	33.12	33.10	33.09	33.07	33.05
	2.5%	32.82	32.57	32.38	32.27	32.11	31.99	31.84	31.76	31.65	31.55	31.46	31.35	31.28	31.15	31.04	30.98	30.89	30.82	30.75	30.71
55-59	97.5%	29.49	29.66	29.80	29.90	30.01	30.06	30.11	30.18	30.24	30.32	30.35	30.41	30.42	30.49	30.57	30.57	30.58	30.66	30.67	30.71
	mean	28.97	28.95	28.94	28.93	28.91	28.90	28.88	28.87	28.85	28.84	28.82	28.80	28.79	28.77	28.75	28.73	28.72	28.70	28.69	28.67
	2.5%	28.45	28.22	28.04	27.94	27.80	27.69	27.54	27.47	27.36	27.28	27.20	27.09	27.03	26.90	26.83	26.75	26.68	26.61	26.56	26.52
60-64	97.5%	25.19	25.34	25.46	25.56	25.67	25.71	25.77	25.81	25.88	25.94	25.98	26.03	26.06	26.11	26.18	26.19	26.20	26.27	26.28	26.31
	mean	24.70	24.69	24.68	24.67	24.65	24.64	24.62	24.61	24.60	24.58	24.57	24.55	24.54	24.52	24.51	24.49	24.48	24.46	24.45	24.43
	2.5%	24.23	24.01	23.85	23.75	23.63	23.52	23.40	23.34	23.24	23.16	23.09	22.99	22.94	22.82	22.75	22.68	22.61	22.55	22.51	22.47
65-69	97.5%	21.03	21.16	21.28	21.37	21.46	21.50	21.54	21.60	21.65	21.70	21.75	21.77	21.81	21.86	21.92	21.93	21.95	22.00	22.01	22.04
	mean	20.59	20.57	20.56	20.55	20.54	20.53	20.51	20.50	20.49	20.48	20.47	20.45	20.44	20.42	20.41	20.39	20.38	20.37	20.36	20.34
	2.5%	20.15	19.96	19.82	19.73	19.62	19.53	19.43	19.35	19.27	19.19	19.13	19.03	18.98	18.88	18.82	18.76	18.67	18.64	18.62	18.56
70-74	97.5%	17.06	17.18	17.28	17.36	17.43	17.48	17.53	17.56	17.61	17.66	17.69	17.74	17.75	17.80	17.85	17.86	17.87	17.92	17.97	17.96
	mean	16.67	16.66	16.65	16.65	16.63	16.62	16.61	16.60	16.59	16.58	16.57	16.56	16.54	16.53	16.52	16.50	16.49	16.48	16.47	16.46
	2.5%	16.29	16.13	16.00	15.92	15.83	15.74	15.65	15.61	15.52	15.47	15.39	15.32	15.27	15.19	15.15	15.08	15.01	14.98	14.95	14.91
75-79	97.5%	13.39	13.50	13.58	13.64	13.71	13.74	13.78	13.81	13.86	13.89	13.93	13.95	13.97	14.01	14.05	14.06	14.07	14.11	14.13	14.15
	mean	13.06	13.05	13.04	13.04	13.03	13.02	13.00	13.00	12.99	12.98	12.97	12.96	12.95	12.94	12.93	12.92	12.91	12.90	12.89	12.88
	2.5%	12.73	12.59	12.50	12.42	12.34	12.28	12.20	12.17	12.09	12.05	11.98	11.91	11.88	11.82	11.78	11.74	11.67	11.63	11.60	11.56
80-84	97.5%	10.07	10.15	10.22	10.27	10.32	10.35	10.38	10.40	10.43	10.46	10.48	10.51	10.53	10.55	10.58	10.60	10.61	10.63	10.66	10.67
	mean	9.80	9.80	9.79	9.78	9.78	9.77	9.76	9.75	9.74	9.73	9.73	9.72	9.71	9.70	9.69	9.68	9.68	9.67	9.66	9.65
	2.5%	9.54	9.43	9.35	9.30	9.23	9.18	9.11	9.09	9.03	8.99	8.93	8.90	8.86	8.82	8.79	8.74	8.71	8.67	8.64	8.61
85-89	97.5%	7.20	7.26	7.31	7.34	7.38	7.40	7.41	7.43	7.45	7.47	7.50	7.51	7.52	7.54	7.55	7.57	7.57	7.59	7.61	7.62
	mean	7.01	7.00	6.99	6.99	6.98	6.98	6.97	6.96	6.96	6.95	6.95	6.94	6.93	6.92	6.92	6.91	6.90	6.90	6.89	6.89
	2.5%</																				

Table A.62. The forecasted U.S.A. male life expectancy by model 3

age 0	97.5%	77.31	77.61	77.85	78.04	78.25	78.30	78.40	78.49	78.60	78.67	78.76	78.86	78.91	78.98	79.09	79.11	79.17	79.26	79.30	79.38
	mean	76.46	76.42	76.39	76.37	76.34	76.30	76.26	76.23	76.20	76.17	76.14	76.10	76.06	76.02	75.99	75.95	75.92	75.89	75.85	75.82
	2.5%	75.56	75.14	74.83	74.60	74.34	74.12	73.90	73.74	73.57	73.35	73.19	73.00	72.85	72.68	72.52	72.36	72.20	72.09	71.94	71.83
1-4	97.5%	76.75	77.04	77.28	77.44	77.62	77.68	77.77	77.86	77.96	78.03	78.12	78.21	78.26	78.32	78.42	78.44	78.49	78.59	78.62	78.71
	mean	75.95	75.92	75.89	75.87	75.84	75.81	75.77	75.74	75.71	75.68	75.65	75.62	75.58	75.55	75.51	75.48	75.45	75.42	75.38	75.35
	2.5%	75.11	74.72	74.43	74.23	73.97	73.77	73.57	73.44	73.27	73.06	72.89	72.75	72.59	72.44	72.27	72.12	72.02	71.89	71.74	71.67
5-9	97.5%	72.82	73.11	73.34	73.51	73.68	73.75	73.84	73.92	74.02	74.09	74.18	74.27	74.32	74.37	74.48	74.50	74.54	74.65	74.67	74.76
	mean	72.04	72.00	71.98	71.95	71.92	71.89	71.85	71.83	71.80	71.77	71.74	71.70	71.67	71.64	71.60	71.56	71.54	71.51	71.48	71.44
	2.5%	71.21	70.82	70.54	70.33	70.08	69.88	69.70	69.56	69.39	69.19	69.02	68.88	68.74	68.57	68.42	68.27	68.17	68.04	67.90	67.81
10-14	97.5%	67.87	68.15	68.38	68.55	68.72	68.78	68.87	68.96	69.05	69.12	69.21	69.30	69.35	69.40	69.50	69.53	69.57	69.67	69.70	69.79
	mean	67.08	67.05	67.02	67.00	66.97	66.94	66.90	66.87	66.84	66.82	66.79	66.75	66.72	66.68	66.65	66.61	66.59	66.56	66.53	66.49
	2.5%	66.26	65.88	65.60	65.39	65.14	64.94	64.76	64.62	64.46	64.26	64.09	63.95	63.82	63.65	63.50	63.35	63.25	63.11	62.98	62.90
15-19	97.5%	62.92	63.20	63.43	63.59	63.76	63.83	63.92	64.00	64.10	64.16	64.25	64.34	64.39	64.45	64.54	64.57	64.61	64.71	64.74	64.82
	mean	62.14	62.10	62.08	62.06	62.03	61.99	61.96	61.93	61.90	61.87	61.84	61.81	61.78	61.74	61.71	61.67	61.64	61.62	61.58	61.55
	2.5%	61.32	60.95	60.66	60.45	60.21	60.02	59.83	59.70	59.53	59.33	59.17	59.04	58.90	58.74	58.58	58.44	58.33	58.21	58.07	57.99
20-24	97.5%	58.10	58.37	58.59	58.75	58.93	58.99	59.08	59.16	59.26	59.32	59.41	59.49	59.54	59.60	59.69	59.72	59.75	59.86	59.89	59.96
	mean	57.33	57.30	57.28	57.25	57.23	57.19	57.16	57.13	57.10	57.08	57.05	57.01	56.98	56.95	56.92	56.88	56.85	56.83	56.80	56.77
	2.5%	56.54	56.17	55.89	55.69	55.46	55.27	55.09	54.95	54.79	54.61	54.44	54.31	54.19	54.02	53.88	53.73	53.62	53.49	53.37	53.27
25-29	97.5%	53.42	53.67	53.90	54.05	54.21	54.28	54.36	54.43	54.54	54.59	54.68	54.76	54.81	54.85	54.94	54.99	55.01	55.12	55.14	55.20
	mean	52.69	52.66	52.64	52.62	52.59	52.56	52.53	52.50	52.47	52.45	52.42	52.39	52.36	52.33	52.30	52.27	52.24	52.22	52.19	52.16
	2.5%	51.93	51.58	51.32	51.13	50.91	50.72	50.56	50.43	50.28	50.10	49.96	49.83	49.70	49.55	49.41	49.28	49.18	49.04	48.93	48.85
30-34	97.5%	48.77	49.01	49.22	49.36	49.52	49.59	49.67	49.74	49.84	49.88	49.97	50.05	50.10	50.15	50.23	50.27	50.29	50.39	50.40	50.47
	mean	48.06	48.03	48.01	47.99	47.97	47.94	47.91	47.88	47.86	47.83	47.81	47.78	47.75	47.72	47.69	47.66	47.64	47.61	47.58	47.56
	2.5%	47.33	47.00	46.74	46.57	46.37	46.19	46.03	45.91	45.77	45.60	45.45	45.33	45.21	45.06	44.93	44.80	44.70	44.56	44.49	44.40
35-39	97.5%	44.11	44.34	44.53	44.67	44.84	44.88	44.95	45.02	45.12	45.16	45.25	45.32	45.37	45.42	45.49	45.53	45.55	45.65	45.65	45.73
	mean	43.43	43.41	43.39	43.37	43.35	43.32	43.29	43.27	43.25	43.22	43.20	43.17	43.15	43.12	43.10	43.07	43.04	43.02	43.00	42.97
	2.5%	42.74	42.43	42.20	42.04	41.87	41.68	41.54	41.43	41.29	41.13	41.00	40.89	40.77	40.66	40.53	40.42	40.32	40.18	40.08	40.04
40-44	97.5%	39.46	39.67	39.85	39.98	40.14	40.19	40.26	40.33	40.43	40.47	40.55	40.62	40.66	40.70	40.78	40.82	40.85	40.94	40.95	41.02
	mean	38.82	38.80	38.78	38.77	38.74	38.72	38.69	38.68	38.65	38.63	38.61	38.59	38.56	38.54	38.52	38.49	38.47	38.45	38.43	38.40
	2.5%	38.18	37.88	37.67	37.53	37.36	37.20	37.05	36.95	36.82	36.68	36.54	36.46	36.36	36.24	36.13	36.01	35.94	35.83	35.75	35.67
45-49	97.5%	34.88	35.09	35.26	35.39	35.52	35.59	35.65	35.71	35.80	35.85	35.92	35.98	36.01	36.06	36.15	36.18	36.19	36.28	36.30	36.35
	mean	34.28	34.26	34.24	34.23	34.21	34.19	34.16	34.14	34.12	34.10	34.09	34.06	34.04	34.02	34.00	33.97	33.95	33.93	33.91	33.89
	2.5%	33.67	33.40	33.19	33.06	32.90	32.76	32.60	32.53	32.40	32.27	32.16	32.07	31.96	31.86	31.73	31.65	31.57	31.50	31.41	31.35
50-54	97.5%	30.46	30.66	30.81	30.93	31.08	31.13	31.19	31.25	31.33	31.37	31.44	31.48	31.53	31.58	31.66	31.69	31.70	31.79	31.81	31.85
	mean	29.90	29.88	29.86	29.85	29.83	29.81	29.79	29.77	29.75	29.73	29.72	29.69	29.67	29.65	29.63	29.61	29.59	29.57	29.55	29.53
	2.5%	29.33	29.07	28.87	28.75	28.58	28.46	28.35	28.25	28.13	28.01	27.92	27.83	27.74	27.62	27.53	27.44	27.38	27.29	27.23	27.17
55-59	97.5%	26.28	26.45	26.61	26.71	26.84	26.90	26.95	27.00	27.07	27.12	27.17	27.23	27.28	27.30	27.39	27.41	27.42	27.50	27.52	27.55
	mean	25.75	25.73	25.72	25.71	25.69	25.67	25.65	25.63	25.61	25.60	25.58	25.56	25.54	25.52	25.50	25.48	25.47	25.45	25.43	25.41
	2.5%	25.22	24.98	24.80	24.69	24.53	24.43	24.30	24.23	24.12	24.01	23.92	23.83	23.75	23.65	23.55	23.49	23.41	23.36	23.27	23.22
60-64	97.5%	22.33	22.49	22.62	22.73	22.84	22.90	22.94	22.98	23.04	23.10	23.15	23.19	23.23	23.26	23.33	23.36	23.37	23.45	23.46	23.50
	mean	21.84	21.83	21.81	21.80	21.79	21.77	21.75	21.74	21.72	21.71	21.69	21.67	21.66	21.64	21.62	21.60	21.59	21.57	21.56	21.54
	2.5%	21.35	21.14	20.98	20.88	20.74	20.63	20.52	20.45	20.35	20.27	20.18	20.11	20.04	19.94	19.85	19.79	19.72	19.66	19.61	19.55
65-69	97.5%	18.59	18.73	18.85	18.94	19.04	19.09	19.13	19.17	19.24	19.26	19.32	19.35	19.40	19.41	19.48	19.51	19.52	19.58	19.60	19.62
	mean	18.15	18.14	18.13	18.12	18.10	18.09	18.07	18.06	18.04	18.03	18.02	18.00	17.99	17.97	17.95	17.94	17.92	17.91	17.90	17.88
	2.5%	17.71	17.53	17.40	17.29	17.17	17.07	16.97	16.92	16.82	16.74	16.67	16.59	16.56	16.46	16.37	16.33	16.28	16.21	16.17	16.13
70-74	97.5%	15.03	15.15	15.25	15.33	15.41	15.46	15.48	15.52	15.57	15.60	15.65	15.68	15.71	15.74	15.79	15.81	15.82	15.88	15.90	15.93
	mean	14.64	14.63	14.62	14.61	14.60	14.58	14.57	14.56	14.54	14.53	14.52	14.51	14.49	14.48	14.47	14.45	14.44	14.43	14.42	14.40
	2.5%	14.25	14.10	13.98	13.89	13.79	13.71	13.62	13.56	13.49	13.43	13.36	13.30	13.24	13.17	13.10	13.06	13.01	12.97	12.92	12.88
75-79	97.5%	11.77	11.86	11.94	12.02	12.08	12.11	12.14	12.17	12.22	12.24	12.28	12.31	12.32	12.35	12.39	12.41				

Table A.63. The forecasted U.S.A. female life expectancy by model 8

age 0	97.5%	82.52	82.82	83.14	83.41	83.72	83.94	84.17	84.43	84.71	84.96	85.25	85.48	85.75	86.01	86.24	86.47	86.76	87.02	87.27	87.47
	mean	81.73	81.89	82.06	82.23	82.41	82.59	82.76	82.94	83.12	83.31	83.48	83.66	83.85	84.04	84.22	84.41	84.60	84.79	84.99	85.18
	2.5%	80.88	80.86	80.91	80.99	81.07	81.12	81.16	81.24	81.35	81.43	81.49	81.57	81.68	81.80	81.90	82.04	82.19	82.23	82.40	82.55
1-4	97.5%	81.92	82.21	82.49	82.76	83.02	83.23	83.46	83.69	83.96	84.19	84.46	84.71	84.94	85.20	85.42	85.65	85.91	86.19	86.44	86.64
	mean	81.16	81.30	81.47	81.63	81.79	81.95	82.11	82.29	82.46	82.63	82.79	82.97	83.14	83.32	83.50	83.68	83.86	84.04	84.23	84.41
	2.5%	80.35	80.31	80.39	80.44	80.53	80.55	80.59	80.66	80.77	80.84	80.88	80.97	81.06	81.20	81.29	81.43	81.56	81.61	81.73	81.87
5-9	97.5%	77.99	78.27	78.54	78.81	79.07	79.27	79.49	79.73	80.00	80.22	80.48	80.74	80.97	81.23	81.44	81.67	81.93	82.20	82.46	82.66
	mean	77.22	77.37	77.53	77.68	77.84	78.01	78.16	78.34	78.51	78.68	78.84	79.01	79.18	79.36	79.53	79.71	79.89	80.07	80.26	80.44
	2.5%	76.41	76.38	76.46	76.51	76.59	76.61	76.66	76.73	76.84	76.91	76.94	77.03	77.12	77.25	77.35	77.48	77.60	77.67	77.77	77.91
10-14	97.5%	73.02	73.30	73.57	73.83	74.09	74.29	74.51	74.75	75.01	75.23	75.50	75.75	75.99	76.24	76.45	76.69	76.94	77.21	77.46	77.67
	mean	72.26	72.40	72.56	72.71	72.87	73.03	73.19	73.36	73.53	73.70	73.86	74.03	74.20	74.38	74.55	74.73	74.91	75.09	75.27	75.45
	2.5%	71.45	71.42	71.50	71.54	71.62	71.65	71.70	71.76	71.87	71.94	71.97	72.06	72.14	72.28	72.38	72.51	72.63	72.69	72.80	72.93
15-19	97.5%	68.05	68.32	68.59	68.86	69.11	69.31	69.54	69.77	70.03	70.26	70.52	70.77	71.01	71.26	71.47	71.70	71.95	72.24	72.48	72.68
	mean	67.29	67.43	67.59	67.74	67.90	68.06	68.22	68.39	68.56	68.72	68.88	69.05	69.22	69.40	69.57	69.75	69.93	70.11	70.29	70.47
	2.5%	66.49	66.45	66.53	66.58	66.66	66.69	66.73	66.80	66.90	66.97	67.00	67.09	67.17	67.31	67.42	67.54	67.66	67.72	67.83	67.97
20-24	97.5%	63.15	63.41	63.69	63.94	64.20	64.39	64.61	64.84	65.10	65.33	65.58	65.83	66.07	66.31	66.53	66.75	67.00	67.27	67.53	67.73
	mean	62.40	62.54	62.69	62.85	63.00	63.16	63.32	63.49	63.65	63.82	63.98	64.15	64.31	64.49	64.66	64.84	65.02	65.19	65.37	65.55
	2.5%	61.60	61.57	61.65	61.70	61.78	61.81	61.85	61.94	62.03	62.08	62.13	62.22	62.30	62.44	62.54	62.66	62.80	62.84	62.96	63.09
25-29	97.5%	58.27	58.52	58.79	59.04	59.30	59.49	59.70	59.93	60.19	60.42	60.67	60.91	61.15	61.39	61.62	61.83	62.08	62.35	62.59	62.80
	mean	57.53	57.67	57.82	57.98	58.13	58.29	58.44	58.61	58.77	58.94	59.10	59.26	59.43	59.60	59.77	59.95	60.13	60.30	60.48	60.66
	2.5%	56.75	56.72	56.80	56.85	56.93	56.98	57.01	57.11	57.20	57.23	57.29	57.39	57.46	57.62	57.68	57.82	57.97	58.02	58.14	58.28
30-34	97.5%	53.41	53.66	53.93	54.16	54.43	54.61	54.81	55.04	55.31	55.52	55.77	56.00	56.24	56.47	56.71	56.92	57.18	57.45	57.68	57.88
	mean	52.69	52.82	52.97	53.12	53.28	53.43	53.59	53.75	53.91	54.08	54.24	54.40	54.56	54.74	54.91	55.08	55.26	55.43	55.61	55.79
	2.5%	51.90	51.89	51.97	52.03	52.11	52.16	52.19	52.30	52.39	52.42	52.48	52.57	52.66	52.82	52.86	53.01	53.17	53.21	53.34	53.48
35-39	97.5%	48.59	48.82	49.09	49.32	49.58	49.77	49.96	50.17	50.44	50.65	50.90	51.13	51.35	51.59	51.83	52.03	52.29	52.55	52.77	52.99
	mean	47.87	48.01	48.16	48.31	48.46	48.61	48.76	48.93	49.09	49.25	49.40	49.57	49.73	49.90	50.07	50.24	50.42	50.59	50.77	50.94
	2.5%	47.10	47.10	47.16	47.25	47.32	47.39	47.43	47.51	47.61	47.63	47.70	47.81	47.90	48.07	48.11	48.25	48.38	48.44	48.54	48.68
40-44	97.5%	43.82	44.03	44.29	44.52	44.76	44.97	45.15	45.36	45.63	45.83	46.06	46.29	46.52	46.76	46.97	47.17	47.45	47.70	47.91	48.11
	mean	43.12	43.25	43.40	43.54	43.69	43.84	43.99	44.15	44.31	44.47	44.63	44.79	44.95	45.12	45.28	45.45	45.62	45.79	45.97	46.14
	2.5%	42.38	42.39	42.45	42.53	42.60	42.67	42.71	42.79	42.92	42.93	43.01	43.09	43.20	43.33	43.39	43.54	43.70	43.73	43.85	43.99
45-49	97.5%	39.09	39.32	39.54	39.76	40.00	40.20	40.39	40.57	40.82	41.03	41.26	41.49	41.72	41.94	42.15	42.36	42.63	42.88	43.07	43.27
	mean	38.40	38.54	38.68	38.82	38.97	39.12	39.26	39.42	39.58	39.73	39.88	40.04	40.20	40.37	40.53	40.69	40.86	41.03	41.20	41.37
	2.5%	37.69	37.70	37.77	37.86	37.92	38.01	38.05	38.13	38.24	38.27	38.37	38.43	38.53	38.68	38.75	38.88	39.01	39.08	39.21	39.36
50-54	97.5%	34.50	34.70	34.91	35.13	35.37	35.53	35.74	35.93	36.16	36.36	36.59	36.80	37.04	37.23	37.44	37.66	37.90	38.15	38.35	38.55
	mean	33.84	33.98	34.12	34.25	34.40	34.54	34.68	34.84	34.99	35.14	35.29	35.45	35.60	35.77	35.93	36.09	36.26	36.42	36.59	36.75
	2.5%	33.15	33.18	33.25	33.36	33.41	33.49	33.55	33.61	33.75	33.79	33.88	33.94	34.08	34.19	34.27	34.39	34.52	34.59	34.74	34.87
55-59	97.5%	30.07	30.26	30.47	30.67	30.89	31.07	31.25	31.45	31.65	31.85	32.06	32.26	32.47	32.70	32.90	33.12	33.34	33.57	33.78	33.96
	mean	29.45	29.58	29.72	29.85	29.99	30.13	30.27	30.42	30.57	30.72	30.87	31.02	31.17	31.33	31.48	31.64	31.81	31.97	32.13	32.29
	2.5%	28.80	28.85	28.93	29.01	29.08	29.15	29.23	29.32	29.43	29.47	29.53	29.63	29.73	29.87	29.96	30.09	30.22	30.28	30.38	30.52
60-64	97.5%	25.77	25.96	26.15	26.35	26.55	26.71	26.90	27.07	27.25	27.46	27.64	27.84	28.05	28.26	28.45	28.65	28.88	29.10	29.27	29.47
	mean	25.20	25.32	25.45	25.58	25.71	25.85	25.98	26.13	26.26	26.41	26.55	26.70	26.84	27.00	27.14	27.30	27.45	27.60	27.76	27.92
	2.5%	24.58	24.63	24.71	24.80	24.85	24.96	25.02	25.11	25.21	25.26	25.34	25.44	25.52	25.66	25.74	25.85	25.96	26.04	26.14	26.28
65-69	97.5%	21.60	21.77	21.94	22.13	22.32	22.45	22.62	22.79	22.94	23.13	23.31	23.48	23.69	23.88	24.04	24.24	24.44	24.65	24.81	24.99
	mean	21.06	21.17	21.29	21.41	21.53	21.65	21.78	21.91	22.03	22.16	22.29	22.43	22.56	22.70	22.84	22.98	23.12	23.26	23.41	23.55
	2.5%	20.46	20.53	20.61	20.68	20.75	20.79	20.90	20.97	21.06	21.12	21.17	21.27	21.35	21.46	21.51	21.66	21.73	21.83	21.92	22.03
70-74	97.5%	17.62	17.76	17.91	18.06	18.24	18.34	18.50	18.66	18.77	18.95	19.10	19.28	19.46	19.62	19.78	19.95	20.14	20.28	20.46	20.62
	mean	17.11	17.21	17.31	17.42	17.52	17.63	17.75	17.86	17.97	18.09	18.20	18.32	18.44	18.57	18.69	18.81	18.95	19.07	19.20	19.33
	2.5%	16.56	16.63	16.69	16.76	16.82	16.87	16.96	17.03	17.11	17.15	17.19	17.29	17.40	17.46	17.54	17.66	17.72	17.80	17.88	17.97
75-79	97.5%	13.89	14.02	14.13	14.27	14.41	14.51	14.64	14.76	14.88	15.04	15.15	15.29	15.48	15.60	15.76	15.90	16.03	16.20	16.33	16.48

Table A.64. The forecasted U.S.A. male life expectancy by model 8

age 0	97.5%	77.74	78.16	78.54	78.98	79.34	79.74	80.14	80.53	80.92	81.35	81.80	82.23	82.66	83.06	83.56	84.00	84.42	84.90	85.34	85.81
	mean	76.83	77.10	77.37	77.65	77.94	78.23	78.53	78.84	79.16	79.48	79.81	80.15	80.50	80.85	81.20	81.58	81.95	82.33	82.72	83.11
	2.5%	75.84	75.98	76.04	76.21	76.40	76.55	76.73	76.93	77.12	77.33	77.56	77.82	78.07	78.28	78.48	78.78	79.01	79.40	79.69	79.98
1-4	97.5%	77.21	77.60	77.96	78.37	78.73	79.10	79.48	79.85	80.24	80.64	81.07	81.49	81.91	82.31	82.79	83.21	83.63	84.09	84.53	84.99
	mean	76.33	76.59	76.85	77.12	77.39	77.67	77.95	78.25	78.56	78.87	79.19	79.51	79.85	80.19	80.53	80.89	81.25	81.63	82.00	82.38
	2.5%	75.37	75.51	75.60	75.75	75.91	76.07	76.24	76.44	76.64	76.84	77.05	77.30	77.55	77.75	77.96	78.25	78.49	78.78	79.16	79.40
5-9	97.5%	73.28	73.66	74.02	74.43	74.78	75.16	75.54	75.90	76.28	76.68	77.11	77.53	77.94	78.34	78.82	79.24	79.65	80.12	80.55	81.01
	mean	72.41	72.67	72.92	73.19	73.46	73.74	74.02	74.31	74.62	74.93	75.24	75.56	75.90	76.24	76.58	76.94	77.30	77.67	78.04	78.42
	2.5%	71.46	71.60	71.68	71.84	71.99	72.16	72.33	72.51	72.73	72.91	73.11	73.37	73.63	73.82	74.02	74.32	74.56	74.85	75.22	75.45
10-14	97.5%	68.32	68.70	69.05	69.46	69.82	70.19	70.56	70.93	71.30	71.70	72.13	72.55	72.96	73.35	73.83	74.26	74.67	75.13	75.57	76.02
	mean	67.46	67.71	67.96	68.23	68.50	68.77	69.05	69.35	69.65	69.96	70.27	70.59	70.93	71.26	71.60	71.96	72.32	72.69	73.06	73.44
	2.5%	66.50	66.64	66.73	66.88	67.04	67.20	67.37	67.55	67.77	67.95	68.16	68.41	68.67	68.85	69.06	69.35	69.59	69.89	70.26	70.48
15-19	97.5%	63.37	63.74	64.10	64.50	64.85	65.22	65.59	65.96	66.33	66.73	67.16	67.57	67.98	68.37	68.85	69.28	69.69	70.15	70.58	71.04
	mean	62.50	62.76	63.00	63.27	63.54	63.81	64.09	64.38	64.68	64.99	65.31	65.62	65.96	66.29	66.63	66.99	67.35	67.72	68.09	68.47
	2.5%	61.55	61.70	61.78	61.93	62.09	62.25	62.43	62.60	62.81	62.99	63.20	63.45	63.71	63.90	64.10	64.38	64.62	64.92	65.29	65.52
20-24	97.5%	58.54	58.91	59.27	59.64	59.99	60.35	60.72	61.09	61.45	61.84	62.27	62.68	63.07	63.46	63.95	64.37	64.77	65.23	65.66	66.11
	mean	57.69	57.94	58.18	58.44	58.71	58.98	59.25	59.54	59.84	60.14	60.45	60.76	61.09	61.42	61.76	62.12	62.47	62.83	63.20	63.58
	2.5%	56.74	56.89	56.97	57.13	57.27	57.45	57.62	57.78	58.00	58.19	58.40	58.65	58.89	59.06	59.27	59.57	59.78	60.10	60.45	60.68
25-29	97.5%	53.85	54.22	54.54	54.91	55.25	55.62	55.96	56.34	56.71	57.08	57.50	57.92	58.31	58.67	59.16	59.55	59.98	60.42	60.85	61.30
	mean	53.03	53.28	53.52	53.77	54.04	54.30	54.57	54.86	55.16	55.46	55.77	56.07	56.40	56.73	57.07	57.42	57.77	58.13	58.50	58.87
	2.5%	52.12	52.29	52.37	52.53	52.69	52.83	53.05	53.21	53.40	53.59	53.84	54.06	54.31	54.51	54.74	55.02	55.24	55.53	55.90	56.11
30-34	97.5%	49.20	49.55	49.86	50.23	50.56	50.93	51.27	51.63	51.98	52.39	52.78	53.17	53.58	53.95	54.40	54.81	55.24	55.66	56.09	56.54
	mean	48.40	48.64	48.88	49.14	49.40	49.67	49.94	50.22	50.52	50.82	51.13	51.44	51.76	52.09	52.43	52.78	53.13	53.49	53.86	54.23
	2.5%	47.52	47.67	47.78	47.93	48.11	48.28	48.48	48.66	48.85	49.07	49.29	49.52	49.76	49.96	50.20	50.50	50.74	51.06	51.37	51.61
35-39	97.5%	44.54	44.88	45.19	45.53	45.85	46.22	46.54	46.91	47.26	47.65	48.04	48.41	48.83	49.19	49.63	50.05	50.47	50.89	51.32	51.77
	mean	43.76	44.00	44.25	44.50	44.77	45.03	45.31	45.59	45.89	46.19	46.50	46.81	47.14	47.47	47.80	48.16	48.51	48.87	49.24	49.61
	2.5%	42.91	43.09	43.20	43.37	43.56	43.74	43.97	44.15	44.37	44.59	44.78	45.03	45.30	45.54	45.76	46.06	46.26	46.63	46.97	47.22
40-44	97.5%	39.88	40.20	40.52	40.85	41.17	41.52	41.84	42.20	42.55	42.92	43.31	43.67	44.08	44.46	44.89	45.30	45.71	46.15	46.56	47.02
	mean	39.14	39.38	39.62	39.88	40.14	40.40	40.67	40.96	41.26	41.56	41.87	42.17	42.50	42.83	43.17	43.52	43.87	44.23	44.60	44.97
	2.5%	38.32	38.51	38.64	38.81	38.99	39.22	39.42	39.60	39.83	40.06	40.27	40.51	40.77	41.01	41.24	41.55	41.82	42.15	42.48	42.73
45-49	97.5%	35.27	35.58	35.88	36.20	36.50	36.87	37.17	37.50	37.86	38.22	38.61	38.96	39.35	39.74	40.15	40.56	40.95	41.40	41.80	42.24
	mean	34.54	34.78	35.01	35.26	35.52	35.78	36.04	36.33	36.62	36.91	37.21	37.52	37.84	38.16	38.49	38.84	39.19	39.54	39.90	40.27
	2.5%	33.73	33.94	34.06	34.25	34.43	34.63	34.82	35.04	35.26	35.49	35.68	35.95	36.16	36.43	36.65	36.96	37.23	37.55	37.85	38.11
50-54	97.5%	30.85	31.14	31.43	31.73	32.03	32.36	32.66	33.00	33.33	33.70	34.05	34.42	34.79	35.18	35.57	35.98	36.37	36.79	37.20	37.60
	mean	30.13	30.37	30.59	30.84	31.09	31.34	31.60	31.88	32.16	32.45	32.74	33.04	33.36	33.67	34.00	34.34	34.68	35.03	35.38	35.74
	2.5%	29.34	29.55	29.68	29.88	30.05	30.26	30.46	30.65	30.85	31.07	31.31	31.53	31.77	32.02	32.24	32.54	32.83	33.10	33.41	33.69
55-59	97.5%	26.71	26.99	27.25	27.54	27.82	28.15	28.44	28.75	29.09	29.44	29.81	30.15	30.50	30.88	31.28	31.67	32.08	32.48	32.85	33.27
	mean	26.03	26.25	26.48	26.71	26.96	27.21	27.46	27.73	28.01	28.29	28.58	28.88	29.18	29.50	29.81	30.15	30.48	30.83	31.18	31.53
	2.5%	25.28	25.47	25.62	25.80	25.97	26.18	26.41	26.59	26.81	27.01	27.22	27.47	27.72	27.94	28.18	28.46	28.72	29.02	29.30	29.61
60-64	97.5%	22.83	23.10	23.36	23.63	23.91	24.22	24.51	24.82	25.13	25.46	25.81	26.17	26.51	26.87	27.24	27.63	28.00	28.43	28.78	29.20
	mean	22.21	22.43	22.65	22.88	23.12	23.36	23.61	23.88	24.15	24.43	24.71	25.00	25.31	25.61	25.92	26.25	26.58	26.92	27.26	27.61
	2.5%	21.53	21.72	21.88	22.07	22.25	22.46	22.67	22.86	23.08	23.29	23.52	23.73	24.00	24.24	24.49	24.77	24.99	25.30	25.58	25.86
65-69	97.5%	19.10	19.35	19.59	19.85	20.11	20.36	20.66	20.95	21.24	21.54	21.86	22.20	22.52	22.87	23.22	23.58	23.92	24.32	24.67	25.06
	mean	18.54	18.75	18.95	19.17	19.39	19.62	19.86	20.11	20.36	20.62	20.89	21.17	21.45	21.74	22.04	22.35	22.66	22.98	23.30	23.63
	2.5%	17.90	18.11	18.26	18.44	18.62	18.83	19.00	19.21	19.41	19.63	19.82	20.04	20.27	20.51	20.72	21.00	21.23	21.52	21.78	22.02
70-74	97.5%	15.52	15.74	15.94	16.17	16.39	16.64	16.89	17.17	17.41	17.67	17.97	18.26	18.56	18.88	19.19	19.53	19.85	20.19	20.53	20.89
	mean	15.01	15.20	15.38	15.57	15.77	15.97	16.19	16.41	16.63	16.87	17.11	17.36	17.61	17.87	18.14	18.42	18.70	18.99	19.28	19.58
	2.5%	14.46	14.62	14.76	14.92	15.09	15.25	15.41	15.59	15.77	15.97	16.15	16.36	16.51	16.74	16.92	17.19	17.38	17.63	17.87	18.11
75-79	97.5%	12.23	12.41	12.60	12.78	12.95	13.17	13.40	13.62	13.85	14.07	14.33	14.57	14.85	15.12	15.42	15.70	1			

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